Avoidable Sets in The Bicyclic Inverse Semigroup

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Abstract

A subset U of a set S with a binary operation is called *avoidable* if S can be partitioned into two subsets A and B such that no element of U can be written as a product of two distinct elements of A or as the product of two distinct elements of B. The avoidable sets of the bicyclic inverse semigroup are classified.

1. Introduction

If $(S, \cdot)$ is a set with a binary operation then a subset U of S is called *avoidable* if S can be partitioned into two subsets A and B such that the partition *avoids* U, that is, no element of U can be written as a product of two distinct elements of A or as the product of two distinct elements of B. Avoidable sets in $(\mathbf{N}, +)$ were first introduced by Alladi, Erdős and Hoggatt [AEH] and further studied in [Eva, Hog, HB, SZ, CL, ZC, Gra, Dum, De1]. Avoidable sets in groups were investigated in [De2].

In this paper we initiate the study of avoidable sets in inverse semigroups. An inverse semigroup is a semigroup such that every element s has a unique adjoint s^* satisfying $ss^*s = s$ and $s^*ss^* = s^*$. Inverse semigroups were first studied by Vagner [Vag] and Preston [Pr1] who considered inverse semigroups as the most promising class of semigroups for study. An inverse semigroup is the next best thing to having an actual group. While a group can be represented as bijections on a set, an inverse semigroup can be represented as partial bijections on a set. In fact, a group is an inverse semigroup with a single idempotent. A comprehensive reference for inverse semigroups is [Pet].

As a starting point of this study, we classify the maximal avoidable sets in the bicyclic inverse semigroup, which is perhaps the most important inverse semigroup. Its role in semigroup theory is similar to the role of $\mathbf{Z}$ in group theory. It is one of the basic building blocks [Pr2] of the monogenic inverse semigroups, that is, inverse semigroups generated by single elements. A possible continuation of our study could consider the other building blocks of monogenic inverse semigroups, in particular, the inverse semigroups generated by finite forward shifts.

The *bicyclic inverse semigroup* $\mathcal{B}$ is the set

$$\mathcal{B} = \{(a, b) \mid a \geq 0, a + b \geq 0\} \subseteq \mathbf{Z} \times \mathbf{Z}$$

equipped with the following multiplication and inverse

$$(a, b)(c, d) = (\max\{c + d, a\} - d, b + d), \qquad (a, b)^* = (a + b, -b).$$

Note that $\mathcal{B}$ can be represented as a semigroup of partial bijections of the nonnegative integers where the element (a, b) is represented by the shift of the set $\{n \in \mathbf{Z} \mid n \geq a\}$ by b. Note that $\mathcal{B}$ is the inverse semigroup generated by the element $(0, 1)$.

Given $U \subseteq S$ the *associated graph* $G_{S,U}$ is the graph whose vertex set is S and two vertices r and s are connected by an edge if $rs \in U$ or $sr \in U$. Then U is avoidable in S exactly when $G_{S,U}$ is bipartite. So to show that a set U is unavoidable in S, it suffices to find a cycle in $G_{S,U}$ with odd length.

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2. Even elements

An element (a,b) of $\mathcal{B}$ is called *even* if b is even, and *odd* if b is odd. Since the operation on $\mathcal{B}$ is written multiplicatively, it might be more appropriate, but less descriptive, to call even elements perfect squares. It is clear that a product is odd exactly when the factors have different parity. This shows that the set of odd elements is avoidable since the partition of $\mathcal{B}$ separating even and odd elements avoids the set off odd elements. Only special even elements of $\mathcal{B}$ can be in an avoidable set as the following proposition shows.

Proposition 2.1. *If $(a,b) \in \mathcal{B}$ is even such that $a \geq 1$, $a+b \geq 1$ and $b \neq 0$ then $U = \{(a,b)\}$ is unavoidable.*

Proof. Let

$$p = \left(\alpha, \frac{b}{2}\right), \quad q = \left(a, \frac{b}{2}\right), \quad r = \left(a + \frac{b}{2}, \frac{b}{2}\right)$$

where α will be chosen later. Then $q \neq r$ since $b \neq 0$. Also

$$rq = \left(a + \frac{b}{2}, \frac{b}{2}\right)\left(a, \frac{b}{2}\right) = \left(\max\left\{a + \frac{b}{2}, a + \frac{b}{2}\right\} - \frac{b}{2}, b\right) = (a,b) \in U.$$

If $b \geq 2$ then let $\alpha = a - 1$. It is clear that $p \in \mathcal{B}$. Then

$$pq = \left(a-1, \frac{b}{2}\right)\left(a, \frac{b}{2}\right) = \left(\max\left\{a + \frac{b}{2}, a - 1\right\} - \frac{b}{2}, b\right) = (a,b) \in U,$$

$$rp = \left(a + \frac{b}{2}, \frac{b}{2}\right)\left(a-1, \frac{b}{2}\right) = \left(\max\left\{a - 1 + \frac{b}{2}, a + \frac{b}{2}\right\} - \frac{b}{2}, b\right) = (a,b) \in U.$$

It is clear that $p \neq q$, and since $b \neq -2$ we have $r \neq p$.

If $b \leq -2$ then let $\alpha = a + \frac{b}{2} - 1$. Then $a \geq -b \geq 2$ and so $a + b \geq 0 \geq -a + 2$ which implies that $\alpha \geq 0$. Since $a + b \geq 1$, we also have $\alpha + \frac{b}{2} \geq 0$ and so $p \in \mathcal{B}$. Then

$$pq = \left(a + \frac{b}{2} - 1, \frac{b}{2}\right)\left(a, \frac{b}{2}\right) = \left(\max\left\{a + \frac{b}{2}, a + \frac{b}{2} - 1\right\} - \frac{b}{2}, b\right) = (a,b),$$

and since $b \leq -2$, we have $a + \frac{b}{2} \geq b - 1$ which implies

$$rp = \left(a + \frac{b}{2}, \frac{b}{2}\right)\left(a + \frac{b}{2} - 1, \frac{b}{2}\right) = \left(\max\left\{a + b - 1, a + \frac{b}{2}\right\} - \frac{b}{2}, b\right) = (a,b).$$

It is clear that $p \neq r$ and since $b \neq 2$ we have $p \neq q$.

In either case $\{p,q,r\}$ forms a triangle in $G_{\mathcal{B},U}$. $\square$

The following figure shows the even elements of $\mathcal{B}$ that are not impossible in an avoidable set:

$$\begin{array}{ccccccc}
 & & & 1_{\mathcal{B}} & \mathit{(0,2)} & \mathit{(0,4)} & \cdots \\
 & & & (1,0) & \times & \times & \cdots \\
 & & \mathbf{(2,-2)} & (2,0) & \times & \times & \\
 & & \times & (3,0) & \times & \times & \\
 & \mathbf{(4,-4)} & \times & (4,0) & \times & \times & \\
 & \times & \times & (5,0) & \times & \times & \\
\mathbf{(6,-6)} & \times & \times & (6,0) & \times & \times & \cdots \\
\times & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

This motivates the following notation:

$$\mathcal{D} := \{(a,-a) \mid a \geq 2,\ a \text{ even}\}, \quad \mathcal{E} := \{(a,0) \mid a \geq 1\}, \quad \mathcal{F} := \{(0,b) \mid b \geq 2,\ b \text{ even}\}.$$

By Proposition 2.1, no even element outside $\mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \{(0,0)\}$ can be in an avoidable set. In the following sections we find the avoidable sets containing each type of these even elements.

3. Sets containing $(a,-a) \in \mathcal{D}$

We investigate the possibility of U being avoidable if $U \cap \mathcal{D} \neq \emptyset$.

Lemma 3.1. *If $(a,-a) \in \mathcal{D}$, $c \geq 0$ and $c+d \geq 1$ then $U = \{(a,-a),(c,d)\} \subseteq \mathcal{B}$ is unavoidable.*

Proof. Let
$$r = \left(\frac{a}{2}, -\frac{a}{2}\right), \quad s = \left(\frac{a}{2}+1, -\frac{a}{2}\right), \quad t = \left(c, d+\frac{a}{2}\right).$$
It is clear that $r, s \in \mathcal{B}$. We also have $t \in \mathcal{B}$ because $c+d+a/2 > c+d \geq 1 > 0$.

It is clear that $r \neq s$. We cannot have $r = t$ because that would imply $1 \leq c+d = a/2 - a = -a/2 < 0$, which is impossible. We cannot have $s = t$ either because that would imply $1 \leq c+d = a/2+1-a = 1-a/2 < 1$.

It is easy to see that $rs = (a,-a) \in U$. We also have
$$st = \left(\frac{a}{2}+1, -\frac{a}{2}\right)\left(c, d+\frac{a}{2}\right) = \left(\max\left\{c+d+\frac{a}{2}, \frac{a}{2}+1\right\} - d - \frac{a}{2}, d\right) = (c,d) \in U,$$
and similar calculation shows that $rt = (c,d) \in U$. So $\{r,s,t\}$ forms a triangle in $G_{\mathcal{B},U}$. □

We now consider the case when $c+d=0$, that is, $d=-c$.

Lemma 3.2. *If $(a,-a) \in \mathcal{D}$, $c \neq a$ and $\frac{a}{2} < c$ then $U = \{(a,-a),(c,-c)\} \subseteq \mathcal{B}$ is unavoidable.*

Proof. Since $a \neq c$ it is easy to see that
$$r = \left(\frac{a}{2}, -\frac{a}{2}\right), \quad s = \left(\frac{a}{2}+1, -\frac{a}{2}\right), \quad t = \left(c - \frac{a}{2}, -c+\frac{a}{2}\right)$$
are different elements of $\mathcal{B}$.

It is also easy to see that $rs = (a,-a) \in U$. We also have
$$ts = \left(c-\frac{a}{2}, -c+\frac{a}{2}\right)\left(\frac{a}{2}+1, -\frac{a}{2}\right) = \left(\max\left\{1, c-\frac{a}{2}\right\} + \frac{a}{2}, -c\right) = (c,-c) \in U,$$
and similar calculation shows that $tr = (c,-c) \in U$ and so $\{r,s,t\}$ forms a triangle in $G_{\mathcal{B},U}$. □

Corollary 3.3. *If $(a,-a),(c,-c) \in \mathcal{D}$ and $a \neq c$ then $U = \{(a,-a),(c,-c)\} \subseteq \mathcal{B}$ is unavoidable.*

Proof. Since $a \neq c$ we can assume, without loss of generality, that $a < c$. But then $\frac{a}{2} < c$ and so the result follows from the previous lemma. □

Proposition 3.4. *If $(a,-a) \in \mathcal{D}$, $0 < c < e \leq \frac{a}{2}$ and c, e are odd then $U = \{(a,-a), (c,-c), (e,-e)\}$ is unavoidable.*

Proof. It is clear that
$$r = \left(0, \frac{a-c-e}{2}\right), \quad s = \left(\frac{a+c-e}{2}, -\frac{a+c-e}{2}\right), \quad t = \left(\frac{a-c+e}{2}, -\frac{a-c+e}{2}\right)$$
are different elements of $\mathcal{B}$. We have
$$\begin{aligned} sr &= \left(\frac{a+c-e}{2}, -\frac{a+c-e}{2}\right)\left(0, \frac{a-c-e}{2}\right) \\ &= \left(\max\left\{\frac{a-c-e}{2}, \frac{a+c-e}{2}\right\} - \frac{a-c-e}{2}, -c\right) = (c,-c) \in U, \end{aligned}$$

$$\begin{aligned}
st &= \left(\frac{a+c-e}{2}, -\frac{a+c-e}{2}\right)\left(\frac{a-c+e}{2}, -\frac{a-c+e}{2}\right) \\
&= \left(\max\left\{0, \frac{a+c-e}{2}\right\} + \frac{a-c+e}{2}, -a\right) = (a,-a) \in U, \\
tr &= \left(\frac{a-c+e}{2}, -\frac{a-c+e}{2}\right)\left(0, \frac{a-c-e}{2}\right) \\
&= \left(\max\left\{\frac{a-c-e}{2}, \frac{a-c+e}{2}\right\} - \frac{a-c-e}{2}, -e\right) = (e,-e) \in U,
\end{aligned}$$

and so $\{r,s,t\}$ forms a triangle in $G_{\mathcal{B},U}$. □

Proposition 3.5. *If $(a,-a) \in \mathcal{D}$ and $0 < c < \frac{a}{2}$ then $U = \{(a,-a),(c,-c),(0,0)\}$ is unavoidable.*

Proof. It is clear that

$$p = (0,0), \quad q = (c,-c), \quad r = (a-c, -(a-c)), \quad s = (0, a-c), \quad t = (a,-a)$$

are different elements of $\mathcal{B}$. It is easy to check that $pq = (c,-c)$, $qr = (a,-a)$, $rs = (0,0)$, $ts = (c,-c)$ and $pt = (a,-a)$. Thus $\{p,q,r,s,t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$. □

We are going to denote the remainder of y modulo m by $[y]_m$.

Proposition 3.6. *Let $(a,-a) \in \mathcal{D}$ and c be odd. If $0 < c < \frac{a}{2}$ then $U = \{(a,-a),(c,-c)\}$ is avoidable. If $c = \frac{a}{2}$ then $U = \{(a,-a), \left(\frac{a}{2}, \frac{a}{2}\right), (0,0)\}$ is avoidable.*

Proof. Let

$$A = \left\{(x,y) \mid \frac{a-2c}{2} \le [y]_{a-c} \le \frac{2a-3c-1}{2}\right\} \setminus \left\{\left(\frac{a}{2} + k(a-c), -\frac{a}{2} - k(a-c)\right) \mid k = 0,1,\ldots\right\}$$

and $B = \mathcal{B} \setminus A$. Note that $[v]_{a-c} = \frac{a-2c}{2}$ if and only if $v = -\frac{a}{2} + l(a-c)$ for some $l \in \mathbf{Z}$. We show that the partition $\{A,B\}$ avoids U. The following figure shows the partition when $a = 8$ and $c = 3$. Note that $\frac{a-2c}{2} = 1$ and $\frac{2a-3c-1}{2} = 3$ in this case.

	-9	-8	-7	-6	-5	$-\frac{a}{2}$	-3	-2	-1	0	1	2	3	4	$a-c$	$\cdots$
0	.	.	.	.	.	.	.	.	.	1	0	0	0	1	1	
1	.	.	.	.	.	.	.	.	1	1	0	0	0	1	1	
2	.	.	.	.	.	.	.	0	1	1	0	0	0	1	1	
3	.	.	.	.	.	.	0	0	1	1	0	0	0	1	1	
$-a/2$	.	.	.	.	.	**1**	0	0	1	1	0	0	0	1	1	
5	.	.	.	.	1	0	0	0	1	1	0	0	0	1	1	
6	.	.	.	1	1	0	0	0	1	1	0	0	0	1	1	
7	.	.	0	1	1	0	0	0	1	1	0	0	0	1	1	
$-a$	.	0	0	1	1	0	0	0	1	1	0	0	0	1	1	
9	**1**	0	0	1	1	0	0	0	1	1	0	0	0	1	1	
10	0	0	0	1	1	0	0	0	1	1	0	0	0	1	1	
$\vdots$																

If $(\alpha,-\alpha) = (x,y)(w,z) = (\max\{w+z,x\} - z, y+z)$ then either $w+z \ge x$ and $w = \alpha$ or $x \ge w+z$ and $x - z = \alpha$. In the first case we have $0 \le x+y \le w+z+y = \alpha - \alpha = 0$ which can only happen if $y = -x$. In the second case we have $y = -\alpha - z = -x$ as well. Thus we only have to show that $(x,-x)(w,z)$ is not in U unless the two factors are in separate classes.

Let $s = (x, y) \neq t = (w, z)$ and suppose that s and t are in the same equivalence class. If $st \in \{(a, -a), (c, -c)\}$ then either $y + z = -a$ or $y + z = -c$. So we need to study the effect of the maps $v \mapsto -v - a$ and $v \mapsto -v - c$ on the congruence classes modulo $a - c$. Since a and c are congruent modulo $a - c$, we only need to study one of the maps. If $\frac{a-2c}{2} \leq [v]_{a-c} \leq a - 2c$ then

$$0 = -(a - 2c) - a + 2(a - c) \leq [-v - a]_{a-c} \leq -\frac{a - 2c}{2} - a + 2(a - c) = \frac{a - 2c}{2}.$$

If $a - 2c < [v]_{a-c} \leq \frac{2a-3c-1}{2}$ then

$$\frac{2a - 3c + 1}{2} = -\frac{2a - 3c - 1}{2} - a + 3(a - c) \leq [-v - a]_{a-c} < -(a - 2c) - a + 3(a - c) = a - c.$$

So

$$[v]_{a-c} \in \left[\frac{a - 2c}{2}, \frac{2a - 3c - 1}{2}\right] \Leftrightarrow [-v - a]_{a-c}, [-v - c]_{a-c} \in \left[0, \frac{a - 2c}{2}\right] \cup \left[\frac{2a - 3c + 1}{2}, a - c\right).$$

First consider the case when $c < \frac{a}{2}$. We must have $[y]_{a-c} = \frac{a-2c}{2} = [z]_{a-c}$ and so $y = -\frac{a}{2} + k(a - c)$ and $z = -\frac{a}{2} + l(a - c)$ for some $k, l \in \mathbf{Z}$. Since $0 \leq x = -y = \frac{a}{2} - k(a - c)$, we must have $k \leq 0$ and so $s \in B$.

If $st = (c, -c)$ then $-c = y + z = -a + (k + l)(a - c)$ and so $k + l = 1$. Hence $l = 1 - k \geq 1$ and so $t \in A$ which is a contradiction. If $st = (a, -a)$ then $-a = y + z = -a + (k + l)(a - c)$ and so $k + l = 0$. If $k = 0$ then $l = 0$ and so $z = y$. Since $s \neq t$ we must have $w \neq x = -y = -z$ and so $t \in A$ which is a contradiction. If $k < 0$ then $l > 0$ and so $t \in A$ which is again a contradiction.

Next consider the case when $c = \frac{a}{2}$. If $st \in \{(c, -c), (a, a)\}$ then again we must have $[y]_{a-c} = \frac{a-2c}{2} = [z]_{a-c}$ and so $y = k\frac{a}{2}$ and $z = l\frac{a}{2}$ for some $k, l \in \mathbf{Z}$. Since $0 \leq x = -y = -k\frac{a}{2}$, we have $k \leq 0$. If $k < 0$ then $s \in B$ and we get a contradiction like we did in the $c < \frac{a}{2}$ case. If $k = 0$ then $s = (0, 0) \in A$ and so $t = (c, -c) \in B$ or $t = (a, -a) \in B$ which is a contradiction.

If $st = (0, 0)$ then we must have $s = (x, -x)$ and $t = (0, x)$ and since $s \neq t$, we also know that $x \geq 1$. Note that in this case $a - c = c$, $\frac{a-2c}{2} = 0$ and $\frac{2a-3c-1}{2} = \frac{a-2}{4}$. If $a = 2$ then $B = \{(x, -x) \mid x = 1, 2, \ldots\}$ and so $s \in B$ while $t \in A$. Since c is odd we cannot have $a = 4$. If $a \geq 6$ then $0 < \frac{a-2}{4} < \frac{a}{2} = a - c$. So we have

$$0 \leq [v]_{a-c} \leq \frac{a - 2}{4}$$

if and only if either

$$[-v]_{a-c} \geq -\frac{a - 2}{4} + (a - c) = \frac{a + 2}{4}$$

or $[-v]_{a-c} = 0$. So since s and t are in the same equivalence class we must have $-x = -\frac{a}{2}k$ for some positive k and so $s \in B$ while $t \in A$. □

Proposition 3.7. *If $U \cap \mathcal{D} \neq \emptyset$ and U is maximal avoidable then U is one of the avoidable sets of Proposition* 3.6.

Proof. Let $(a, -a) \in U \cap \mathcal{D}$. If $(a, -a) \neq (x, y) \in U$ then by Lemma 3.1, we must have $y = -x$. By Lemma 3.2 and Corollary 3.3 we know that x cannot be even and $x \leq \frac{a}{2}$. If $0 < y < \frac{a}{2}$ then by Propositions 3.4, 3.5 and 3.6 $U = \{(a, -a), (x, -x)\}$ is maximal avoidable. If $y = 0$ or $y = \frac{a}{2}$ then by Propositions 3.4 and 3.5, U cannot have yet another element $(w, -w)$ unless $w = 0$ or $w = \frac{a}{2}$. This fact and Proposition 3.6 implies that $U = \{(a, -a), (\frac{a}{2}, -\frac{a}{2}), (0, 0)\}$ is maximal avoidable. We considered all the possibilities so these are the only maximal avoidable sets intersecting $\mathcal{D}$. □

4. Sets containing $(0,b) \in \mathcal{F}$

We investigate the possibility of U being avoidable if $U \cap \mathcal{F} \neq \emptyset$. Our main tool is the fact that $\mathcal{F} = \mathcal{D}^*$, which allows us to transform the results of the previous section.

Proposition 4.1. *The set U is avoidable if and only if U^* is avoidable. Furthermore, U is maximal avoidable if and only if U^* is maximal avoidable.*

Proof. First, assume U is avoidable. Then U can be partitioned into two subsets A and B such that the partition avoids U. Now $\{A^*, B^*\}$ is a partition of U^*. If $x, y \in A^*$ and $x \neq y$ then $x^*, y^* \in A$ and so $y^* x^* \notin U$, which means $xy = (y^* x^*)^* \notin U^*$. Similar argument shows that no element of U^* is the product of two different elements of B^*.

Now if U^* is avoidable then $U = U^{**}$ is also avoidable by the previous argument.

The second part of the proposition follows from the fact that if U and V are subsets of $\mathcal{B}$ then $U \subseteq V$ exactly when $U^* \subseteq V^*$. □

Proposition 4.2. *If $U \cap \mathcal{F} \neq \emptyset$ and U is maximal avoidable then either $U = \{(0,b),(0,d)\}$ where $(0,b) \in \mathcal{F}$, d is odd and $0 < d < \frac{b}{2}$, or $U = \{(0,b), \left(0, \frac{b}{2}\right), (0,0)\}$ where $(0,b) \in \mathcal{F}$ and $\frac{b}{2}$ is odd.*

Proof. By the previous proposition, U is maximal avoidable exactly when U^* is maximal avoidable. Since $U \cap \mathcal{F} \neq \emptyset$ and $\mathcal{F}^* = \mathcal{D}$, we must have $U^* \cap \mathcal{D} \neq \emptyset$. Hence U^* is one of the maximal avoidable sets of Proposition 3.6. □

5. Sets containing $(a,0) \in \mathcal{E}$

We investigate the possibility of U being avoidable if $U \cap \mathcal{E} \neq \emptyset$.

Lemma 5.1. *If $(a,0) \in \mathcal{E}$ and $\max\{c, c+d\} \geq a$ then $U = \{(a,0),(c,d)\} \subseteq \mathcal{B}$ is unavoidable.*

Proof. If $c \geq a$ then $(c,d)(a,0) = (c,d)$. If $c+d \geq a$ then $(a,0)(c,d) = (c,d)$. In either case $\{(0,0), (a,0), (c,d)\}$ forms a triangle in $G_{\mathcal{B},U}$. □

Corollary 5.2. *If $a \neq c$ then $U = \{(a,0),(c,0)\} \subseteq \mathcal{E}$ is unavoidable.*

Proof. Without loss of generality we can assume that $a < c$ and so the result follows from Lemma 5.1. □

Lemma 5.3. *If $(a,0) \in \mathcal{E}$, d and f are odd, $d \neq f$, $\max\{c, c+d, e, e+f\} < a$ then $U = \{(a,0),(c,d),(e,f)\} \subseteq \mathcal{B}$ is unavoidable.*

Proof. Without loss of generality we can assume that $d < f$. Let $x = \frac{f-d}{2}$, $y = \frac{f+d}{2}$ and

$$p = (\alpha, x), \quad q = (\beta, y), \quad r = (c+y, -x), \quad s = (a, x), \quad t = (a, -x).$$

where α and β will be chosen later. Since $y - x = d$ and $x + y = f$ are odd and so not zero, we have $p \neq q \neq r$. Since $x > 0$, we also have $r \neq s \neq t \neq p$.

Since

$$c + y = c + \frac{f+d}{2} = \frac{c+f+c+d}{2} \geq \frac{c+d+c+d}{2} = c+d \geq 0$$

and $c + y - x = c + d \geq 0$, we have $r \in \mathcal{B}$. Since $a + x > a \geq 0$, we have $s \in \mathcal{B}$. We have $t \in \mathcal{B}$ because

$$a - x = a - \frac{f-d}{2} = \frac{(a-f)+a+d}{2} > \frac{e+a-c}{2} > \frac{e}{2} \geq 0.$$

Since $a + x > c + d + x = c + y$ we have

$$rs = (c+y, -x)(a,x) = (\max\{a+x, c+y\} - x, 0) = (a,0) \in U.$$

Also

$$ts = (a, -x)(a, x) = (\max\{a + x, a\} - x, 0) = (a, 0) \in U.$$

First, assume that $e + x < a$. Let $\alpha = e$ and $\beta = \min\{c, e + x\} \geq 0$. Then clearly $p \in \mathcal{B}$ and since

$$\begin{aligned}\min\{c, e + x\} + y = \min\{c + y, e + x + y\} &= \min\left\{\frac{c + f + c + d}{2}, e + f\right\}\\ &\geq \min\{c + d, e + f\} \geq 0,\end{aligned}$$

we must have $q \in \mathcal{B}$. Now we have

$$pt = (e, x)(a, -x) = (\max\{a - x, e\} + x, 0) = (a, 0) \in U,$$

$$qp = (\min\{c, e + x\}, y)(e, x) = (\max\{e + x, \min\{c, e + x\}\} - x, y + x) = (e, f) \in U,$$

$$rq = (c + y, -x)(\min\{c, e + x\}, y) = (\max\{\min\{c, e + x\} + y, c + y\} - y, d) = (c, d) \in U,$$

and so $\{p, q, r, s, t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$.

Next, assume that $e + x \geq a$. We need to consider two subcases. In the first subcase we assume that $e < c$. Let $\alpha = 0$ and $\beta = e$. It is clear that $p \in \mathcal{B}$. Since $e + y = e + x + y - x \geq a + d > c + d \geq 0$, we have $q \in \mathcal{B}$. It is clear that $pt = (0, x)(a, -x) = (a, 0) \in U$ and $pq = (0, x)(e, y) = (e, f) \in U$. We also have

$$rq = (c + y, -x)(e, y) = (\max\{e + y, c + y\} - y, d) = (c, d) \in U,$$

and so $\{p, q, r, s, t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$.

In the second subcase we assume that $e \geq c$. This implies that $e + d \geq c + d \geq 0$. Let $\alpha = e + y$ and $\beta = c$. Since

$$e + y = e + \frac{f + d}{2} = \frac{e + f + e + d}{2} \geq 0$$

and $e + y + x = e + f \geq 0$, we have $p \in \mathcal{B}$. Since

$$c + y = c + \frac{f + d}{2} = \frac{c + f + c + d}{2} > \frac{c + d + c + d}{2} = c + d \geq 0,$$

we have $q \in \mathcal{B}$. Since $a - x > e + f - x = e + y + x - x = e + y$ we have

$$pt = (e + y, x)(a, -x) = (\max\{a - x, e + y\} + x, 0) = (a, 0) \in U.$$

Also

$$pq = (e + y, x)(c, y) = (\max\{c + y, e + y\} - y, f) = (e, f) \in U,$$

$$rq = (c + y, -x)(c, y) = (\max\{c + y, c + y\} - y, d) = (c, d) \in U,$$

and so $\{p, q, r, s, t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$. □

Proposition 5.4. *If $(a, 0) \in \mathcal{E}$, d is odd and $d < a$ then*

$$U = \{(a, 0), (0, 0)\} \cup \{(c, d) \in \mathcal{B} \mid \max\{c, c + d\} < a\}$$

is avoidable.

Proof. First assume that $d > 0$. For $y \in \mathbf{Z} \setminus \{0\}$ define

$$\phi(y) = \begin{cases} 0 & \text{if } 0 < [y]_d \leq \frac{d-1}{2} \\ 0 & \text{if } [y]_d = 0 \text{ and } y < 0 \\ 1 & \text{otherwise.} \end{cases}$$

Note that if $y \neq 0$ then $\phi(y) \neq \phi(-y)$. Let

$$A = \{(x, y) \in \mathcal{B} \mid y \neq 0 \text{ and } \phi(y) = 0\} \cup \{(x, y) \in \mathcal{B} \mid y = 0 \text{ and } x < a\}$$

and $B = \mathcal{B} \setminus A$. We show that the partition $\{A, B\}$ avoids U. The following figure shows the partition when $a = 6$ and $d = 5$:

	-7	-6	$-d$	-4	-3	-2	-1	0	1	$\frac{d-1}{2}$	3	4	d	6	7	$\cdots$
0	.	.	.	.	.	.	.	0	0	0	1	1	**1**	0	0	
1	.	.	.	.	.	.	1	0	0	0	1	1	**1**	0	0	
2	.	.	.	.	.	1	1	0	0	0	1	1	**1**	0	0	
3	.	.	.	.	0	1	1	0	0	0	1	1	**1**	0	0	
4	.	.	.	0	0	1	1	0	0	0	1	1	**1**	0	0	
5	.	.	**0**	0	0	1	1	0	0	0	1	1	**1**	0	0	
a	.	1	**0**	0	0	1	1	1	0	0	1	1	**1**	0	0	
7	1	1	**0**	0	0	1	1	1	0	0	1	1	**1**	0	0	
$\vdots$																

Let $s = (x, y) \neq t = (w, z)$ and suppose that s and t are in the same equivalence class. If $st = (u, 0)$ then $y + z = 0$ and so $y = 0 = z$. Hence if $st = (0, 0)$ then we must have $s = (0, 0) = t$ which is a contradiction. If $st = (a, 0)$ then without loss of generality we can assume that $x > w$ and so $x = a$ and $w < a$. Thus $s \in B$ and $t \in A$ which is a contradiction.

Now assume $(c, d) \in U$ and (c, f) can be written as st or ts. Then $y + z = d$ and so we either have $y = 0$ and $z = d$ or we have $y = d$ and $z = 0$. If $y = 0$ and $z = d$ then $t \in B$ and so we must have $s \in B$ which implies that $x \geq a$. Hence $(c, d) = st = (x, 0)(w, d) = (\max\{w + d, x\} - d, d)$ and so $c = \max\{w+d, x\} - d \geq x - d \geq a - d$ which is a contradiction. If $y = d$ and $z = 0$ then $s \in B$ and so we must have $t \in B$ which implies $w \geq a$. Hence $(c, d) = (x, d)(w, 0) = (\max\{w, x\}, d)$ and so $c = \max\{w, x\} \geq w \geq a$ which is a contradiction.

In case $d < 0$ the proof is similar but we need to replace the definition of ϕ by the following:

$$\phi(y) = \begin{cases} 0 & \text{if } 0 < [y]_d \leq \frac{d-1}{2} \\ 0 & \text{if } [y]_d = 0 \text{ and } y > 0 \\ 1 & \text{otherwise.} \end{cases}$$

□

Proposition 5.5. *If $U \cap \mathcal{E} \neq \emptyset$ and U is maximal avoidable then U is the avoidable set of Proposition* 5.4.

Proof. Let $(a, 0) \in U \cap \mathcal{E}$. If $(a, 0) \neq (c, d) \in U$ then by Lemma 3.1, $(c, d) \notin \mathcal{D}$, by Lemma 4.2, $(c, d) \notin \mathcal{F}$ and by Corollary 5.2, $(c, d) \notin \mathcal{E}$. If $(c, d) \neq (0, 0)$ then d is odd and by Lemma 5.1, $\max\{c, c + d\} < a$. If (w, z) is also in U and $(c, d) \neq (w, z) \neq (0, 0)$ then again $\max\{w, w + z\} < a$ and by Lemma 5.3, $z = d$. □

6. Sets containing $(0, 0)$

Proposition 6.1. *If d and f are odd and $d \neq f$ then $U = \{(0, 0), (c, d), (e, f)\} \subseteq \mathcal{B}$ is unavoidable.*

Proof. Without loss of generality we can assume that $d < f$. Let $x = \frac{f-d}{2}$ and $y = \frac{f+d}{2}$. Note that $x \neq y \neq -x$ since $y - x = d$ and $x + y = f$ are odd. Also note that $c + y = c + d + x \geq x \geq 0$.

First, we consider the case when $c \leq e$. If

$$p = (0, x), \quad q = (x, -x), \quad r = (c, y), \quad s = (e + y, x), \quad t = (e, y)$$

then $p \neq q \neq r \neq s \neq t \neq p$. Since $c + y \geq 0$, $r \in \mathcal{B}$. Since

$$e + y = \frac{e + f + e + d}{2} \geq \frac{e + f + c + d}{2} \geq 0$$

we have $s, t \in \mathcal{B}$. Since $e + y + x = e + f \geq 0$ we have $s \in \mathcal{B}$. It is easy to see that $qp = (0, 0) \in U$ and $pt = (e, f) \in U$. We have

$$\begin{aligned} qr &= (x, -x)(c, y) = (\max\{c + y, x\} - y, y - x) = (c, d) \in U, \\ sr &= (e + y, x)(c, y) = (\max\{c + y, e + y\} - y, f) = (e, f) \in U, \\ st &= (e + y, x)(e, y) = (\max\{e + y, e + y\} - y, f) = (e, f) \in U \end{aligned}$$

and so $\{p, q, r, s, t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$.

Next, we consider the case when $e < c$. We need to consider two subcases. In the first subcase we assume that $e + y \geq 0$. It is clear that

$$p = (0, x), \quad q = (x, -x), \quad r = (c, y), \quad s = (c + y, -x), \quad t = (e, y)$$

are in $\mathcal{B}$ and $p \neq q \neq r \neq s \neq t \neq p$. As before, we have $qp = (0, 0) \in U$, $pt = (e, f) \in U$ and $rq = (c, d) \in U$. We also have

$$\begin{aligned} qr &= (x, -x)(c, y) = (\max\{c + y, x\} - y, d) = (c, d) \in U, \\ sr &= (c + y, -x)(c, y) = (\max\{c + y, c + y\} - y, d) = (c, d) \in U, \end{aligned}$$

and so $\{p, q, r, s, t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$.

In the second subcase we assume that $e + y < 0$. If

$$p = (0, x), \quad q = (e + x, y), \quad r = (c + y, -x), \quad s = (c - x, y), \quad t = (x, -x)$$

then $p \neq q \neq r \neq s \neq t \neq p$. It is clear that $p, q, t \in \mathcal{B}$. Since $c + y \geq x \geq 0$ and $c - x = c + y - x - y = c + d - y > c + d + e \geq e \geq 0$, we have $r, s \in \mathcal{B}$. It is easy to see that $tp = (0, 0) \in U$ and $qp = (e, f) \in U$. Since $c \geq e + x$ we have

$$rq = (c + y, -x)(e + x, y) = (\max\{e + x + y, c + y\} - y, d) = (c, d) \in U.$$

We also have

$$\begin{aligned} rs &= (c + y, -x)(c - x, y) = (\max\{c - x + y, c + y\} - y, d) = (c, d) \in U, \\ st &= (c - x, y)(x, -x) = (\max\{0, c - x\} + x, d) = (c, d) \in U, \end{aligned}$$

and so $\{p, q, r, s, t\}$ forms a cycle with odd length in $G_{\mathcal{B},U}$. □

Proposition 6.2. *If d is odd then $U = \{(0, 0)\} \cup \{(e, f) \in \mathcal{B} \mid f = d\}$ is avoidable.*

Proof. Let

$$A = \left\{(x, y) \mid [x]_{|d|} \leq \frac{|d| - 1}{2}\right\} \setminus \{y = kd \mid k = 1, 2, \ldots\}$$

and $B = \mathcal{B} \setminus A$. We show that the partition $\{A, B\}$ avoids U. The following figure shows the partition when $d = 5$. Note that $\frac{|d|-1}{2} = 2$ in this case.

	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	d	6	7	8	9	$2d$	11	12	13	$\cdots$
0	.	.	.	.	.	.	.	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
1	.	.	.	.	.	.	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
2	.	.	.	.	.	1	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
3	.	.	.	.	0	1	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
4	.	.	.	0	0	1	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
5	.	.	0	0	0	1	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
6	.	1	0	0	0	1	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
7	1	1	0	0	0	1	1	0	0	0	1	1	**1**	0	0	1	1	**1**	0	0	1	
$\vdots$																						

Let $s = (x, y) \neq t = (w, z)$ and suppose that s and t are in the same equivalence class. If $st = (0, 0)$ then $s = (x, -x)$ and $t = (0, x)$. But s and t cannot be in the same equivalence class unless $y = 0 = z$. So $s = t$ which is a contradiction. If $st \neq (0, 0)$ then either we have $y = 0$ and $z = d$ or we have $y = d$ and $z = 0$. Since $y = -x$ and so negative, $y = 0$ and $z = d$ and so $s \in A$ but $t \in B$ which is a contradiction. □

Proposition 6.3. *If d is odd then $U = \{(0,0)\} \cup \{(e,f) \in \mathcal{B} \mid f = d\}$ is maximal avoidable.*

Proof. Clearly U has infinitely many elements. In particular, if $d > 0$ then $(0,d) \in U$ and if $d < 0$ then $(-d,d) \in U$. If $(x,y) \in U$ and y is odd then by Proposition 6.1 $y = d$. If $(x,y) \neq (0,0)$ but y is even then $(x,y) \in \mathcal{D} \cup \mathcal{F} \cup \mathcal{E}$. This is impossible because then Propositions 3.7, 4.2 and 5.4 would imply that U is finite. $\square$

7. Conclusion

We are in position to give a full classification of the maximal avoidable sets of $\mathcal{B}$.

Theorem 7.1. *The maximal avoidable sets in $\mathcal{B}$ are the following:*
(a) $\{(a,b) \in \mathcal{B} \mid b \text{ is odd}\}$*;*
(b) $\{(0,0)\} \cup \{(e,f) \in \mathcal{B} \mid f = d\}$ *where d is a fixed odd number;*
(c) $\{(a,0),(0,0)\} \cup \{(c,d) \in \mathcal{B} \mid \max\{c, c+d\} < a\}$ *where a and d are fixed such that $d < a > 0$ and d is odd;*
(d) $\{(a,-a),(c,-c)\}$ *where $a > 0$, a is even, $0 < c < \frac{a}{2}$ and c is odd;*
(e) $\{(a,-a),\left(\frac{a}{2},-\frac{a}{2}\right),(0,0)\}$ *where $a > 0$, a is even and $\frac{a}{2}$ is odd;*
(f) $\{(0,b),(0,d)\}$ *where $b > 0$, b is even, $0 < d < \frac{b}{2}$ and d is odd;*
(g) $\{(0,b),\left(0,\frac{b}{2}\right),(0,0)\}$ *where $b > 0$, b is even and $\frac{b}{2}$ is odd.*

Proof. Suppose U is maximal avoidable. If U has no even elements then U is the set of part (a). So suppose U has an even element s. Then by Proposition 2.1, $s = (0,0)$ or $s \in \mathcal{E} \cup \mathcal{D} \cup \mathcal{F}$. If $s \in \mathcal{E} \cup \mathcal{D} \cup \mathcal{F}$ then by Propositions 3.7, 4.2 and 5.4, U is one of the sets in parts (c,d,e,f,g). Assume $s = (0,0)$ and $s \neq t \in U$. If t is even then again $t \in \mathcal{E} \cup \mathcal{D} \cup \mathcal{F}$ and so U is one of the sets in parts (c,e,g). If U does not have any more even elements then by Proposition 6.1, U is the set in part (b). $\square$

The notion of avoidable sets can be generalized. We could call a set U of S n-avoidable if there is a partition of S into n subsets such that no element of U can be written as a product of two distinct elements of the same subset. It would be interesting to know if there are sets that are not 3-avoidable in a group or an inverse semigroup.

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