Exam 2 MAT661

Work 6 Problems, clearly marking which problems are to be graded.

(1) Consider the "transport" PDE $u_t + cu_x = 0$, $x \in \mathbb{R}$, $t > 0$, for a given constant $c > 0$. Write down a general solution. Find *and verify* a solution to the corresponding IVP given by $u(0, x) = f(x)$.

(2) State *carefully* (style points!) the Heat Equation in time and one space variable $x \in (0, 1)$ with zero boundary conditions and a generic initial temperature distribution f.

(a) Write down the general solution to the BVP (no initial condition).

(b) Write down the exact solution when the initial temperature is the L_2-normalized eigenfunction corresponding to the eigenvalue $\lambda = 9\pi^2$.

(3) Consider the explicit method for approximating solutions to the Heat Equation.

(a) Briefly and conveniently, write down the method in *matrix form.* Do define all symbols and notations adequately.

(b) Divide the space interval $(0,1)$ into $n = 3$ subintervals, and take the initial temperature distribution to be given by $f(x) = c \sin \pi x$ (chose c conveniently). Perform one step of the explicit method to approximate the temperature u at time $t = \Delta t = 1$.

(4) Consider the implicit method for approximating solutions to the Heat Equation.

(a) Briefly and conveniently, write down the method in *matrix form.* Do define all symbols and notations adequately.

(b) Similarly to above, perform one step of the implicit method to approximate the temperature u at time $t = \Delta t = 1$.

(5) Consider the 2-D Heat Equation on the Square.

(a) Write down equation with appropriate boundary and initial conditions.

(b) Write down the corresponding general solution using linear combinations of the eigenfunctions.

(c) Write down the exact solution if the initial temperature is the L_2-normalized eigenfunction corresponding to the eigenvalue $\lambda = 8\pi^2$.

(6) Consider the 2-D Heat Equation on the Disk.

(a) Write down the equation, with appropriate boundary and initial conditions, in terms of polar coordinates.

(b) *Assuming* the basis is nicely ordered with eigenfunctions $\{\psi_i\}$ and corresponding eigenvalues $\{\lambda_i\}$, write down the corresponding general solution using linear combinations of the eigenfunctions.

(c) Write down the exact solution if the initial temperature is *a* non normalized eigenfunction corresponding to the eigenvalue which corresponds to the 2^{nd} root of the 3^{rd} Bessel Function.

(7) Consider the least-squares fit of a line $y = mx + b$ to a collection of data $\{(x_i, y_i)\}_{i=1,\ldots,n}$.

(a) Derive the 2×2 linear system for the least-squares fit of a line $y = mx + b$ to a collection of data $\{(x_i, y_i)\}_{i=1,\ldots,n}$ by writing down an overdetermined linear system and multiplying both sides by the appropriate matrix.

(b) Solve the system and plot relevant objects for the data $\{(0,0),(1,2),(2,0)\}$.

(8) Consider doing a Least-Squares fit of functions $y = ax^2 + bx + c$ to the data $\{(0,0),(1,1),(2,4)\}$.

(a) Write down the Error function and compute the gradient. Is the resulting system linear or nonlinear? Overdetermined or Consistent?

(b) Think about the type of curve you are fitting, reflect on the data, and write down the solution to this system (hopefully with no real arithmetic work!).

(9) State the wave equation in one time variable and two space variables on the unit square, together with reasonable initial and boundary conditions.

(a) Sketch the separation of variables argument just far enough so that you have one or more eigenvalue problems set up.

(b) Given the appropriate initial conditions, explain two methods for computing an approximation to the displacement at the *first* time step. Why do you need this special case in order to implement an explicit solver?

(10) Consider an $n = 4$ divisions by $n = 4$ divisions grid of the unit square and functions that are potentially nonzero at the nodes $\{(1, 2), (2, 1), (2, 2), (3, 2), (2, 3)\}$ (lying on an Plus-shape), but zero at the other 20 grid points.

(a) Write down the *negative Laplacian matrix* representing the negative second derivative operator on this region.

(b) Verify that $(1, -1, 0, 1, -1)$ is an eigenvector, give the corresponding eigenvalue, and give a "contourplot" of the eigenfunction approximation.