# MORITA EQUIVALENCE OF $C^*$-CROSSED PRODUCTS BY INVERSE SEMIGROUP ACTIONS AND PARTIAL ACTIONS

Nándor Sieben


## Abstract

Morita equivalence of twisted inverse semigroup actions and discrete twisted partial actions are introduced. Morita equivalent actions have Morita equivalent crossed products.


## 1. Introduction

Morita equivalence of group actions on $C^*$-algebras was studied by Combes [Com], Echterhoff [Ech], Curto, Muhly and Williams [CMW] and Kaliszewski [Kal]. We adapt this notion for both Busby-Smith and Green type inverse semigroup actions, introduced in [Si1] and [Si2]. We show that Morita equivalence is an equivalence relation and that Morita equivalent actions have Morita equivalent crossed products. The close connection between inverse semigroup actions and partial actions [Si1], [Ex3], [Si2] makes it easy to find the notion of Morita equivalence for discrete twisted partial actions. In Section 4 we work out some of the details of discrete twisted partial crossed products, continuing the work started in [Ex2]. The fact that Morita equivalent twisted partial actions have Morita equivalent crossed products will then follow from the connection with semigroup actions. In [AEE] Abadie, Eilers and Exel introduced Morita equivalence of crossed products by Hilbert bimodules. We show that this definition is equivalent to our definition of Morita equivalence on the common special case of partial actions by $\mathbf{Z}$.

The research for this paper was carried out while the author was a student at Arizona State University under the supervision of John Quigg. I thank Professor Quigg for his help during the writing of this paper.

## 2. Preliminaries

In this section we recall some basic definitions to fix our terminology and notation. Our references for Hilbert modules are [JT] and [Lan].

Let $B$ be a $C^*$-algebra. A *(right) $B$-module* is a complex vector space $X$ with a bilinear map $(x,b) \mapsto x \cdot b : X \times B \to X$ such that $(x \cdot b) \cdot c = x \cdot (b \cdot c)$ for all $x \in X$ and $b, c \in B$. A *(right) inner-product $B$-module* is a $B$-module with a map $\langle \cdot, \cdot \rangle_B : X \times X \to B$, called a *$B$-valued inner product*, such that for all $\lambda, \mu \in \mathbf{C}$, $x, y, z \in X$ and $b \in B$ we have

(a) $\langle x, \lambda y + \mu z \rangle_B = \lambda \langle x, y \rangle_B + \mu \langle x, z \rangle_B$;
(b) $\langle x, y \cdot b \rangle_B = \langle x, y \rangle_B b$;
(c) $\langle x, y \rangle_B^* = \langle y, x \rangle_B$;
(d) $\langle x, x \rangle_B \geq 0$;
(e) $\langle x, x \rangle = 0$ only if $x = 0$.

In an inner product $B$-module we have a norm $\|x\|_B = \|\langle x, x \rangle_B\|^{1/2}$ satisfying $\|x \cdot b\|_B \leq \|x\|_B \|b\|$ and $\|\langle x, y \rangle_B\| \leq \|x\| \, \|y\|$ for all $x, y \in X$ and $b \in B$. A *(right) Hilbert $B$-module* is an inner-product $B$-module, which is complete in the norm $\|\cdot\|_B$. A Hilbert $B$-module $X$ satisfying

$$\overline{\operatorname{span}}\{\langle x, y \rangle_B : x, y \in X\} = B$$

is called *full*.

Left modules are defined similarly, with the left inner product linear in the first variable. For a left inner-product $A$-module we use the notation ${}_A\langle \cdot, \cdot \rangle$ for the $A$-valued inner product.

---

Date. 5/15/1997
1991 *Mathematics Subject Classification.* Primary 46L55.
This material is based upon work supported by the National Science Foundation under Grant No. DMS9401253.

**Lemma 2.1.** *Let $X$ be a full right Hilbert $B$-module and $b \in B$. If $x \cdot b = 0$ for all $x \in X$, then $b = 0$.*

*Proof.* For all $x, y \in X$ we have $\langle x, y\rangle_B b = \langle x, y \cdot b\rangle_B = 0$ which implies that $b = 0$ by fullness. □

Let $A$ and $B$ be $C^*$-algebras. An *$A - B$-bimodule* ${}_AX_B$ is a right $B$-module $X$ which is also a left $A$-module satisfying

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b$$

for all $a \in A$, $x \in X$ and $b \in B$. Note that a bimodule satisfies $(\lambda a) \cdot (x \cdot b) = a \cdot (x \cdot (\lambda b))$ for all $\lambda \in \mathbf{C}$.

**Definition 2.2.** Let $A$ and $B$ be $C^*$-algebras. A *Hilbert $A - B$-bimodule* is a bimodule ${}_AX_B$ which is a left Hilbert $A$-module and a right Hilbert $B$-module such that

$${}_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B$$

for all $x, y, z \in X$.
A Hilbert bimodule which is also full on both sides is called an *imprimitivity bimodule.*

Note that for any Hilbert bimodule ${}_AX_B$ there is a corresponding imprimitivity bimodule ${}_{A_0}X_{B_0}$ where $A_0 = \overline{\text{span}}\,{}_A\langle X, X\rangle$ and $B_0 = \overline{\text{span}}\,\langle X, X\rangle_B$.

**Lemma 2.3.** *If ${}_AX_B$ is a Hilbert bimodule then*
(a) $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$;
(b) ${}_A\langle x \cdot b, y\rangle = {}_A\langle x, y \cdot b^*\rangle$,
*for all $a \in A$, $b \in B$ and $x, y \in X$.*

*Proof.* Part (a) follows from the following calculation:

$$\begin{aligned}
&\|\langle a \cdot x, y\rangle_B - \langle x, a^* \cdot y\rangle_B\|^2 \\
&\quad = \|(\langle a \cdot x, y\rangle_B - \langle x, a^* \cdot y\rangle_B)^*(\langle a \cdot x, y\rangle_B - \langle x, a^* \cdot y\rangle_B)\| \\
&\quad = \|\langle y, a \cdot x\rangle_B\langle a \cdot x, y\rangle_B + \langle a^* \cdot y, x\rangle_B\langle x, a^* \cdot y\rangle_B \\
&\qquad\quad - \langle y, a \cdot x\rangle_B\langle x, a^* \cdot y\rangle_B - \langle a^* \cdot y, x\rangle_B\langle a \cdot x, y\rangle_B\| \\
&\quad = \|\langle y, a \cdot x \cdot \langle a \cdot x, y\rangle_B\rangle_B + \langle a^* \cdot y, x \cdot \langle x, a^* \cdot y\rangle_B\rangle_B \\
&\qquad\quad - \langle y, a \cdot x \cdot \langle x, a^* \cdot y\rangle_B\rangle_B - \langle a^* \cdot y, x \cdot \langle a \cdot x, y\rangle_B\rangle_B\| \\
&\quad = \|\langle y, a{}_A\langle x, a \cdot x\rangle y\rangle_B + \langle a^* \cdot y, {}_A\langle x, x\rangle a^* \cdot y\rangle_B \\
&\qquad\quad - \langle y, a{}_A\langle x, x\rangle a^* \cdot y\rangle_B - \langle a^* \cdot y, {}_A\langle x, a \cdot x\rangle \cdot y\rangle_B\| = 0\,.
\end{aligned}$$

Part (b) can be proved similarly. □

**Definition 2.4.** The triple $(\phi_A, \phi, \phi_B)$ is called an *isomorphism between the Hilbert bimodules* ${}_AX_B$ and ${}_CY_D$ if $\phi_A : A \to C$ and $\phi_B : B \to D$ are $C^*$-algebra isomorphisms and $\phi : X \to Y$ is a map such that for all $x, y \in X$ and $a \in A$, $b \in B$ we have
(a) $\phi(x \cdot b) = \phi(x) \cdot \phi_B(b)$;
(b) $\phi_B(\langle x, y\rangle_B) = \langle \phi(x), \phi(y)\rangle_D$;
(c) $\phi(a \cdot x) = \phi_A(a) \cdot \phi(x)$;
(d) $\phi_A({}_A\langle x, y\rangle) = {}_C\langle \phi(x), \phi(y)\rangle$;
(e) $\phi$ is surjective.

The following lemma shows that we can relax some of these conditions. Note that part (ii) is an improvement of [Kal, Lemma 1.1.3].

**Lemma 2.5.** *With the notations of Definition 2.4 we have*
(i) *if $\phi$ satisfies* (b) *then it is a linear isometry;*
(ii) *if $\phi$ satisfies* (b) *then it also satisfies* (a);
(iii) *if $\phi$ satisfies* (b) *and* (c) *and ${}_CY_D$ is an imprimitivity bimodule then $\phi$ also satisfies* (d) *and* (e) *so that it is an isomorphism between $X$ and $Y$.*

*Proof.* An easy calculation using (b) and the linearity of $\phi_B$ shows that $\|\phi(\lambda a+\mu b)-\lambda\phi(a)-\mu\phi(b)\|^2=0$. It is also an isometry since

$$\|\phi(x)\|^2=\|\langle\phi(x),\phi(x)\rangle_D\|=\|\phi_B(\langle x,x\rangle_B)\|=\|\langle x,x\rangle_B\|=\|x\|^2\,.$$

Part (ii) follows from the following calculation:

$$\begin{aligned}&\|\phi(x\cdot b)-\phi(x)\cdot\phi_B(b)\|^2\\ &\quad=\|\langle\phi(x\cdot b)-\phi(x)\cdot\phi_B(b),\phi(x\cdot b)-\phi(x)\cdot\phi_B(b)\rangle_D\|\\ &\quad=\|\langle\phi(x\cdot b),\phi(x\cdot b)\rangle_D-\langle\phi(x\cdot b),\phi(x)\cdot\phi_B(b)\rangle_D\\ &\qquad-\langle\phi(x)\cdot\phi_B(b),\phi(x\cdot b)\rangle_D+\langle\phi(x)\cdot\phi_B(b),\phi(x)\cdot\phi_B(b)\rangle_D\|\\ &\quad=\|\phi_B(\langle x\cdot b,x\cdot b\rangle_B)-\phi_B(\langle x\cdot b,x\rangle_B)\phi_B(b)\\ &\qquad-\phi_B(b^*)\phi_B(\langle x,x\cdot b\rangle_B)+\phi_B(b^*)\phi_B(\langle x,x\rangle_B)\phi_B(b)\|=0\,.\end{aligned}$$

To show (iii) let $Z=\overline{\phi(X)}$. Then we have

$$\begin{aligned}D=\phi_B(B)&=\phi_B(\overline{\operatorname{span}}\,\langle X,X\rangle_B)\subset\overline{\operatorname{span}}\,\phi_B(\langle X,X\rangle_B)\\ &=\overline{\operatorname{span}}\,\langle\phi(X),\phi(X)\rangle_D\subset\overline{\operatorname{span}}\,\langle Z,Z\rangle_D\,,\end{aligned}$$

and so $D=\overline{\operatorname{span}}\,\langle Z,Z\rangle_D$. $Z$ is a left $C$-module since

$$\begin{aligned}C\cdot Z=\phi_A(A)\cdot\overline{\phi(X)}&\subset\overline{(\phi_A(A)\cdot\phi(X))}\\ &=\overline{\phi(A\cdot X)}=\overline{\phi(X)}=Z\,.\end{aligned}$$

$Z$ is also a right $D$-module since

$$\begin{aligned}Z\cdot D&\subset\overline{\phi(X)}\cdot\overline{\operatorname{span}}\,\langle\phi(X),\phi(X)\rangle_D\\ &=\overline{\operatorname{span}}\,({}_C\langle\phi(X),\phi(X)\rangle\cdot\phi(X))\subset Z\,.\end{aligned}$$

Hence $Z$ is a closed subbimodule of $Y$ with full right inner product, and so $Z=Y$ by the Rieffel correspondence. This shows that $\phi(X)=Y$ since $\phi$ is an isometry between Banach spaces. For $x,y,z\in X$ we have

$$\begin{aligned}\phi_A({}_A\langle x,y\rangle)\phi(z)&=\phi({}_A\langle x,y\rangle\cdot z)=\phi(x\cdot\langle y,z\rangle_B)\\ &=\phi(x)\phi_B(\langle y,z\rangle_B)=\phi(x)\langle\phi(y),\phi(z)\rangle_D\\ &={}_C\langle\phi(x),\phi(y)\rangle\cdot\phi(z)\,,\end{aligned}$$

which implies condition (d) by Lemma 2.1. □

Note that the proof of (iii) shows that if $\phi$ satisfies (b) and (c) then $\phi$ is an isomorphism of $X$ onto a $C-D$ Hilbert subbimodule of $Y$. Also note that the statements of the lemma remain true if we interchange condition (b) with (d) and condition (c) with (a).

An equivalent characterization of isomorphisms between the imprimitivity bimodules ${}_AX_B$ and ${}_CY_D$ is a Banach space isomorphism $\phi:X\to Y$ satisfying the *ternary homomorphism identity*, that is,

$$\phi(x\cdot\langle y,z\rangle_B)=\phi(x)\cdot\langle\phi(y),\phi(z)\rangle_D$$

for all $x,y,z\in X$.

**Lemma 2.6.** *$(\mathrm{id}, \phi, \mathrm{id})$ is an isomorphism between the Hilbert bimodules ${}_AX_B$ and ${}_AY_B$ if and only if $(\mathrm{id}, \phi, \mathrm{id})$ is an isomorphism between the corresponding imprimitivity bimodules ${}_{A_0}X_{B_0}$ and ${}_{C_0}Y_{D_0}$.*

*Proof.* If $(\mathrm{id}, \phi, \mathrm{id})$ is an isomorphism between ${}_AX_B$ and ${}_AY_B$ then

$$A_0 = \overline{\operatorname{span}}\,{}_A\langle X, X\rangle = \overline{\operatorname{span}}\,{}_A\langle \phi(X), \phi(X)\rangle = \overline{\operatorname{span}}\,{}_A\langle Y, Y\rangle = C_0 .$$

Similarly, $B_0 = D_0$ and so $(\mathrm{id}, \phi, \mathrm{id})$ is an isomorphism between ${}_{A_0}X_{B_0}$ and ${}_{C_0}Y_{D_0}$. Now suppose that $(\mathrm{id}, \phi, \mathrm{id})$ is an isomorphism between ${}_{A_0}X_{B_0}$ and ${}_{C_0}Y_{D_0}$. If $a \in A$ and $x \in X$ then $x = i \cdot x'$ for some $i \in A_0$ and $x' \in X$ and so

$$\phi(a \cdot x) = \phi(a \cdot (i \cdot x')) = ai \cdot \phi(x') = a \cdot \phi(i \cdot x') = a \cdot \phi(x) .$$

Hence $(\mathrm{id}, \phi, \mathrm{id})$ is an isomorphism between ${}_AX_B$ and ${}_AY_B$ by Lemma 2.5. □

## 3. Morita equivalent twisted actions

Recall from [Rie] that if ${}_AX_B$ is an imprimitivity bimodule then there is a bijective correspondence (often called the *Rieffel correspondence*) between closed subbimodules of $X$ and closed ideals of $A$. If $I$ is a closed ideal of $A$ then $I \cdot X$ is a closed subbimodule of $X$. Note that by the Cohen-Hewitt factorization theorem we do not have to take the closure of $I \cdot X$. Similarly $X \cdot J$ is a closed subbimodule of $X$ if $J$ is a closed ideal of $B$. On the other hand if $Y$ is a closed subbimodule of $X$ then ${}_IY_J$ is an imprimitivity bimodule, where $I$ is the closed span of ${}_A\langle Y, Y\rangle$ and $J$ is the closed span of $\langle Y, Y\rangle_B$. We call ${}_IY_J$ an *imprimitivity subbimodule* of $X$.

**Definition 3.1.** A *partial automorphism* of the imprimitivity bimodule ${}_AX_B$ is an isomorphism between two imprimitivity subbimodules of $X$. We denote the set of partial automorphisms by $\mathrm{PAut}(X)$.

Let $A$ be a $C^*$-algebra, and let $S$ be a unital inverse semigroup with idempotent semilattice $E$, and unit $e$. Recall from [Si2] that a *Busby-Smith twisted action* of $S$ on $A$ is a pair $(\beta, v)$, where for all $s \in S$, $\beta_s : A_{s^*} \to A_s$ is a partial automorphism, that is, an isomorphism between closed ideals of $A$, and for all $s, t \in S$, $v_{s,t}$ is a unitary multiplier of $A_{st}$, such that for all $r, s, t \in S$ we have
(a) $A_e = A$;
(b) $\beta_s\beta_t = \mathrm{Ad}\ v_{s,t} \circ \beta_{st}$;
(c) $v_{s,t} = 1_{M(A_{st})}$ if $s$ or $t$ is an idempotent;
(d) $\beta_r(av_{s,t})v_{r,st} = \beta_r(a)v_{r,s}v_{rs,t}$ for all $a \in A_{r^*}A_{st}$.
We refer to condition (d) as the cocycle identity.

Also recall that a *covariant representation* of a Busby-Smith twisted action $(A, S, \beta, v)$ is a triple $(\pi, V, H)$, where $\pi$ is a nondegenerate representation of $A$ on the Hilbert space $H$ and $V_s$ is a partial isometry for all $s \in S$, such that for all $r, s \in S$ we have
(a) $V_s$ has initial space $\pi(A_{s^*})H$ and final space $\pi(A_s)H$;
(b) $V_rV_s = \pi(v_{r,s})V_{rs}$;
(c) $\pi(\beta_s(a)) = V_s\pi(a)V_s^*$ for $a \in A_{s^*}$.

**Definition 3.2.** The Busby-Smith twisted actions $(A, S, \alpha, u)$ and $(B, S, \beta, v)$ are *Morita equivalent* if there is an imprimitivity bimodule ${}_AX_B$ and a map $s \mapsto (\alpha_s, \phi_s, \beta_s) : S \to \mathrm{PAut}(X)$, such that $\phi_s : X_{s^*} \to X_s$ where $X_s := A_s \cdot X = X \cdot B_s$ and for all $s, t \in S$ we have

$$\phi_s\phi_t = u_{s,t} \cdot \phi_{st}(\cdot) \cdot v_{s,t}^* .$$

We say that $(X, \phi)$ is a Morita equivalence between $(\alpha, u)$ and $(\beta, v)$, and we write

$$(A, S, \alpha, u) \sim_{X,\phi} (B, S, \beta, v) .$$

Note that $\phi_s\phi_t$ and $\phi_{st}$ have the same range $X_{st}$ and so $X_{st} \subset X_s$.

**Lemma 3.3.** *Using the notations of Definition 3.2 we have*
(a) $\phi_s(X_{s^*}\cdot B_t)=X_{st}$;
(b) $\phi_s(A_{s^*}\cdot X_t)=X_{st}$;
(c) $\overline{\text{span}}\,\alpha_s({}_A\langle X_{s^*},X_t\rangle)=A_{st}$,
*for all* $s,t\in S$.

*Proof.* We know from [Si2] that $\beta_s(B_{s^*}B_t)=B_{st}$ and so we have

$$\phi_s(X_{s^*}\cdot B_t)=\phi_s(X_{s^*}\cdot B_{s^*}B_t)=\phi_s(X_{s^*})\cdot\beta_s(B_{s^*}B_t)=X_s\cdot B_{st}=X_{st}\ ,$$

showing (a). A similar calculation shows (b). Finally (c) follows from the calculation:

$$\begin{aligned}\overline{\text{span}}\,\alpha_s({}_A\langle X_{s^*},X_t\rangle)&=\overline{\text{span}}\,\alpha_s({}_A\langle A_{s^*}\cdot X_{s^*},X_t\rangle)=\overline{\text{span}}\,\alpha_s({}_A\langle X_{s^*},A_{s^*}\cdot X_t\rangle)\\&=\overline{\text{span}}\,{}_A\langle\phi_s(X_{s^*}),\phi_s(A_{s^*}\cdot X_t)\rangle=\overline{\text{span}}\,{}_A\langle X_s,X_{st}\rangle=A_{st}\ .\end{aligned}$$

□

**Proposition 3.4.** *Morita equivalence of Busby-Smith twisted actions is an equivalence relation.*

*Proof.* It is easy to see that $(A,S,\alpha,u)\sim_{A,\alpha}(A,S,\alpha,u)$. It is also easy to check that if $(A,S,\alpha,u)\sim_{X,\phi}(B,S,\beta,v)$ then $(B,S,\beta,v)\sim_{\tilde X,\tilde\phi}(A,S,\alpha,u)$, where $\tilde\phi(\tilde x)=\phi(x)\tilde{}$. To show transitivity, suppose

$$(A,S,\alpha,u)\sim_{X,\phi}(B,S,\beta,v)\sim_{Y,\psi}(C,S,\gamma,w)\,.$$

Let $Z$ be the balanced tensor product $X\otimes_B Y$, that is, the Hausdorff completion of $X\odot Y$ in the $C$-valued inner product determined by

$$\langle x_1\otimes y_1,x_2\otimes y_2\rangle_C:=\langle y_1,\langle x_1,x_2\rangle_B\cdot y_2\rangle_C\ .$$

It is well known that $Z$ is an $A-B$ imprimitivity bimodule. We are going to define a map $\theta$ such that $(A,S,\alpha,u)\sim_{Z,\theta}(C,S,\gamma,w)$. For all $s\in S$ we have

$$\begin{aligned}Z_s&=(X\otimes_B Y)\cdot C_s=X\otimes_B(Y\cdot C_s)\\&=X\otimes_B(B_s\cdot Y_s)=(X\cdot B_s)\otimes_B Y_s=X_s\otimes_B Y_s\ .\end{aligned}$$

For all $s\in S$ the map $\theta':X_{s^*}\times Y_{s^*}\to Z_s$ defined by $\theta'(x,y)=\phi_s(x)\otimes\psi_s(y)$ is bilinear, and so we have a linear map $\theta''_s:X_{s^*}\odot Y_{s^*}\to Z_s$ satisfying $\theta''_{s^*}(x\otimes y)=\theta'(x,y)$. The following computation suffices to check that $\theta''$ is isometric:

$$\begin{aligned}\langle\theta''_s(x_1\otimes y_1),\theta''_s(x_2\otimes y_2)\rangle_C&=\langle\phi_s(x_1)\otimes\psi_s(y_1),\phi_s(x_2)\otimes\psi_s(y_2)\rangle_C\\&=\langle\psi_s(y_1),\langle\phi_s(x_1),\phi_s(x_2)\rangle_B\cdot\psi_s(y_2)\rangle_C\\&=\langle\psi_s(y_1),\psi_s(\langle x_1,x_2\rangle_B\cdot y_2)\rangle_C\\&=\gamma_s(\langle y_1,\langle x_1,x_2\rangle_B\cdot y_2\rangle_C)\\&=\gamma_s(\langle x_1\otimes y_1,x_2\otimes y_2\rangle_C)\,.\end{aligned}$$

So $\theta''_s$ extends uniquely to an isometric linear map $\theta_s:Z_{s^*}\to Z_s$. The above calculation also shows that $\theta_s$ satisfies Definition 2.4(b), and it is routine to check Definition 2.4(c). Finally for all $s,t\in S$ we have

$$\begin{aligned}\theta_s\theta_t&=\phi_s\phi_t\otimes\psi_s\psi_t=u_{s,t}\cdot\phi_{st}(\cdot)\cdot v^*_{s,t}\otimes v_{s,t}\cdot\psi_{st}(\cdot)\cdot w^*_{s,t}\\&=u_{s,t}\cdot\phi_{st}(\cdot)\otimes v^*_{s,t}v_{s,t}\cdot\psi_{st}(\cdot)\cdot w^*_{s,t}=u_{s,t}\cdot\theta_{st}\cdot w^*_{s,t}\ .\end{aligned}$$

□

Recall [BGR] that two projections $p$ and $q$ in the multipliers of a $C^*$-algebra $C$ are called complementary if $p+q=1$. The two corners $pCp$ and $qCq$ are also called complementary. The projection $p$ is called full if the corner $pCp$ is full, which means $pCp$ is not contained in any proper ideal of $C$ or equivalently $CpC$ is dense in $C$. If the $C^*$-algebras $A$ and $B$ are Morita equivalent then they are isomorphic to complementary full corners of the linking $C^*$-algebra

$$C = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$$

of ${}_AX_B$, where $\tilde{X}$ is the reverse module of $X$ and the operations on $C$ are defined by

$$\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} \begin{pmatrix} c & z \\ \tilde{w} & d \end{pmatrix} = \begin{pmatrix} ac + {}_A\langle x, w\rangle & a\cdot z + x\cdot d \\ \tilde{y}\cdot c + b\cdot \tilde{w} & \langle y, z\rangle_B + bd \end{pmatrix}$$

$$\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}^* = \begin{pmatrix} a^* & y \\ \tilde{x} & b^* \end{pmatrix}.$$

In fact, we can identify $A$ and $B$ with $pCp$ and $qCq$ respectively, where

$$p = \begin{pmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{pmatrix}.$$

Here we identified the multiplier algebra $M(C)$ with

$$\begin{pmatrix} M(A) & M(X) \\ M(\tilde{X}) & M(B) \end{pmatrix}$$

as in [ER, Appendix]. On the other hand if two $C^*$-algebras are isomorphic to complementary full corners of a $C^*$-algebra, then they are Morita equivalent.

Note that if the actions $(A,S,\alpha,u)$ and $(B,S,\beta,w)$ are Morita equivalent then the $C^*$-algebras $A$ and $B$ are also Morita equivalent. We have a natural action of $S$ on the linking algebra of $A$ and $B$:

**Proposition 3.5.** *If $(A,S,\alpha,u) \sim_{X,\phi} (B,S,\beta,v)$ then the formulas*

$$\gamma_s \begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} = \begin{pmatrix} \alpha_s(a) & \phi_s(x) \\ \phi_s(y)\tilde{} & \beta_s(b) \end{pmatrix}, \quad w_{s,t} = \begin{pmatrix} u_{s,t} & 0 \\ 0 & v_{s,t} \end{pmatrix}$$

*define a Busby-Smith twisted action $(C,S,\gamma,w)$ on the linking algebra $C$ of ${}_AX_B$. Moreover, $(Y,\gamma(\cdot)|Y)$ implements a Morita equivalence between $(C,S,\gamma,w)$ and $(B,S,\beta,v)$, where $Y = \left(\begin{smallmatrix} 0 & X \\ 0 & B \end{smallmatrix}\right) \subset C$.*

*Proof.* It is well-known that ${}_CY_B$ is an imprimitivity bimodule if $Y$ inherits the inner products from the $C^*$-algebra $C$, that is, ${}_C\langle y_1, y_2\rangle = y_1y_2^*$ for all $y_1, y_2 \in Y$ and $\langle \left(\begin{smallmatrix} 0 & x \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z \\ 0 & d \end{smallmatrix}\right)\rangle_B = \langle x, z\rangle_B + b^*d$ for all $x, z \in X$ and $b, d \in B$. It is easy to check that

$$C_s = \begin{pmatrix} A_s & X_s \\ \tilde{X}_s & B_s \end{pmatrix}$$

is the closed ideal of $C$ which is in Rieffel correspondence with $B_s$ via the imprimitivity bimodule ${}_CY_B$. The calculation

$$\begin{aligned} \gamma_s\left(\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}\begin{pmatrix} c & z \\ \tilde{w} & d \end{pmatrix}\right) &= \begin{pmatrix} \alpha_s(ac) + \alpha_s({}_A\langle x, w\rangle) & \phi_s(a\cdot z + x\cdot d) \\ \phi_s(c\cdot y + w\cdot b)\tilde{} & \beta_s(\langle y, z\rangle_B + bd) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_s(a) & \phi_s(x) \\ \phi_s(y)\tilde{} & \beta_s(b) \end{pmatrix}\begin{pmatrix} \alpha_s(c) & \phi_s(z) \\ \phi_s(w)\tilde{} & \beta_s(d) \end{pmatrix} \\ &= \gamma_s\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}\gamma_s\begin{pmatrix} c & z \\ \tilde{w} & d \end{pmatrix} \end{aligned}$$

shows that $\gamma_s$ is a homomorphism for all $s \in S$. It is easy to verify that $\gamma_s$ preserves adjoints and is bijective, hence is an isomorphism between $C_{s^*}$ and $C_s$. We only check the cocycle identity in the definition of Busby-Smith twisted actions. It suffices to show that for $a \in A_{r^*}A_{st}$, $b \in B_{r^*}B_{st}$ and $x, y \in X_{r^*} \cap X_{st}$,

$$\gamma_r\left(\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} w_{s,t}\right) = \begin{pmatrix} \alpha_r(au_{s,t})u_{r,st} & \phi_r(x \cdot v_{s,t})v_{r,st} \\ \phi_r(u_{s,t} \cdot y)^{\tilde{}} \cdot u_{r,st} & \beta_r(bv_{s,t})v_{r,st} \end{pmatrix}$$

and

$$\gamma_r \begin{pmatrix} a & x \\ y^* & b \end{pmatrix} w_{r,s}w_{rs,t} = \begin{pmatrix} \alpha_r(a)u_{r,s}u_{rs,t} & \phi_r(x)v_{r,s}v_{rs,t} \\ \phi_r(y)^{\tilde{}} \cdot u_{r,s}u_{rs,t} & \beta_r(b)v_{r,s}v_{rs,t} \end{pmatrix}$$

are the same. The diagonals are clearly equal. We check the upper right corner. Since $x = x_r \cdot a_r$ for some $x_r \in X_{r^*}$ and $a_r \in A_{r^*}$ we have

$$\begin{aligned} \phi_r(xv_{s,t})v_{r,st} &= \phi_r(x_r \cdot a_r v_{s,t}) = \phi_r(x_r)\beta_r(a_r v_{s,t}) \\ &= \phi_r(x_r)\beta_r(a_r)v_{r,s}v_{rs,t} = \phi_r(x)v_{r,s}v_{rs,t} \,. \end{aligned}$$

The equality of the lower left corners follows similarly. For the other part, the conditions of Definition 3.2 for the pair $(Y, \gamma(\cdot)|Y)$ follow from routine calculations. □

A similar proof shows that in the previous theorem $(C, S, \gamma, w)$ and $(A, S, \beta, v)$ are also Morita equivalent. Recall [Si2] that two Busby-Smith twisted actions $(\alpha, u)$ and $(\beta, w)$ of $S$ on $A$ are *exterior equivalent* if for all $s \in S$ there is a unitary multiplier $V_s$ of $E_s$ such that for all $s, t \in S$
(a) $\beta_s = \mathrm{Ad}\ V_s \circ \alpha_s$;
(b) $w_{s,t} = V_s \alpha_s(1_{M(E_{s^*})}V_t)u_{s,t}V_{st}^*$.

**Theorem 3.6.** *If the twisted actions $(A, S, \alpha, u)$ and $(A, S, \beta, w)$ are exterior equivalent, then they are also Morita equivalent.*

*Proof.* Let $V$ implement an exterior equivalence between $(\alpha, u)$ and $(\beta, w)$. We show that $(A, \phi)$ implements the Morita equivalence, where $\phi_s : A_{s^*} \to A_s$ is defined by $\phi_s(a) = \alpha_s(a)V_s^*$. For $a, b, x \in A_{s^*}$ we have

$$\phi_s(x \cdot b) = \alpha_s(x)\alpha_s(b)V_s^* = \alpha_s(x)V_s^*\beta_s(b) = \phi_s(x) \cdot \phi_s(x)$$

verifying Definition 2.4(a). If $x, y \in X_{s^*} = A_{s^*}$, then we have

$$\alpha_s({}_A\langle x, y\rangle) = \alpha_s(xy^*) = \alpha_s(x)V^*(\alpha_s(y)V^*)^* = {}_A\langle \phi_s(x), \phi_s(y)\rangle,$$

which verifies Condition 2.4(d). By the note after Lemma 2.5, it remains to observe that if $x \in X_{(st)^*} = A_{(st)^*}$ then

$$\begin{aligned} (\phi_s\phi_t)(x) &= \alpha_s(\alpha_t(x)V_t^*)V_s^* \\ &= \alpha_s(\alpha_t(x))u_{s,t}V_{st}^*V_{st}u_{s,t}^*\alpha_s(1_{M(A_{s^*})}V_t)^*V_s^* \\ &= u_{s,t}\phi_{st}(x)w_{s,t}^* \,. \end{aligned}$$

□

Recall [Rie] that if ${}_AX_B$ is an imprimitivity bimodule then every representation $\pi$ of $B$ on a Hilbert space $H$ induces a representation $\pi^X$ of $A$ on the Hilbert space $H^X$ defined by $\pi^X(a)(x \otimes \xi) = (a \cdot x) \otimes \xi$, where $H^X$ is the Hausdorff completion $X \otimes_B H$ of the algebraic tensor product $X \odot H$ in the seminorm generated by the semi-inner product

$$(x \otimes \xi \mid y \otimes \eta) := (\pi(\langle y, x\rangle_B)\xi \mid \eta)_H = (\xi \mid \pi(\langle x, y\rangle_B)\eta))_H \,.$$

Note that $(x \cdot b) \otimes \xi = x \otimes \pi(b)\xi$ for all $x \in X$, $b \in B$ and $\xi \in H$. The following is the semigroup version of [Com, Section 2].

**Theorem 3.7.** *If $(A,S,\alpha,u)\sim_{X,\phi}(B,S,\beta,v)$ then every covariant representation $(\pi,V,H)$ of $(\beta,v)$ induces a covariant representation $(\pi^X,V^X,H^X)$ of $(\alpha,u)$ on the Hilbert space $H^X=X\otimes_B H$, where $\pi^X$ is as above and*

$$V_s^X(x\otimes\xi)=\phi_s(x)\otimes V_s(\xi)$$

*for all elementary tensors $x\otimes\xi\in H_{s^*}^X=X_{s^*}\otimes_B H$.*

*Proof.* First note that if $x\in X_s$ and $\xi\in H$ then $x=y\cdot b$ for some $y\in X_s$ and $b\in B_s$, hence $x\otimes\xi=(y\cdot b)\otimes\xi=y\otimes\pi(b)\xi$. So $H_s^X=X_s\otimes_B H_s$ where $H_s=\pi(B_s)H=V_sH$. To show the existence of $V_s^X$, define $T:X_{s^*}\times H\to X_s\otimes_B H_s$ by $T(x,\xi)=\phi_s(x)\otimes V_s\xi$. $T$ is clearly bilinear so there is a unique linear map $T':X_{s^*}\odot H_{s^*}\to X_{s^*}\otimes_B H_{s^*}$ such that $T'(x\otimes\xi)=T(x,\xi)$. We check that $T'$ is isometric. For $x,y\in X_{s^*}$ and $\xi,\eta\in H_s$ we have

$$\begin{aligned}(T'(x\otimes\xi)\mid T'(y\otimes\eta))_{H^X}&=(\phi_s(x)\otimes V_s\xi\mid\phi_s(y)\otimes V_s\eta)_{H^X}\\&=(\pi(\langle\phi_s(y),\phi_s(x)\rangle_B)V_s\xi\mid V_s\eta)_H\\&=(\pi(\beta_s(\langle y,x\rangle_B))V_s\xi\mid V_s\eta)_H\\&=(V_s\pi(\langle y,x\rangle_B)\xi\mid V_s\eta)_H\\&=(\pi(\langle y,x\rangle_B)\xi\mid\eta)_H=(x\otimes\xi\mid y\otimes\eta)_{H^X}\,.\end{aligned}$$

So $T'$ determines an isometry $T''$ from $H_{s^*}^X$ to $H_s^X$. If we define $V_s^X$ to be $T''$ on $H_{s^*}^X$ and 0 on $(H_{s^*}^X)^\perp$ then $V_s^X$ is a partial isometry with initial space $H_{s^*}^X=(A_{s^*}\cdot X)\otimes_B H=\pi^X(A_{s^*})H^X$. It follows that the final space of $V_s^X$ is $\pi^X(A_s)H^X$.

We can check the covariance condition for elementary tensors. Let $a\in A_{s^*}$ and $x\otimes\xi\in X\odot H$. Since $H=H_s\oplus H_s^\perp$, we only need to consider the two cases $\xi\in H_s$ and $\xi\in H_s^\perp$. If $\xi\in H_s$ then $\xi=\pi(ab)\eta$ for some $a,b\in A_s$ and $\eta\in H$. Hence $x\otimes\xi=x\cdot a\otimes\pi(b)\eta$ and so we can assume that $x\in X_s$. Thus,

$$\begin{aligned}V_s^X\pi^X(a)(V_s^X)^*(x\otimes\xi)&=\phi_s(a\cdot\phi_s^{-1}(x))\otimes V_sV_s^*(\xi))\\&=(\alpha_s(a)\cdot x)\otimes\xi=\pi^X(\alpha_s(a))(x\otimes\xi)\,.\end{aligned}$$

On the other hand if $\xi\in(H_s)^\perp$ then for all $y\in X_s$ and $\eta\in H$ we have

$$(x\otimes\xi\mid y\otimes_B\eta)_H=(\xi\mid\pi(\langle x,y\rangle_B)\eta)_H=0$$

and so $x\otimes\xi\in(H_s^X)^\perp$. This means $(V_s^X)^*(x\otimes\xi)=0$. Since $\alpha_s(a)\cdot x$ is in $X_s$ it is of the form $y\cdot b$ for some $y\in X$ and $b\in B_s$. Thus,

$$\pi^X(\alpha_s(a))(x\otimes\xi)=(\alpha_s(a)\cdot x)\otimes\xi=y\otimes\pi(b)\xi=0$$

as well. □

Of course the inducing process works the other way too, that is, every covariant representation of $\alpha$ induces a covariant representation of $\beta$.

Recall [Si2] that the crossed product $A\times_{\alpha,u}S$ of a Busby-Smith twisted action $(A,S,\alpha,u)$ is the Hausdorff completion of the Banach $*$-algebra

$$L_\alpha=\{x\in l^1(S,A):x(s)\in A_s\text{ for all }s\in S\}$$

with operations

$$(x*y)(s)=\sum_{rt=s}\alpha_r(\alpha_r^{-1}(x(r))y(t))u_{r,t}\quad\text{and}\quad x^*(s)=u_{s,s^*}\alpha_s(x(s^*)^*)$$

in the $C^*$-seminorm $\|\cdot\|_\alpha$ defined by

$$\|x\|_\alpha = \sup\{\|(\pi \times V)(x)\| : (\pi, V) \text{ is a covariant representation of } (A, S, \alpha, u)\}.$$

Alternatively, generalizing Paterson's approach [Pat] to the twisted case, we could define

$$\|x\|_\alpha = \sup\{\|\phi(x)\| : \phi \text{ is a coherent representation of } L_\alpha\}$$

where a representation $\phi$ of $L_\alpha$ is coherent if it satisfies $\phi(a\chi_{ss^*}) = \phi(a\chi_e)$ for all $s \in S$. So if $I_\alpha$ is the closed ideal generated by elements of the form $a\chi_{ss^*} - a\chi_e$, then the crossed product $A \times_\alpha S$ is the enveloping $C^*$-algebra of $L_\alpha / I_\alpha$.

If $\chi_s$ denotes the characteristic function of $\{s\}$, then $a\chi_s$ is an element of $L_\alpha$ for all $a \in A_s$. The canonical image of $a\chi_s$ in $A \times_{\alpha,u} S$ will be denoted by $a\delta_s$. Then $A \times_{\alpha,u} S$ is the closed span of $\{a\delta_s : a \in A_s,\ s \in S\}$. Note that we have the following formulas:

$$a_s\delta_s * a_t\delta_t = \alpha_s(\alpha_s^{-1}(a_s)a_t)u_{s,t}\delta_{st}$$
$$(a\delta_s)^* = \alpha_s^{-1}(a^*)u_{s^*,s}^*\delta_{s^*}.$$

The idea of the proof of the following theorem comes from [Com], [CMW] and [Kal].

**Theorem 3.8.** *If $(A, S, \alpha, u)$ and $(B, S, \beta, v)$ are Morita equivalent actions, then the crossed products $A \times_{\alpha,u} S$ and $B \times_{\beta,v} S$ are also Morita equivalent.*

*Proof.* Let $(X, \phi)$ be a Morita equivalence, and let $(\gamma, w)$ be the Busby-Smith twisted action of $S$ on the linking algebra $C$ of ${}_AX_B$ as in Proposition 3.5. It suffices to show that $A \times_{\alpha,u} S$ and $B \times_{\beta,v} S$ are complementary full corners of $C \times_{\gamma,w} S$. Let

$$p = \begin{pmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{pmatrix}.$$

It is clear that $p\delta_e$ and $q\delta_e$ are complementary projections in $M(C \times_{\gamma,w} S)$. We show that $q\delta_e$ is a full projection. If

$$c = \begin{pmatrix} a_s & x_s \\ \tilde{y}_s & b_s \end{pmatrix} \in C_s \quad \text{and} \quad d = \begin{pmatrix} a_t & x_t \\ \tilde{y}_t & b_t \end{pmatrix} \in C_t$$

then

$$c\delta_s * q\delta_e * d\delta_t = \begin{pmatrix} u_{s,t}\alpha_s({}_A\langle \phi_s^{-1}(x_s), y_t\rangle) & \phi_s(\phi_s^{-1}(x_s) \cdot b_t) \cdot v_{s,t} \\ \phi_s(\beta_s^{-1}(b_s) \cdot y_t)\tilde{} \cdot u_{s,t} & \beta_s(\beta_s^{-1}(b_s)b_t)v_{s,t} \end{pmatrix} \delta_{st}.$$

We can check fullness on the four corners, and this can be done easily using Lemma 3.3. A similar calculation shows that $p\delta_e$ is also full.

Now we show that $B \times_{\beta,v} S = q\delta_e * (C \times_{\gamma,w} S) * q\delta_e$. We use the fact that $B \times_{\beta,v} S$ is the Hausdorff completion $\overline{L_\beta}^{\|\cdot\|_\beta}$ of $L_\beta$ in the greatest $C^*$-seminorm $\|\cdot\|_\beta$ coming from covariant representations of $(\beta, v)$, while $C \times_{\gamma,w} S$ is the Hausdorff completion $\overline{L_\gamma}^{\|\cdot\|_\gamma}$ of $L_\gamma$ in the greatest $C^*$-seminorm $\|\cdot\|_\gamma$ coming from covariant representations of $(\gamma, w)$. Since

$$q\delta_e(C \times_{\gamma,w} S)q\delta_e = q\delta_e(\overline{L_\gamma}^{\|\cdot\|_\gamma})q\delta_e = \overline{q\chi_e * L_\gamma * q\chi_e}^{\|\cdot\|_\gamma} = \overline{L_\beta}^{\|\cdot\|_\gamma},$$

it suffices to show that the seminorms $\|\cdot\|_\beta$ and $\|\cdot\|_\gamma$ are the same on $L_\beta$, where we regard $L_\beta$ as a subspace of $L_\gamma$. If $(\pi, V)$ is a covariant representation of $(\gamma, w)$ then $(\pi|B, \pi(q)V)$ is a covariant representation of $(\beta, v)$ and so $\|\cdot\|_\gamma \leq \|\cdot\|_\beta$ on $L_\beta$. On the other hand, a covariant representation

$(\pi, V, H)$ of $(\beta, v)$ induces a covariant representation $(\pi^Y, V^Y, H^Y)$ of $(\gamma, w)$, where $Y = \left(\begin{smallmatrix} 0 & X \\ 0 & B \end{smallmatrix}\right)$, $H^Y = Y \otimes_B H$ and

$$\pi^Y \begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} \left( \begin{pmatrix} 0 & z \\ 0 & d \end{pmatrix} \otimes \xi \right) = \begin{pmatrix} 0 & a \cdot z + x \cdot d \\ 0 & \langle y, z \rangle_B + bd \end{pmatrix} \otimes \xi \,,$$

$$V_s^Y \left( \begin{pmatrix} 0 & z \\ 0 & d \end{pmatrix} \otimes \xi \right) = \begin{pmatrix} 0 & \phi_s(z) \\ 0 & \beta_s(d) \end{pmatrix} \otimes V_s \xi \,.$$

The image of $\left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right) \chi_s \in L_\beta$ under $\pi^Y \times V^Y$ evaluated at an elementary tensor $\left(\begin{smallmatrix} 0 & z \\ 0 & d \end{smallmatrix}\right) \otimes \xi$ of $H^Y$ is

$$\begin{aligned} \pi^Y \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} V_s^Y \left( \begin{pmatrix} 0 & z \\ 0 & d \end{pmatrix} \otimes \xi \right) &= \begin{pmatrix} 0 & 0 \\ 0 & b\beta_s(d) \end{pmatrix} \otimes V_s \xi \\ &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right) \otimes \pi(\beta_s(d)) V_s \xi \\ &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right) \otimes V_s \pi(d) \xi \,. \end{aligned}$$

If $d \in B$ and $\xi \in H$ then

$$\begin{aligned} \| \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \otimes \xi \|_{H^Y}^2 &= ( \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \otimes \xi \mid \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \otimes \xi )_{H^Y} \\ &= (\pi(\langle \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \rangle_B) \xi \mid \xi)_H \\ &= (\pi(d^* d) \xi \mid \xi)_H = (\pi(d)\xi \mid \pi(d)\xi)_H \\ &= \|\pi(d)\xi\|_H^2 \,. \end{aligned}$$

Hence if $b_i \in B_{s_i}$ for all $i = 1, \ldots, n$ and $f = \sum_{i=1}^n b_i \chi_{s_i} \in L_\beta$ then

$$\|\pi^Y \times V^Y(f) ( \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \otimes \xi )\|_{H^Y} = \|\pi \times V(f) \pi(d) \xi\|_H \,.$$

On the other hand

$$\begin{aligned} \|\pi \times V(f)\| &= \sup\{ \frac{\|\pi \times V(f)\pi(d)\xi\|_H}{\|\pi(d)\xi\|_H} : d \in B, \xi \in H \} \\ &= \sup\{ \frac{\|\pi^Y \times V^Y(f)( \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \otimes \xi )\|_{H^Y}}{\| \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \otimes \xi \|_{H^Y}} : d \in B, \xi \in H \} \\ &\leq \|\pi^Y \times V^Y(f)\| \end{aligned}$$

which implies that $\|\cdot\|_\gamma \geq \|\cdot\|_\beta$ on $L_\beta$. A similar argument shows that $A \times_{\alpha,u} S = p\delta_e * (C \times_{\gamma,w} S) * p\delta_e$. □

The proof also shows that if we use the notation $X \times_{u,\phi,v} S := p\delta_e (C \times_{\gamma,w} S) q\delta_e$ or simply $X \times S$, then

$$A \times_{\alpha,u} S \sim_{X \times S} B \times_{\beta,v} S \,.$$

We now have two different ways to induce representations of $A \times_\alpha S$ from representations of $B \times_\beta S$. The next result shows that they are essentially the same. For simplicity we only state the untwisted version of the result because that is all we need later. The proof closely follows that of similar results in [Ech] and [Kal], and goes back ultimately to [Com].

**Proposition 3.9.** *If $(A,S,\alpha)\sim_{X,\phi}(B,S,\beta)$ and $\pi\times V$ is a representation of $B\times_\beta S$ on $H$, then the induced representations $\pi^X\times V^X$ and $(\pi\times V)^{X\times_\phi S}$ are unitarily equivalent.*

*Proof.* Let $Y=X\times_\phi S$. The map

$$T':X\times H\to H^Y\quad\text{defined by}\quad T'(x,\xi)=\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\delta_e\otimes\xi$$

is bilinear and so there is a unique linear map $T'':X\odot H\to H^Y$ such that $T''(x\otimes\xi)=T'(x,\xi)$. We check that $T''$ is isometric. For $x,y\in X$ and $\xi,\eta\in H$ we have

$$\begin{aligned}(T''(x\otimes\xi)\mid T''(y\otimes\eta))_{H^Y}&=\left(\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\delta_e\otimes\xi\mid\left(\begin{smallmatrix}0&y\\0&0\end{smallmatrix}\right)\delta_e\otimes\eta\right)_{H^Y}\\&=\left((\pi\times V)\left(\left\langle\left(\begin{smallmatrix}0&y\\0&0\end{smallmatrix}\right)\delta_e,\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\delta_e\right\rangle_{B\times_\beta S}\right)\xi\mid\eta\right)_H\\&=\big((\pi\times V)(\langle y,x\rangle_B\delta_e)\xi\mid\eta\big)_H=\big(\pi(\langle y,x,\rangle_B)\xi\mid\eta\big)_H\\&=(x\otimes\xi\mid y\otimes\eta)_{H^X}.\end{aligned}$$

So we have an isometry $T:H^X\to H^Y$ such that $T(x\otimes\xi)=\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\delta_e\otimes\xi$. $T$ is onto since if $x_s\in X_s$ and $\xi\in H$ then there are $x\in X$ and $b\in B_s$ such that $x_s=x\cdot b$ and so

$$\begin{aligned}\begin{pmatrix}0&x_s\\0&0\end{pmatrix}\delta_s\otimes\xi&=\begin{pmatrix}0&x\\0&0\end{pmatrix}\delta_e\begin{pmatrix}0&0\\0&b\end{pmatrix}\delta_s\otimes\xi=\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\delta_e\cdot b\delta_s\right)\otimes\xi\\&=\begin{pmatrix}0&x\\0&0\end{pmatrix}\delta_e\otimes\pi\times V(b\delta_s)\xi=T(x\otimes\pi(b)V_s\xi).\end{aligned}$$

Carrying the above calculation a little further, we have

$$\begin{pmatrix}0&x_s\\0&0\end{pmatrix}\delta_s\otimes\xi=T((x\cdot b)\otimes V_s\xi)=\begin{pmatrix}0&x_s\\0&0\end{pmatrix}\delta_e\otimes V_s\xi,$$

and we need this fact in the verification that $T$ intertwines $\pi^X\times V^X$ and $(\pi\times V)^Y$: for $a\in A_s$ and $x\otimes\xi\in X\odot H$ we have

$$\begin{aligned}(\pi\times V)^Y(a\delta_s)T(x\otimes\xi)&=\left(a\delta_s\cdot\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\delta_e\right)\otimes\xi\\&=\left(\left(\begin{smallmatrix}a&0\\0&0\end{smallmatrix}\right)\delta_s\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\delta_e\right)\otimes\xi=\gamma_s\left(\gamma_s^{-1}\left(\begin{smallmatrix}a&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)\right)\delta_s\otimes\xi\\&=\begin{pmatrix}0&\phi_s(\alpha_s^{-1}(a)\cdot x)\\0&0\end{pmatrix}\delta_s\otimes\xi=\begin{pmatrix}0&\phi_s(\alpha_s^{-1}(a)\cdot x)\\0&0\end{pmatrix}\delta_e\otimes V_s\xi\\&=T(\phi_s(\alpha_s^{-1}(a)\cdot x)\otimes V_s\xi)=TV_s^X\pi^X(\alpha_s^{-1}(a))(x\otimes\xi)\\&=T\pi^X(a)V_s^X(x\otimes\xi)=T(\pi^X\times V^X)(a\delta_s)(x\otimes\xi).\end{aligned}$$

□

Let $A$ be a $C^*$-algebra, let $S$ be a unital inverse semigroup with idempotent semilattice $E$, and let $N$ be a *normal Clifford subsemigroup* of $S$. Recall from [Si2] that a subsemigroup $N$ of $S$ is a normal Clifford subsemigroup if it is normal, that is, $E\subset N$ and $sNs^*\subset N$, and it is also Clifford, that is, $n^*n=nn^*$ for all $n\in N$. Also recall from [Si2] that a *Green twisted action* of $(S,N)$ on $A$ is a pair $(\gamma,\tau)$, where $\gamma$ is an inverse semigroup action of $S$ on $A$ (that is, a semigroup homomorphism $s\mapsto(\gamma_s,A_{s^*},A_s):S\to\mathrm{PAut}(A)$ with $A_e=A$) and $\tau_n$ is a unitary multiplier of $A_n$ for all $n\in N$, such that for all $n,l\in N$ we have
(a) $\gamma_n=\mathrm{Ad}\,\tau_n$;
(b) $\gamma_s(\tau_n)=\tau_{sns^*}$ for all $s\in S$ with $n^*n\le s^*s$;
(c) $\tau_n\tau_l=\tau_{nl}$.
The following is the semigroup version of [Ech, Definition 1].

**Definition 3.10.** The Green twisted actions $(A,S,N,\alpha,\tau)$ and $(B,S,N,\beta,\rho)$ are *Morita equivalent* if there is a Morita equivalence $(X,\phi)$ between the untwisted actions $(A,S,\alpha)$ and $(B,S,\beta)$ such that $\tau_n \cdot x = \phi_n(x)\cdot \rho_n$ for all $n \in N$ and $x \in X_n$. We say that $(X,\phi)$ is a *Morita equivalence* between $(A,S,N,\alpha,\tau)$ and $(B,S,N,\beta,\rho)$, and we write $(A,S,N,\alpha,\tau) \sim_{X,\phi} (A,S,N,\beta,\rho)$.

The proof of the following theorem is modeled on Echterhoff's proof [Ech] in the group case.

**Theorem 3.11.** *If $(A,S,N,\alpha,\tau)$ and $(B,S,N,\beta,\rho)$ are Morita equivalent Green twisted actions then the crossed products $A\times_{\alpha,\tau} S$ and $A\times_{\beta,\rho} S$ are also Morita equivalent.*

*Proof.* Let $(X,\phi)$ be a Morita equivalence. Suppose $(\pi,V,H)$ is a covariant representation of $\beta$ which preserves the twist, that is, $\pi(\rho_n) = V_n$ for all $n \in N$. The induced representation $(\pi^X, V^X, H^X)$ of $\alpha$ also preserves the twist, since if $x,y \in X_n$ and $\xi,\eta \in H_n$ then

$$\begin{aligned}(\pi^X(\tau_n)(x\otimes\xi) \mid y\otimes\eta)_{H^X} &= (\tau_n\cdot x\otimes\xi \mid y\otimes\eta)_{H^X} = (\pi(\langle y,\tau_n\cdot x\rangle_B)\xi \mid \eta)_H\\ &= (\pi(\langle y,\phi_n(x)\rangle_B\rho_n)\xi \mid \eta)_H = (\pi(\langle y,\phi_n(x)\rangle_B)V_n\xi \mid \eta)_H\\ &= (V_n^X(x\otimes\xi) \mid y\otimes\eta)_{H^X}\end{aligned}$$

and so $\pi^X(\tau_n) = V_n^X$. A similar calculation shows that if $(\pi^X, V^X)$ preserves the twist then so does $(\pi,V)$. By [Rie, Proposition 3.3] the kernels of $\pi\times V$ and $(\pi\times V)^{X\times_\phi S}$ are in Rieffel correspondence. By Proposition 3.9, $\pi^X\times V^X$ and $(\pi\times V)^{X\times_\phi S}$ have the same kernel. Hence the twisting ideals of $I_\tau$ and $I_\rho$ are in Rieffel correspondence and so the quotients are Morita equivalent by [Rie, Corollary 3.2].

## 4. Connection with twisted partial actions

The close connection between partial actions and inverse semigroup actions [Si1], [Ex3], [Si2] makes it possible to get quick results about the Morita equivalence of crossed products of twisted partial actions. First recall the definition of a twisted partial action from [Ex2].

**Definition 4.1.** A (discrete) *twisted partial action* of a group $G$ on a $C^*$-algebra $A$ is a pair $(\alpha,u)$, where for all $s\in G$, $\alpha_s : A_{s^{-1}} \to A_s$ is a partial automorphism of $A$, and for all $r,\ s\in G$, $u_{r,s}$ is a unitary multiplier of $A_rA_{rs}$, such that for all $r,\ s,\ t \in G$ we have
(a) $A_e = A$, and $\alpha_e$ is the identity automorphism of $A$;
(b) $\alpha_r(A_{r^{-1}}A_s) = A_rA_{rs}$;
(c) $\alpha_r(\alpha_s(a)) = u_{r,s}\alpha_{rs}(a)u_{r,s}^*$ for all $a \in A_{s^{-1}}A_{s^{-1}r^{-1}}$;
(d) $u_{e,t} = u_{t,e} = 1_{M(A)}$;
(e) $\alpha_r(au_{s,t})u_{r,st} = \alpha_r(a)u_{r,s}u_{rs,t}$ for all $a \in A_{r^{-1}}A_sA_{st}$;

**Definition 4.2.** The twisted partial actions $(A,G,\alpha,u)$ and $(B,G,\mu,w)$ are *Morita equivalent* if there is an imprimitivity bimodule ${}_AX_B$ and a map $s\mapsto(\alpha_s,\phi_s,\mu_s) : G \to \mathrm{PAut}(X)$, such that $\phi_s : X_{s^*} \to X_s$ where $X_s = A_s\cdot X = X\cdot B_s$ and for all $s,t\in G$ and $x \in X\cdot B_{t^{-1}s^{-1}}B_{t^{-1}}$ we have

$$\phi_s\phi_t(x) = u_{s,t}\cdot\phi_{st}(x)\cdot w_{s,t}^*\,.$$

We say that $(X,\phi)$ is a Morita equivalence between $(A,G,\alpha,u)$ and $(B,G,\theta,w)$, and we write

$$(A,G,\alpha,u) \sim_{X,\phi} (B,G,\mu,w)\,.$$

Recall from [Ex3] that for a group $G$, the *associated inverse semigroup* $S(G)$ has elements written in canonical form $[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s]$, where $g_1,\ldots,g_n,s\in G$, and the order of the $[g_i][g_i^{-1}]$ terms is irrelevant. Multiplication and inverses are defined by

$$\begin{aligned}&[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s]\cdot[h_1][h_1^{-1}]\cdots[h_m][h_m^{-1}][t]\\&\quad=[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s][s^{-1}][sh_1][(sh_1)^{-1}]\cdots[sh_m][(sh_m)^{-1}][st]\end{aligned}$$

and

$$([g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s])^*=[s^{-1}g_m][(s^{-1}g_m)^{-1}]\cdots[s^{-1}g_1][(s^{-1}g_1)^{-1}][s^{-1}]\,.$$

Thus [e] is an identity element for $S(G)$ if $e$ is the identity of $G$, so we can write $[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}]$ for $[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][e]$, and these are the idempotents of $S(G)$. Recall from [Si2, Section 4] that if $(A,G,\alpha,u)$ is a twisted partial action, then the corresponding Busby-Smith twisted action $(A,S(G),\beta,v)$ is defined by

$$\begin{aligned}A_p&=A_{g_1}\cdots A_{g_m}A_s\\\beta_p&=\alpha_{g_1}\alpha_{g_1}^{-1}\cdots\alpha_{g_m}\alpha_{g_m}^{-1}\alpha_s\\v_{p,q}&=1_{M(A_{pq})}u_{s,t}\,,\end{aligned}$$

where

$$p=[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s],\quad q=[h_1][h_1^{-1}]\cdots[h_n][h_n^{-1}][t]\,.$$

**Theorem 4.3.** *The twisted partial actions $(A,G,\alpha,u)$ and $(B,G,\mu,w)$ are Morita equivalent if and only if the corresponding Busby-Smith twisted actions $(A,S(G),\beta,v)$ and $(B,S(G),\nu,z)$ are Morita equivalent.*

*Proof.* Suppose $(A,S(G),\beta,v)\sim_{X,\phi}(A,S(G),\nu,z)$. If we identify the element $s\in G$ with $[s]\in S(G)$, then $\phi:G\to\mathrm{PAut}(X)$. For $s,t\in G$ and $x\in X\cdot B_{t^{-1}s^{-1}}B_{t^{-1}}$ we have

$$\begin{aligned}\phi_s\phi_t(x)&=\phi_{[s]}\phi_{[t]}(x)=v_{[s],[t]}\cdot\phi_{[s][t]}(x)\cdot z^*_{[s],[t]}\\&=v_{[s],[t]}\cdot\phi_{[st][t^{-1}][t]}(x)\cdot z^*_{[s],[t]}\\&=v_{[s],[t]}v^*_{[st],[t^{-1}][t]}\cdot\phi_{[st]}\phi_{[t^{-1}][t]}(x)\cdot z_{[st],[t^{-1}][t]}z^*_{[s],[t]}\\&=u_{s,t}\cdot\phi_{[st]}(x)\cdot w^*_{s,t}\qquad\text{since, e.g., } v_{[st],[t^{-1}][t]}=1_{M(A_{[s][t]})}\\&=u_{s,t}\cdot\phi_{st}(x)\cdot w^*_{s,t}\,.\end{aligned}$$

Now suppose $(A,G,\alpha,u)\sim_{X,\phi}(B,G,\mu,w)$. We can extend $\phi$ to $S(G)$ by defining

$$\phi_p=\phi_{g_1}\phi_{g_1}^{-1}\cdots\phi_{g_m}\phi_{g_m^{-1}}\phi_s$$

for $p=[g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s]\in S(G)$. We verify Definition 2.4(d). If

$$x,y\in X_{p^*}=X\cdot B_{p^*}=B_sB_{g_m^{-1}}B_{g_m^{-1}g_{m-1}^{-1}}\cdots B_{g_m^{-1}\cdots g_1^{-1}}\,,$$

then

$$\begin{aligned}\beta_p({}_A\langle x,y\rangle)&=\alpha_{g_1}\alpha_{g_1}^{-1}\cdots\alpha_{g_m}\alpha_{g_m}^{-1}\alpha_s({}_A\langle x,y\rangle)\\&={}_A\langle\phi_{g_1}\phi_{g_1}^{-1}\cdots\phi_{g_m}\phi_{g_m}^{-1}\phi_s(x),\phi_{g_1}\phi_{g_1}^{-1}\cdots\phi_{g_m}\phi_{g_m}^{-1}\phi_s(y)\rangle\\&={}_A\langle\phi_p(x),\phi_p(y)\rangle\,.\end{aligned}$$

Similar calculations show that Definition 2.4(a) is also satisfied, which is enough by Lemma 2.5. □

Starting with a twisted partial action $(A, G, \alpha, u)$, Exel [Ex3] builds a semidirect product $C^*$-algebraic bundle $B$ over $G$ in the sense of Fell. He defines [Ex3, Introduction] the crossed product $A \times_{\alpha,u} G$ as the enveloping $C^*$-algebra of the cross sectional algebra $L^1(B)$. We show that the corresponding Busby-Smith twisted action has an isomorphic crossed product:

**Proposition 4.4.** *If the Busby-Smith twisted action $(A, S(G), \beta, w)$ corresponds to the twisted partial action $(A, G, \alpha, u)$ then the crossed products $A \times_{\alpha,u} G$ and $A \times_{\beta,w} S(G)$ are isomorphic.*

*Proof.* We are going to show that the Banach $*$-algebras $L_\beta / I_\beta$ and $L^1(B)$ are isomorphic, which suffices since the crossed products are the enveloping $C^*$-algebras. The formula

$$\phi'(a\chi_{[g_1]\cdots[g_n^{-1}][s]}) := a\chi_s$$

defines a bounded $*$-homomorphism $\phi' : L_\beta \to L^1(B)$. Since

$$\phi'(a\chi_{[g_1]\cdots[g_n^{-1}][e]} - a\chi_{[e]}) = a\chi_e - a\chi_e = 0\,,$$

$\phi'$ takes $I_\beta$ to 0 and hence determines a bounded $*$-homomorphism $\phi : L_\beta/I_\beta \to L^1(B)$. Going the other way, the formula

$$\psi(a\chi_s) := a\chi_{[s]} + I_\beta$$

defines a bounded $*$-homomorphism $\psi : L^1(B) \to L_\beta$. It is clear that $\psi \circ \phi$ is the identity map. To show that $\psi \circ \phi$ is also the identity map, consider $\psi \circ \phi(a\chi_{[g_1]\cdots[g_n^{-1}][s]} + I_\beta) = a\chi_{[s]} + I_\beta$. We can choose elements $b, c \in A_{[g_1]\cdots[g_n^{-1}][s]}$ such that $a = bc$. Hence

$$a\chi_{[g_1]\cdots[g_n^{-1}][s]} - a\chi_{[s]} = (b\chi_{[g_1]\cdots[g_n^{-1}]} - b\chi_{[e]}) * c\chi_{[s]} \in I_\beta\,. \qquad \square$$

Using Theorems 3.8 and 4.3 we now have:

**Corollary 4.5.** *Morita equivalent twisted partial actions have Morita equivalent crossed products.*

We now develop the basic theory of covariant representations for twisted partial actions.

**Definition 4.6.** A *covariant representation* of a twisted partial action $(A, G, \alpha, u)$ is a triple $(\pi, U, H)$, where $\pi$ is a nondegenerate representation of $A$ on the Hilbert space $H$ and for all $s \in G$, $U_s$ is a partial isometry on $H$ such that

(a) $U_s$ has initial space $\pi(A_{s^{-1}})H$ and final space $\pi(A_s)H$;
(b) $U_s U_t = \pi(u_{s,t})U_{st}$ for all $s, t \in G$;
(c) $\pi(\alpha_s(a)) = U_s \pi(a) U_s^*$ for all $a \in A_{s^{-1}}$.

Note that we have $U_{s^*} = \pi(u_{s^*,s})U_s^*$ for all $s \in G$. Every covariant representation gives a representation of the cross sectional algebra:

**Definition 4.7.** The *integrated form* $\pi \times U : L_1(B) \to B(H)$ of the covariant representation $(\pi, U)$ is defined by

$$(\pi \times U)(x) = \sum_{s \in G} \pi(x(s))U_s\,,$$

where the series converges in norm.

The proof of the following proposition is essentially the same as that of [Si2, Proposition 3.5].

**Proposition 4.8.** *$\pi \times U$ is a nondegenerate representation of $L_1(B)$.*

**Lemma 4.9.** *Let $(A, S(G), \beta, v)$ be a Busby-Smith twisted action corresponding to the twisted partial action $(A, G, \alpha, u)$. If $(\pi, V)$ is a covariant representation of $(\beta, v)$ then $(\pi, U)$ is a covariant representation of $(\alpha, u)$, where $U_s := V_{[s]}$ for all $s \in G$. Conversely, if $(\pi, U)$ is a covariant representation of $(\alpha, U)$ then $(\pi, V)$ is a covariant representation of $(\beta, v)$, where*

$$V_{[g_1][g_1^{-1}]\cdots[g_n][g_n^{-1}][s]} := P_{g_1}\cdots P_{g_n} U_s$$

*and $P_t$ denotes $\pi(1_{M(A_t)})$ for all $t \in G$. Moreover this correspondence between covariant representations of $(\alpha, u)$ and $(\beta, v)$ is bijective.*

*Proof.* The only nontrivial condition to check for the first part is Definition 4.6(b):

$$\begin{aligned} U_s U_t = V_{[s]} V_{[t]} &= \pi(v_{[s],[t]}) V_{[s][s^{-1}][st]} = \pi(1_{M(A_{[s][s^{-1}][t]})} u_{s,t}) V_{[s][s^{-1}][st]} \\ &= \pi(u_{s,t}) \pi(1_{M(A_{[s][s^{-1}][t]})}) V_{[s][s^{-1}][st]} = \pi(u_{s,t}) \pi(v_{[s][s^{-1}],[st]}) V_{[s][s^{-1}]} V_{[st]} \\ &= \pi(u_{s,t}) V_{[s][s^{-1}]} V_{[st]} = \pi(u_{s,t}) U_{st} \,. \end{aligned}$$

To show the second part first notice that the $P_t$'s commute since the $1_{M(A_t)}$'s are central projections in the double dual of $A$. Therefore $V_{[g_1]\cdots[g_n^{-1}][s]}$ is well defined since $P_{g_1}\cdots P_{g_n}$ does not depend on the order of the idempotents $[g_1][g_1^{-1}], \ldots, [g_n][g_n^{-1}]$. It is clear that $V_{[g_1]\cdots[g_n^{-1}][s]}$ is a partial isometry. This partial isometry has the required final space since

$$\begin{aligned} \pi(V_{[g_1]\cdots[g_n^{-1}][s]})H &= P_{g_1}\cdots P_{g_n} U_s H = P_{g_1}\cdots P_{g_n} \pi(A_s) H \\ &= P_{g_1}\cdots P_{g_n} P_s H = \pi(A_{g_1}\cdots A_{g_n} A_s) H \\ &= \pi(A_{[g_1]\cdots[g_n^{-1}][s]}) H \,. \end{aligned}$$

We can show that it also has the required initial space by taking conjugates. To check multiplicativity, let $p = [g_1]\cdots[g_m^{-1}][s]$ and $q = [h_1]\cdots[h_n^{-1}][t]$. Then we have

$$V_p V_q = P_{g_1}\cdots P_{g_m} U_s P_{h_1}\cdots P_{h_n} U_t \;.$$

We first simplify a piece of this expression:

$$\begin{aligned} U_s P_{h_1} &= U_s U_s^* U_s U_{h_1} U_{h_1}^* \\ &= P_s \pi(u_{s,h_1}) U_{sh_1} U_{h_1}^* \\ &= P_s \pi(u_{s,h_1}) P_{sh_1} U_{sh_1} U_{h_1^{-1}} \pi(u_{h_1,h_1^{-1}})^* \\ &= P_s P_{sh_1} \pi(u_{s,h_1}) \pi(u_{sh_1,h_1^{-1}}) U_s \pi(u_{h_1,h_1^{-1}})^* \\ &= P_s P_{sh_1} \pi(u_{s,h_1}) U_s U_s^* \pi(u_{sh_1,h_1^{-1}}) U_s \pi(u_{h_1,h_1^{-1}})^* \\ &= \lim_\lambda P_s P_{sh_1} \pi(u_{s,h_1}) U_s \pi(\alpha_s^{-1}(e_\lambda u_{sh_1,h_1^{-1}})) \pi(u_{h_1,h_1^{-1}})^* \,, \\ &\qquad \text{where } e_\lambda \text{ is an approximate identity for } A_s A_{sh_1} \\ &= \lim_\lambda P_s P_{sh_1} \pi(u_{s,h_1}) U_s \pi(u_{s^{-1},s}^* \alpha_{s^{-1}}(e_\lambda u_{sh_1,h_1^{-1}}) u_{s^{-1},s}) \pi(u_{h_1,h_1^{-1}})^* \\ &= \lim_\lambda P_s P_{sh_1} \pi(u_{s,h_1}) U_s \pi(u_{s^{-1},s}^* \alpha_{s^{-1}}(e_\lambda) u_{s^{-1},sh_1} u_{h_1,h_1^{-1}}) \pi(u_{h_1,h_1^{-1}})^* \\ &= P_s P_{sh_1} \pi(u_{s,h_1}) U_s \pi(u_{s^{-1},s}^* u_{s^{-1},sh_1}) \\ &= \lim_\mu P_s P_{sh_1} U_s U_s^* \pi(e_\mu u_{s,h_1}) U_s \pi(u_{s^{-1},s}^* u_{s^{-1},sh_1}) \\ &= \lim_\mu P_s P_{sh_1} U_s \pi(\alpha_s^{-1}(e_\mu u_{s,h_1})) \pi(u_{s^{-1},s}^* u_{s^{-1},sh_1}) \\ &= \lim_\mu P_s P_{sh_1} U_s \pi(u_{s^{-1},s}^* \alpha_{s^{-1}}(e_\mu u_{s,h_1}) u_{s^{-1},s}) \pi(u_{s^{-1},s}^* u_{s^{-1},sh_1}) \\ &= \lim_\mu P_s P_{sh_1} U_s \pi(u_{s^{-1},s}^* \alpha_{s^{-1}}(e_\mu) u_{s^{-1},s} u_{e,h_1} u_{s^{-1},sh_1}^*) \pi(u_{s^{-1},sh_1}) \\ &= P_s P_{sh_1} U_s \,. \end{aligned}$$

Repeating this calculation $n-1$ times we have

$$\begin{aligned} V_pV_q &= P_{g_1}\cdots P_{g_m}P_sP_{sh_1}P_{sh_2}\cdots P_{sh_n}U_sU_t \\ &= P_{g_1}\cdots P_{g_m}P_sP_{sh_1}\cdots P_{sh_m}\pi(u_{s,t})U_{st} \\ &= \pi(v_{p,q})V_{[g_1]\cdots[g_m^{-1}][s][s^{-1}][sh_1]\cdots[sh_m^{-1}][st]} \\ &= \pi(v_{p,q})V_{pq}\,. \end{aligned}$$

Finally we check the covariance condition. If $p=[g_1]\cdots[g_m^{-1}][s]$ and $a\in A_{p^*}$ then

$$\begin{aligned} \pi(\beta_p(a)) &= \pi(\alpha_s(a)) \\ &= \pi(\alpha_{g_1}\alpha_{g_1}^{-1}\cdots\alpha_{g_n}\alpha_{g_n}^{-1}\alpha_s(a)) \\ &= U_{g_1}\pi(u^*_{g_1^{-1},g_1}\alpha_{g_1^{-1}}\cdots\alpha_{g_n}\alpha_{g_n}^{-1}\alpha_s(a)u_{g_1^{-1},g_1})U^*_{g_1} \\ &= U_{g_1}\pi(u^*_{g_1^{-1},g_1})U_{g_1^{-1}}\pi(\alpha_{g_2}\cdots\alpha_{g_n}\alpha_{g_n}^{-1}\alpha_s(a))U^*_{g_1^{-1}}\pi(u_{g_1^{-1},g_1})U^*_{g_1} \\ &= U_{g_1}U^*_{g_1}\pi(\alpha_{g_2}\cdots\alpha_{g_n}\alpha_{g_n}^{-1}\alpha_s(a))U^*_{g_1^{-1}}U_{g_1^{-1}} \\ &= \cdots \\ &= P_{g_1}\cdots P_{g_n}U_s\pi(a)U_s^*P^*_{g_n}\cdots P^*_{g_1} \\ &= V_p\pi(a)V_p^*\,. \end{aligned}$$

It is clear from the construction that the correspondence is bijective. □

**Proposition 4.10.** *If $(A,G,\alpha,u)$ is a twisted partial action then $(\pi,U)\mapsto\pi\times U$ is a bijective correspondence between covariant representations of $(\alpha,u)$ and nondegenerate representations of the crossed product $A\times_{\alpha,u}G$.*

*Proof.* We know that there is an isomorphism $\phi$ between $A\times_{\beta,v}S(G)$ and $A\times_{\alpha,u}G$ where $(A,S(G),\beta,v)$ is the corresponding semigroup action. We also know that there is a bijective correspondence $\Psi\mapsto(\pi^\Psi,V^\Psi)$ between nondegenerate representations of $A\times_{\beta,v}S(G)$ and covariant representations of $(\beta,v)$ such that $\Psi=\pi^\Psi\times V^\Psi$. We define a bijective correspondence $\Phi\mapsto(\pi^\Phi,U^\Phi)$ between nondegenerate representations of $A\times_{\alpha,u}G$ and covariant representations of $(\alpha,u)$ satisfying $\Phi=\pi^\Phi\times U^\Phi$ using the following diagram:

$$\begin{array}{ccccccc} \mathrm{Rep}\,(A\times_{\alpha,u}G) & \leftrightarrow & \mathrm{Rep}\,(A\times_{\beta,v}S(G)) & \qquad & \Phi & \leftrightarrow & \Psi=\Phi\circ\phi \\ & & \updownarrow & & & & \updownarrow \\ \mathrm{CovRep}\,(\alpha,u) & \leftrightarrow & \mathrm{CovRep}\,(\beta,v) & & (\pi^\Phi,U^\Phi) & \leftrightarrow & (\pi^\Psi,V^\Psi) \end{array}$$

If $\Phi$ is a nondegenerate representation of $A\times_{\alpha,u}G$ then $\Psi=\Phi\circ\phi$ is a nondegenerate representation of $A\times_{\beta,v}S(G)$ and so $\Psi=\pi^\Psi\times V^\Psi$. Let $(\pi^\Phi,U^\Phi)$ be the covariant representation of $(\alpha,u)$ corresponding to $(\pi^\Psi,V^\Psi)$ as in Lemma 4.9. If $a\in A_s$ then

$$\pi^\Phi\times U^\Phi(a\delta_s)=\pi^\Phi(a)U^\Phi_s=\pi^\Psi(a)V^\Psi_{[s]}=\Psi(a\delta_{[s]})=\Phi(a\delta_s)$$

and so $\pi^\Phi\times U^\Phi=\Phi$. □

## 5. Connection with crossed products by Hilbert bimodules

Recall from [AEE] that the crossed product $A \times_\alpha \mathbf{Z}$ of the partial action $(A, \mathbf{Z}, \alpha)$ is isomorphic to the crossed product $A \times_X \mathbf{Z}$ of $A$ by the Hilbert bimodule ${}_A X_A$, where $X$ is the vector space $A_1$ with module structure

$$a \cdot j := aj, \quad j \cdot a := \alpha_1(\alpha_1^{-1}(j)a)$$

and inner products

$${}_A\langle j, k\rangle := jk^*, \quad \langle j, k\rangle_A := \alpha_1^{-1}(j^*k)$$

for $j, k \in A_1$ and $a \in A$. In other words, we can get ${}_A X_A$ by converting the standard $A_1 - A_1$ imprimitivity bimodule $A_1$ into an $A_1 - A_{-1}$ imprimitivity bimodule via the isomorphism $\alpha_1$, then extending it canonically to a Hilbert $A - A$ bimodule.

**Definition 5.1.** The Hilbert bimodules ${}_A X_A$ and ${}_B Y_B$ are called *Morita equivalent* if there is an isomorphism $(\mathrm{id}, \phi, \mathrm{id})$ between the Hilbert bimodules $X \otimes_A M$ and $M \otimes_B Y$ for some imprimitivity bimodule ${}_A M_B$.

Abadie, Eilers and Exel show that if ${}_A X_A$ and ${}_B Y_B$ are Morita equivalent bimodules then the crossed products $A \times_X \mathbf{Z}$ and $B \times_Y \mathbf{Z}$ are Morita equivalent. They note that Hilbert bimodules corresponding to Morita equivalent actions of $\mathbf{Z}$ are Morita equivalent. We show that the Morita equivalence of Hilbert bimodules corresponding to partial actions of $\mathbf{Z}$ is equivalent to the Morita equivalence of the partial actions, in the sense of Definition 4.2.

Suppose we have two partial actions $(A, \alpha, \mathbf{Z})$ and $(B, \beta, \mathbf{Z})$ with corresponding Hilbert bimodules ${}_A X_A$ and ${}_B Y_B$. We show that the two notions of Morita equivalence of the actions coincide.

**Proposition 5.2.** *The partial actions $(A, \alpha, \mathbf{Z})$ and $(B, \beta, \mathbf{Z})$ are Morita equivalent if and only if the corresponding Hilbert bimodules ${}_A X_A$ and ${}_B Y_B$ are Morita equivalent.*

*Proof.* If ${}_A M_B$ is an imprimitivity bimodule then

$$\begin{aligned}\overline{\mathrm{span}}\,\langle M \otimes_B Y, M \otimes_B Y\rangle_B &= \overline{\mathrm{span}}\,\langle Y, \langle M, M\rangle_B \cdot Y\rangle_B \\ &= \overline{\mathrm{span}}\,\beta_1^{-1}(B_1^*\langle M, M\rangle_B B_1) = B_{-1}\,,\end{aligned}$$

hence the imprimitivity bimodule corresponding to $M \otimes_B Y$ is of the form ${}_D(M \otimes_B Y)_{B_{-1}}$ for some closed ideal $D$ of $A$. Similarly, the imprimitivity bimodule corresponding to ${}_A(X \otimes_A M)_B$ is of the form ${}_{A_1}(X \otimes_A M)_C$ for some closed ideal $C$ of $B$. It is routine to check that the map $m \otimes l \mapsto m \cdot l$ for $m \in M$ and $l \in B_1$ extends to a map $\nu : M \otimes_B Y \to M \cdot B_1$ such that $(\mathrm{id}, \nu, \beta_1)$ is an isomorphism between ${}_D(M \otimes_B Y)_{B_{-1}}$ and the imprimitivity subbimodule ${}_D(M \cdot B_1)_{B_1}$ of ${}_A M_B$. Similarly, the map $j \otimes m \mapsto \alpha_{-1}(j) \cdot m$ for $j \in A_1$ and $m \in M$ extends to a map $\mu : X \otimes_A M \to A_{-1} \cdot M$ such that $(\alpha_{-1}, \mu, \mathrm{id})$ is an isomorphism between ${}_{A_1}(X \otimes_A M)_C$ and the imprimitivity subbimodule ${}_{A_{-1}}(A_{-1} \cdot M)_C$ of ${}_A M_B$.

Suppose now that the Hilbert bimodules $X$ and $Y$ are Morita equivalent. Then by Lemma 2.6 and the above there exists an imprimitivity bimodule ${}_A M_B$ and an isomorphism $(\mathrm{id}, \psi, \mathrm{id})$ between the imprimitivity bimodules ${}_{A_1}(X \otimes_A M)_C$ and ${}_D(M \otimes_B Y)_{B_{-1}}$. Then $A_1 = D$ and $C = B_{-1}$ and so $(\alpha_1, \nu \circ \psi \circ \mu^{-1}, \beta_1)$ is an isomorphism between ${}_{A_{-1}}(A_{-1} \cdot M)_{B_{-1}}$ and ${}_{A_1}(M \cdot B_1)_{B_1}$. This implies that $(A, X, \mathbf{Z}) \sim_{X,\phi} (B, Y, \mathbf{Z})$, where $\phi_n := (\nu \circ \psi \circ \mu^{-1})^n$ for $n \in \mathbf{Z} \setminus \{0\}$. The situation can be visualized by the following diagram:

$$\begin{array}{ccccc}
{}_A(X \otimes_A M)_B & & {}_{A_1}(X \otimes_A M)_C & \xrightarrow{(\alpha_{-1},\mu,\mathrm{id})} & {}_{A_{-1}}(A_{-1} \cdot M)_C \\
\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id},\psi,\mathrm{id})} & & \big\downarrow{\scriptstyle (\alpha_1,\phi_1,\beta_1)} \\
{}_A(M \otimes_A Y)_B & & {}_D(M \otimes_B Y)_{B_{-1}} & \xrightarrow{(\mathrm{id},\nu,\beta_1)} & {}_D(M \cdot B_1)_{B_1}
\end{array}$$

Going the other way, if $(A, \alpha, \mathbf{Z}) \sim_{M,\phi} (B, \beta, \mathbf{Z})$ then $\phi_1$ is an isomorphism between ${}_{A_{-1}}(A_{-1} \cdot M)_{B_{-1}}$ and ${}_{A_1}(M \cdot B_1)_{B_1}$. So $C = B_{-1}$, $D = A_1$ and $(\mathrm{id}, \nu^{-1} \circ \phi_1 \circ \mu, \mathrm{id})$ is an isomorphism between ${}_{A_1}(X \otimes_A M)_C$ and ${}_D(M \otimes_B Y)_{B_{-1}}$ and so the Hilbert bimodules $X$ and $Y$ are Morita equivalent by Lemma 2.6. □

## References

[AEE] B. Abadie, S. Eilers and R. Exel, *Morita equivalence for crossed products by Hilbert $C^*$-bimodules*, preprint.

[BGR] L. Brown, P. Green and M. Rieffel, *Stable isomorphism and strong Morita equivalence of $C^*$-algebras*, Pac. J. Math. **71**(2) (1977), 349-363.

[Com] F. Combes, *Crossed products and Morita equivalence*, Proc. London. Math. Soc. **49** (1984), 289-306.

[CMW] R. Curto, P. Muhly and D. Williams, *Cross products of strongly Morita equivalent $C^*$-algebras*, Proc. Amer. Math. Soc. **90**(4) (1984), 528-530.

[Ech] S. Echterhoff, *Morita equivalent twisted actions and a new version of the Packer-Raeburn stabilization trick*, J. London Math. Soc. (2) **50** (1994), 170-186

[ER] S. Echterhoff and I. Raeburn, *Multipliers of imprimitivity bimodules and Morita equivalence of crossed products*, Math. Scand. **76** (1995), 289-309.

[Ex1] R. Exel, *Circle actions on $C^*$-algebras, partial automorphisms and a generalized Pimsner-Voiculescu exact sequence*, J. Funct. Anal. **122** (1994), 361-401.

[Ex2] R. Exel, *Twisted partial actions a classification of stable $C^*$-algebraic bundles*, preprint.

[Ex3] R. Exel, *Partial actions of groups and actions of inverse semigroups*, preprint.

[JT] K. Jensen and K. Thomsen, *Elements of KK-theory*, Birkhäuser, Boston, 1991.

[Kal] S. Kaliszewski, *Morita equivalence methods for twisted $C^*$-dynamical systems*, Ph.D. thesis, Dartmouth College, 1994.

[Lan] E. Lance, *Hilbert $C^*$-modules, a toolkit for operator algebraists*, Cambridge University Press, Cambridge, 1995.

[Pat] A. Paterson, *r-Discrete $C^*$-algebras as covariant $C^*$-algebras*, Groupoid Fest lecture notes, Reno, 1996.

[Rie] M. Rieffel, *Unitary representations of group extensions: an algebraic approach to the theory of Mackey and Blattner*, Advances in Mathematics Supplementary Studies, **4** (1979), 43-81.

[Si1] N. Sieben, *$C^*$-crossed products by partial actions and actions of inverse semigroups*, J. Australian Math. Soc. (to appear).

[Si2] N. Sieben, *$C^*$-crossed products by twisted inverse semigroup actions*, preprint.