Elements of Style

Anders O.F. Hendrickson

Years of elementary school math taught us incorrectly that the answer to a math problem is just a single number, “the right answer.” It is time to unlearn those lessons: those days are over. From here on out, mathematics is about discovering proofs and writing them clearly and compellingly.

The following rules apply whenever you write a proof. I will refer to them, by number, in my comments on your homework and tests. You will find them summarized on the last page. Take them to heart; repeat them to yourself before breakfast; bind them to your forehead; ponder their wisdom with your friends over lunch.

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader’s job to guess them from a few hints. You are trying to convince a skeptical reader who doesn’t believe you, so you need to argue with airtight logic in crystal clear language; otherwise he will continue to doubt. If you didn’t write something on the paper, then (a) you didn’t communicate it, (b) the reader didn’t learn it, and (c) the grader has to assume you didn’t know it in the first place.

2. **Tell the reader what you’re proving.** The reader doesn’t necessarily know or remember what “Problem 5c” is. Even a professor grading a stack of papers might lose track from time to time. Therefore the statement you are proving should be on the same page as the beginning of your proof. For an exam this won’t be a problem, of course, but on your homework, recopy the claim you are proving. This has the additional advantage that when you study for tests by reviewing your homework, you won’t have to flip back in the textbook to know what you were proving.

3. **Use English words.** Although there will usually be equations or symbolic statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your textbook, you’ll see that each proof consists mostly of English words.

   In particular, use a word, not a symbol, after each punctuation mark and at the beginning of each sentence; otherwise the punctuation mark looks too much like part of the mathematical formulas. The sentence “For all $x \in \mathbb{Z}$, $x^2 \in \mathbb{Z}$,” for example, is slightly harder to parse than the equivalent sentence “For all $x \in \mathbb{Z}$, its square $x^2 \in \mathbb{Z}$.”

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use grammatically correct sentences, complete with verbs, to convey your logical train of thought. A good way to test whether your proof has complete sentences is to read the proof aloud.

   Some complete sentences can be written purely in mathematical symbols, such as equations (like $a^3 = b^{-1}$), inequalities (like $o(a) < 5$), and other relations (like $5 \big| 10$ or $7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like numbers (e.g., 7), vectors (e.g., $\vec{v}$ and $\vec{w}$), or sets (e.g., V and $\mathbb{R}$).

   If your verb is a symbol, the objects it joins together must be given in symbols too, not in words. For example, “Thus $x \in A \cap B$” and “Thus x lies in the intersection of A and B” are good style, but “Thus $x \in$ the intersection of A and B” looks funny.

5. **Show the logical connections among your sentences.** Use phrases like “Therefore” or “because” or “if. . . then. . . ” or “if and only if” or “we see that” to connect your sentences.

6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a group, an element, a vector space, a number, an ordered pair, etc. Objects either exist or don't exist. Statements, on the other hand, are mathematical *sentences*: they can be true or false.

   When you see or write a cluster of math symbols, be sure you know whether it's an object (like "$x^2 + 3$") or a statement (like "$x^2 + 3 < 7$"). One way to tell is that every mathematical statement includes a verb, such as $=$, $\implies$, $\iff$, $<$, $>$, $\equiv$, $\in$, $\subseteq$, $\big|$, $\approx$, $\cong$, and $\lhd$.

7. **Don't interchange $=$ and $\implies$.** The equals sign connects two *objects*, as in "$x^2 = b$"; the symbol "$\implies$" connects two *statements*, as in "$ab = a \implies b = 1$." And please, please don't just use a generic "$\to$" to connect two lines. That symbol has no meaning in that context; it doesn't tell the reader anything.

8. **Use whitespace.** Don't cram your proof into a few lines of the paper, filled from left margin to right margin. Let your proof breathe! When you start a new thought, start a new line. Use indentation to organize your sentences. This helps the reader understand your thought much better, and it also encourages you to be more clear.

9. **Use multiple sheets of paper.** Some people write tiny, trying to cram everything onto a single sheet of paper, with the result that their proofs are so terse as to be incomprehensible. An extra tree will gladly sacrifice its life to help your proofs be legible and understandable.

10. **Use scratch paper.** Finding your proof will be a long, messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully. *Do not hand in your scratch work!*

    Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a "brain dump," throwing everything you know onto the paper before trying to find logical steps that prove the conclusion. *That is what scratch paper is for.*

11. **"=" means equals.** Don't write $A = B$ unless you mean that A actually equals B. This rule seems obvious, but there is a great temptation to be sloppy. In linear algebra, for example, some people might write

    $$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 6 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$$

    (which is obviously false), when they really mean that the first matrix can be row-reduced to the second. Likewise they might write "$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$," when they really mean that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V.

12. **Say exactly what you mean.** Just as the symbol "=" is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that a_{ij} is an entry in matrix A. Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.

13. **Don't write anything unproven.** Every statement on your paper should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you don't yet know is true, you *must* preface it with words like "assume," "suppose," or "if" (if you are temporarily assuming it), or with words like "we need to show that" or "we claim that" (if it is your goal). Otherwise the reader will think he's missed part of your proof.

14. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like

$$A = B \leq C = D,$$

he expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and he expects the *point* of the entire line to be the more complicated fact that $A \leq D$. For example, if you were computing the distance d of the point $(12, 5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean Theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write "$\sqrt{12^2 + 5^2} = 13 = d$," your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

15. **Don't beg the question.** Be sure that no step in your proof makes use of the conclusion! That is called "begging the question" or "circular logic," and it makes your proof invalid.

For example, consider this student's "proof" that $\vec{0} + \vec{0} = \vec{0}$:

Claim: $\vec{0} + \vec{0} = \vec{0}$.
Proof:
$$\begin{aligned} \vec{0} + \vec{0} &= 1\vec{0} + 1\vec{0} \\ &= 1(\vec{0} + \vec{0}) \\ &= 1\vec{0} \\ &= \vec{0}. \quad \square \end{aligned}$$

To move from the second line of her proof to the third, the student replaced $\vec{0} + \vec{0}$ with $\vec{0}$ because they're equal—*but that is exactly what needs to be proven*! This argument only shows that $\vec{0} + \vec{0} = \vec{0}$ is true if we *already* know it to be true, so it's a worthless argument.

16. **Don't write the proof backwards.** For some reason, beginning students tend to write "proofs" like this "proof" that $\tan^2 x = \sec^2 x - 1$:

Claim: $\tan^2 x = \sec^2 x - 1$.
Proof:
$$\begin{aligned} \tan^2 x &= \sec^2 x - 1 \\ \left(\tfrac{\sin x}{\cos x}\right)^2 &= \tfrac{1}{\cos^2 x} - 1 \\ \tfrac{\sin^2 x}{\cos^2 x} &= \tfrac{1-\cos^2 x}{\cos^2 x} \\ \sin^2 x &= 1 - \cos^2 x \\ \sin^2 x + \cos^2 x &= 1 \\ 1 &= 1 \end{aligned}$$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement "$1 = 1$." In other words, he has proved "If $\tan^2 x = \sec^2 x - 1$, then $1 = 1$," which is true but highly uninteresting.

Now this isn't a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it's time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

17. **Be concise.** Most students err by writing their proofs too short, so that the reader can't understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it's probably because you don't really have a proof, but just an intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.

18. **Avoid weasel words.** There are some notorious phrases that advertise that you don't really understand the logic you need. Be wary of phrases like "clearly," "obviously," and "the only way this can happen is. . . ."

19. **Introduce every symbol you use.** If you use the letter "k," the reader should know exactly what k is. Good phrases for introducing symbols include "Let $x \in G$," "Let k be the least integer such that. . . ," "For every real number a. . . ," and "Suppose that X is a counterexample."

20. **Use appropriate quantifiers, once.** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean "for all $x \in S$" or merely "for some $x \in S$." If you just say something like "$y = x^2$ where $x \in S$," the word "where" doesn't indicate which of the two you mean.

    Phrases indicating the quantifier "for all" ($\forall$) include "Let $x \in S$"; "for all $x \in S$"; "for every $x \in S$"; "for each $x \in S$"; etc. Phrases indicating the quantifier "there exists" ($\exists$) include "for some $x \in S$"; "there exists an $x \in S$"; "for a suitable choice of $x \in S$"; etc.

    On the other hand, don't introduce a variable more than once! Once you have said "Let $x \in S$," the letter x has its meaning defined. You don't *need* to say "for all $x \in S$" again, and you definitely should *not* say "let $x \in S$" again.

21. **Use a symbol to mean only one thing.** Once you use the letter x once, its meaning is fixed for the duration of your proof. You cannot use x to mean anything else.

22. **Don't "prove by example."** Most problems ask you to prove that something is true "for all"—for all $x \in G$, say, or for all vector spaces V. You *cannot* prove this by giving a single example. Your answer will need to be a logical argument that holds for *every example there could possibly be*.

23. **Write "Let $x = \ldots$," not "Let $\ldots = x$."** When you have an existing expression, say $\vec{u} - r\vec{v}$, and you want to give it a new, simpler name like $\vec{x}$, you should write "Let $\vec{x} = \vec{u} - r\vec{v}$," which means, "Let the new symbol $\vec{x}$ mean $\vec{u} - r\vec{v}$." This convention makes it clear to the reader that $\vec{x}$ is the brand-new symbol and $\vec{u} - r\vec{v}$ is the old expression he already understands.

    If you were to write it backwards, saying "Let $\vec{u} - r\vec{v} = \vec{x}$," then your startled reader would ask, "What if $\vec{u} - r\vec{v} \neq \vec{x}$? And for that matter, what is $\vec{x}$?"

24. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but disproofs—counterexamples—should be absolutely concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must name its elements, and for a function you must give its rule. Do not say things like "θ could be one-to-one but not onto"; instead, produce an actual function θ that *is* one-to-one but not onto.

25. **Shun pronouns, especially "it."** Pronouns do have a proper place in the English language, of course. When you write a proof, however, you are usually juggling several mathematical objects at once, and when you use a pronoun like "it," too often the reader won't be able to tell which of them is the antecedent.

    Never use a pronoun unless the antecedent is crystal clear from the grammar itself.

26. **Don't include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it doesn't need an example to back it up. If your logic is bad, a dozen examples won't help it (see rule 22). There are only two legitimate reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.