Test 2 316

Work 6 of the first 8 problems *and* either Problem A or Problem B. *No notes, books, or calculators.*

(1) State *carefully* 5 of the 10 *axioms* for a collection of objects V to be vector space.

(2) Let V and W be vector spaces. Define what it means for:

(a) A transformation $T : V \to W$ to be *linear.*

(b) A set $\{v_1, \ldots, v_p\}$ of vectors in V to be *linearly independent.*

(c) A collection of vectors in V to be a *basis* for a subspace H of V.

(d) The spaces V and W to be *isomorphic.*

(3) Compute the determinants (support your answer) *and determine if the given matrix A is invertible*:

(a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

(4) Let P_2 denote the set of all polynomials of degree less than or equal to 2. Determine if the following sets H are a subspace of P_2, and if so, determine a basis for H and the dimension of H.

(a) $H = \{p \in P_2 | p(-t) = -p(t)\}$

(b) $H = \{p \in P_2 | p'(1) = 0\}$

(c) $H = \{p \in P_2 | \int_{-1}^{1} tp(t)\, dt = 1\}$

(5) Consider the parallelogram P determined by $u = (1, 2)^T$ and $v = (-2, 1)^T$.

(a) Determine the area of P.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ has standard matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, compute the area of the parallelogram $T(P)$.

(6) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$. Given the system $Ax = b$, where $x = (x_1, x_2, x_3)^T$, solve for x_1 using Cramer's Rule.

(7) Let $T : P_1 \to P_1$ be defined by $T(a + bx) = b$.

(a) Find a basis for $\ker(T)$.

(b) What is the standard basis for P_1? Write down the standard matrix for T.

(c) If possible, compute $T^{-1}(a + bx)$.

(8) Let V be the vector space of all upper triangular 2×2 matrices and let $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

(a) Find a basis B for V.

(b) Given the linear transformation $T : V \to V$ defined by $T(M) = AM$, find the matrix $[T]_B$.

Work either Problem A or Problem B.

A Given a linear transformation $T : V \to W$, show that $H = \text{range}(T)$ is a subspace of W.

B True or False? *If False, make a reasonable and related true statement; if True, support the claim briefly.*

(a) If $B = \{b_1, \ldots, b_p\}$ is a basis for a subspace W of a vector space V and $u \in V$, then there exists unique scalars $c_1, \ldots, c_p$ such that $u = \sum_{i=1}^{p} c_i b_i$.

(b) The number of linearly independent rows in a matrix equals the number of linearly independent columns in that same matrix.

(c) P_2 is isomorphic to $\mathbb{R}^2$.

(d) If a basis for a vector space W consists of n vectors, then every basis of W consists of n vectors.

(e) If two vector spaces V and W are isomorphic, then there exists an invertible linear map $T : V \to W$.