Test 3 316

Work Problems 1 and 2, and 4 of Problems 3-9. Clearly mark through omitted problems. No notes, books, or calculators allowed.

(1) Define and/or State:

(a) x is an *eigenvector* of $A \in M_{n \times n}$ corresponding to an *eigenvalue* λ

(b) The *algebraic* and *geometric multiplicities* of an eigenvalue

(c) Given $T(x) = Ax$ where the $n \times n$ matrix A has n eigenvectors in V, give the relationships between T and the diagonal matrix D of corresponding eigenvalues, and between $[x]_V$, and $T(x)$.

(d) The Power Method for finding the largest (in magnitude) eigenvalue of a matrix A.

(e) The Inverse Power Method with Shifting for finding the nearest eigenvalue to r of a matrix A.

(2) Define and/or Discuss:

(a) The formula for $P_{\perp v}u$, the projection of u orthogonal to v

(b) The angle between two vectors u and v

(c) The stadard norm, inner product, and distance notion for vectors in $\mathbb{R}^n$

(d) $\{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$

(e) UU^T, where the columns of U form an orthonormal basis for an m-dimensional subspace W of $\mathbb{R}^n$

(f) The Orthogonal Decompostion Theorem

(3) Let $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Compute $V^{-1}AV$.

(4) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Iterate the dynamical system $x^0 = (1,0)^T$, $x^{k+1} = Ax^k$ and plot $\{x_0, x_1, x_2, x_3, x_4\}$.

(5) Consider an $n \times n$ matrix A with an eigenvector x.

(a) Show that x is also an eigenvector for A^{-1}. What is the relationship between the corresponding eigenvalues of A and A^{-1}?

(b) Show that x is also an eigenvector for A^2. What is the relationship between the corresponding eigenvalues of A and A^2?

(c) If $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ (see Problem 3), write down a formulae for A^k.

(6) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -1 \\ 0 & -1 & 6 \end{bmatrix}$.

(a) Show that λ is an eigenvalue of A if and only if $\lambda - 2$ is an eigenvalue of $B = A - 2I$.

(b) Using the initial guess $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, perform one iteration of the shifted inverse power method with $r = 2$, that is, compute $x_1 = \frac{B^{-1}x_0}{|B^{-1}x_0|_\infty}$.
Hint: Solve $By = x_0$ *to compute* $y = B^{-1}x_0$.

(c) Compare your approximate eigenvalue of B^{-1} to the actual eigenvalue of A nearest to 2, as well as the corresponding approximate and actual eigenvectors.

(7) Perform the Gram-Schmidt Process (orthogonalize only, no need to normalize) on the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(8) Let A be a matrix and consider the homogeneous system $Ax = 0$. Explain the statement $(\text{Row})^{\perp} = \text{Nul}\, A$.

(9) Let W be the plane through the origin in $\mathbb{R}^3$ consisting of all vectors orthogonal to $(1, 2, 1)^T$.

(a) Find an orthogonal basis for W.

(b) Compute the projection $P_W \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) Compute the projection $P_{\perp W} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.