The Jacobian Matrix and the Chain Rule

Let $\mathbb{R}^n = \{(x_1, \ldots, x_n) | x_i \in \mathbb{R}\}$ be the $n$-dimensional Euclidean space. If $f : \mathbb{R}^n \to \mathbb{R}^m$ then we write $f(x) = (f_1(x), \ldots, f_m(x))$ where $f_i : \mathbb{R}^n \to \mathbb{R}$ is the $i$-th coordinate function of $f$.

Example 1. If $f : \mathbb{R} \to \mathbb{R}^2$, $f(t) = (t^2, t - 1)$ then $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ where $f_1(t) = t^2$ and $f_2(t) = t - 1$.

Definition 2. A linear transformation is a function $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $L(\alpha a) = \alpha L(a)$ and $L(a + b) = L(a) + L(b)$ for all $\alpha \in \mathbb{R}$ and $a, b \in \mathbb{R}^n$.

Example 3. $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x, y) = (x - y, 2y)$ is a linear transformation. We have

$$L(\alpha(x, y)) = L(\alpha x, \alpha y) = (\alpha x - \alpha y, 2\alpha y) = \alpha(x - y, 2y) = \alpha L(x, y)$$

and the reader can easily verify the other condition.

Note that every linear transformation takes the zero vector to the zero vector. In this example $L(0, 0) = (0 - 0, 20) = (0, 0)$. This means that shifting the space is not a linear transformation.

Example 4. $L : \mathbb{R} \to \mathbb{R}^2$, $L(x) = (2x, x - 1)$ is not a linear transformation because for example

$$L(2x) = (2(2x), 2x - 1) \neq (4x, 2x - 2) = 2(2x, x - 1) = 2L(x).$$

The problem is the $-1$ in the second coordinate function, which is a shift. Note that $L(0) = (0, -1)$ which is not the zero vector.

Example 5. $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x, y) = (-2y, 2x)$ is the linear transformation that rotates the plain by 90 degrees around the origin and stretches it away from the origin by a factor of 2.

Definition 6. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then the coordinate vector of $x$ in the standard basis is the column vector $[x] = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Example 7. If $a = (2, -1) \in \mathbb{R}^2$ then $[a] = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. And if $a = (3) \in \mathbb{R}^1$ then $[a] = (3)$ is a $1 \times 1$ matrix.
Theorem 8. Every linear transformation \( L \) is determined by a matrix \([L]\) such that \([L(x)] = [L][x]\). That is, the coordinate vector of \( L(x) \) is the matrix product of \([L]\) and the coordinate vector \([x]\) of \(x\).

Example 9. The matrix of the linear transformation \( L \) in Example 3 is \([L] = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\), because

\[
[L(x, y)] = [(x - y, 2y)] = \begin{pmatrix} x - y \\ 2y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = L[(x, y)].
\]

Example 10. If \( L : \mathbb{R}^1 \to \mathbb{R}^2 \) and \([L] = \begin{pmatrix} 4 \\ 3 \end{pmatrix}\) then \([L(t)] = [L][(t)] = \begin{pmatrix} 4t \\ 3t \end{pmatrix}\) and so \(L(t) = (4t, 3t)\).

Remark 11. If \( L : \mathbb{R}^n \to \mathbb{R}^n \) is a linear transformation then the determinant of the matrix tells us how the size of a region \(R\) in the domain will change when we apply the linear transformation \( L \)

\[
\text{size}(L(R)) = \det[L] \cdot \text{size}(R).
\]

This is why the Jacobian, which is the determinant of the Jacobian matrix, is showing up in the multivariable version of the change of variable formula for integrals.

Definition 12. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be differentiable at \(a \in \mathbb{R}^n\). The differential \(d_a f\) of \(f\) at \(a\) is the linear transformation determined by the matrix

\[
J_a f = \begin{pmatrix} D_1 f_1(a) & \cdots & D_m f_1(a) \\
\vdots & \ddots & \vdots \\
D_1 f_m(a) & \cdots & D_m f_m(a) \end{pmatrix}.
\]

\(J_a f\) is called the Jacobian matrix of \(f\) at \(a\).

Example 13. If \( f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (xy, x + y)\) then

\[
J_{(1,2)} f = \begin{pmatrix} \frac{\partial}{\partial x} xy \bigg|_{(1,2)} & \frac{\partial}{\partial y} xy \bigg|_{(1,2)} \\
\frac{\partial}{\partial x} (x + y) \bigg|_{(1,2)} & \frac{\partial}{\partial y} (x + y) \bigg|_{(1,2)} \end{pmatrix}
\]

\[
= \begin{pmatrix} y \bigg|_{(1,2)} & x \bigg|_{(1,2)} \\
1 \bigg|_{(1,2)} & 1 \bigg|_{(1,2)} \end{pmatrix}
\]

\[
= \begin{pmatrix} 2 & 1 \\
1 & 1 \end{pmatrix}.
\]
Example 14. If \( f : \mathbb{R} \to \mathbb{R} \) then the Jacobian matrix is a \( 1 \times 1 \) matrix

\[
J_x f = (D_1 f_1(x)) = \left( \frac{\partial}{\partial x} f(x) \right) = (f'(x))
\]

whose only entry is the derivative of \( f \). This is why we can think of the differential and the Jacobian matrix as the multivariable version of the derivative.

The differential gives the local linearization of a function:

\[
f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) - f(x_1, \ldots, x_n) \approx d(x_1, \ldots, x_n)(\Delta x_1, \ldots, \Delta x_n)
\]

That is, the change in the output can be approximated by the value of the differential at the change in the input.

Exercise 15. Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), \( f = (f_1, f_2) \) where \( f_1(x, y) = 2x - y \), \( f(2, 1) = (3, 2) \), \( \frac{\partial}{\partial x} f_2(x, y) \bigg|_{(2, 1)} = -2 \) and \( \frac{\partial}{\partial y} f_2(x, y) \bigg|_{(2, 1)} = 3 \). Approximate \( f(2.1, 0.8) \).

Solution. We have \( f(2.1, 0.8) - f(2, 1) \approx d(2, 1)(2.1 - 2, 0.8 - 1) \). Using the Jacobian matrix we have

\[
[d(2, 1)(0.1, -0.2)] = J_{(2, 1)} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ -0.8 \end{pmatrix}.
\]

Thus, \( f(2.1, 0.8) \approx f(2, 1) + (0.4, -0.8) = (3, 2) + (0.4, -0.8) = (3.4, 1.2) \). \( \square \)

Theorem 16. (Chain Rule) If \( f : \mathbb{R}^m \to \mathbb{R}^n \) is differentiable at \( a \in \mathbb{R}^m \) and \( g : \mathbb{R}^m \to \mathbb{R}^l \) is differentiable at \( f(a) \), then \( h = g \circ f \) is differentiable at \( a \) and

\[
d_a h = d_{f(a)} g \circ d_a f.
\]

With matrix notation

\[
J_a h = J_{f(a)} g \cdot J_a f.
\]

Example 17. If \( f : \mathbb{R} \to \mathbb{R} \), \( g : \mathbb{R} \to \mathbb{R} \) and \( h = g \circ f \) then

\[
(h'(x)) = J_x h \quad \text{by Example 14}
\]

\[
= J_{f(x)} g \cdot J_x f \quad \text{by the Chain Rule}
\]

\[
= (D_1 g(f(x))) \cdot (D_1 f(x))
\]

\[
= (g'(f(x))) \cdot (f'(x))
\]

\[
= (g'(f(x))f'(x))
\]

This is the well-known 1-dimensional version of the chain rule.
Example 18. Let \( f : \mathbb{R} \to \mathbb{R}^2 \), \( f(x) = (x, x^2) \) and \( g : \mathbb{R}^2 \to \mathbb{R} \), \( g(x, y) = x^3 + 2xy \). Then
\[
h(x) = (g \circ f)(x) = g(f(x)) = g(x, x^2) = x^3 + 2x^3 = 3x^3
\]
and so \( h'(x) = 9x^2 \). We get the same answer using the chain rule.

\[
(h'(x)) = J_x h \quad \text{by Example 14}
\]
\[
= J_{f(x)}g \cdot J_x f \quad \text{by the Chain Rule}
\]
\[
= \left( \frac{\partial}{\partial x}(x^3 + 2xy) \right|_{f(x)} \frac{\partial}{\partial y}(x^3 + 2xy) \right|_{f(x)} \cdot \left( \begin{array}{c} 1 \\ 2x \end{array} \right)
\]
\[
= \left( 3x^2 + 2y \right|_{(x,x^2)} \frac{2}{2x} \right|_{(x,x^2)} \cdot \left( \begin{array}{c} 1 \\ 2x \end{array} \right)
\]
\[
= (5x^2 \ 2x) \cdot \left( \begin{array}{c} 1 \\ 2x \end{array} \right)
\]
\[
= (9x^2).
\]

Example 19. Let \( z = x^2 + 2y \), \( x = 3t \) and \( y = \sin(t) \). We can find \( \frac{dz}{dt} \) in two different ways. A direct calculation gives \( z = (3t)^2 + 2\sin(t) \) and so \( \frac{dz}{dt} = 18t + 2\cos(t) \). We can get the same result using the chain rule. If we define \( g(x, y) = x^2 + 2y \) and \( f(t) = (3t, \sin(t)) \) then \( z = g(f(t)) \) and so by the chain rule

\[
(\frac{dz}{dt}) = J_{f(t)}g \cdot J_t f
\]
\[
= (2x|_{f(t)} \ 2|_{f(t)}) \cdot \left( \begin{array}{c} 3 \\ \cos(t) \end{array} \right)
\]
\[
= (6t \ 2) \cdot \left( \begin{array}{c} 3 \\ \cos(t) \end{array} \right) = (18t + 2\cos(t)).
\]

Of course the first way was much simpler in this particular case. Sometimes we don’t have the exact formula for our functions and the chain rule is the only way to go.

The situation in this example is a very important special case. It is useful to remember that the chain rule in this case is in the form

\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt}.
\]

This is because \( J_{f(t)}g = \left( \frac{dx}{dx} \ \frac{dy}{dy} \right) \) and \( J_t f = \left( \frac{dx}{dt} \ \frac{dy}{dt} \right) \).
Exercise 20. Let \( f : \mathbb{R}^2 \to \mathbb{R} \), \( g : \mathbb{R}^2 \to \mathbb{R}^2 \), \( g(x, y) = (x^2y, x - y) \) and \( h = f \circ g \). Find \( h_x(1, 2) \) if \( f_x(2, -1) = 3 \) and \( f_y(2, -1) = -2 \).

Solution. We have

\[
(h_x(1, 2) \quad h_y(1, 2)) = J_{(1,2)}h \\
= J_{g(1,2)}f \cdot J_{(1,2)}g \quad \text{(by the chain rule)} \\
= (f_x(g(1,2)) \quad f_y(g(1,2))) \left( \begin{array}{cc}
D_1g_1(1,2) & D_2g_1(1,2) \\
D_1g_2(1,2) & D_2g_2(1,2)
\end{array} \right) \\
= (3 \quad -2) \left( \begin{array}{cc}
2xy\big|_{(1,2)} & x^2\big|_{(1,2)} \\
1\big|_{(1,2)} & -1\big|_{(1,2)}
\end{array} \right) \\
= (3 \quad -2) \left( \begin{array}{cc}
4 & 1 \\
1 & -1
\end{array} \right) = (10 \quad 5)
\]

and so \( h_x(1, 2) = 10 \). \( \square \)

Exercise 21. Let \( x = (2, t, -1) \). Find the coordinate vector \([3x]\) of \(3x\).

Exercise 22. Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \), \( L(x, y) = (3x - y, x + 2y) \). Find the matrix \([L]\) of \(L\).

Exercise 23. Let \( L : \mathbb{R}^2 \to \mathbb{R} \) and \([L] = (-1 \quad 2)\). Find \(L(x, y)\) and \(L(3, -2)\).

Exercise 24. What is the size of the Jacobian matrix of \( f : \mathbb{R}^3 \to \mathbb{R}^2\)?

Exercise 25. Let \( g : \mathbb{R}^3 \to \mathbb{R} \), \( g(x, y, z) = x^2y - yz \). Find the Jacobian matrix \(J_{(1,t,2)}g\).

Exercise 26. Let \( h : \mathbb{R} \to \mathbb{R}^2 \), \( h(x) = (\sin(2x), -x^2) \). Find \(J_{\pi}h\).

Exercise 27. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( f(1, 2) = (3, 4) \). Approximate \( f(1.2, 1.8) \) if \( J_{(1,2)}f = \left( \begin{array}{cc}
1 & -1 \\
2 & -3
\end{array} \right) \).