FIXED POINT CONVERGENCE AND NEWTON'S METHOD

Definition 0.1. Given $g : \mathbb{R} \to \mathbb{R}$, if $g(x) = x$ for some $x \in \mathbb{R}$ we say that x is a *fixed point* of g.

Problem 0.2. Given $g : \mathbb{R} \to \mathbb{R}$, define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x - g(x)$. Show that g has a fixed point x if and only if x is a root of h.

Fact 0.3. *The Intermediate Value Theorem asserts that given a continuous function $f : [a, b] \to \mathbb{R}$ with $f(a)f(b) < 0$, there exists $p \in (a, b)$ with $f(p) = 0$.*

Problem 0.4. (Theorem) Let $g : [a, b] \to [a, b]$ be a continuous function. Show that g has a fixed point.

Hint: Consider the Intermediate Value Theorem.

Fact 0.5. *Recall that the Mean Value Theorem states that given a continous function $f : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), there exists $c \in (a, b)$ so that $f(b) - f(a) = f'(c)(b - a)$.*

Problem 0.6. (Theorem) Let g be as in Theorem 0.4 and assume also that g is differentiable on (a, b) with $|g'(x)| \leq k$ for all $x \in (a, b)$, for some $k < 1$. Show that the fixed point is unique.

Hint: Suppose there are two distinct fixed points and apply the Mean Value Theorem.

Problem 0.7. (Theorem) Let g be as in Theorem 0.6 and assume that $p_0 \in [a, b]$. Define a sequence (p_k) by $p_k = g(p_{k-1})$. Show that (p_k) converges to the unique fixed point p of g.

Hint: Apply the Mean Value Theorem repeatedly (to $|p_{k+1} - p_k| = |g(p_k) - g(p)|$).

Definition 0.8. In general a function g generates an *iterated function sequence* $p_k = g(p_{k-1})$. Observe that if the sequence converges, then it converges to a fixed point of g; we call such a process *fixed point iteration.*

Problem 0.9. (Algorithm) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, and $(p_0, f(p_0))$ is a point on the graph of f. Let p_k be the x-intercept of the tangent line of f at the previous point $(p_{k-1}, f(p_{k-1}))$. Show that (p_k) is an iterated function sequence and find the function that generates this sequence. That is, find a recursive formula for p_k.

Problem 0.10. Show that if the above iterated function sequence converges then fixed point is a root of f.

Definition 0.11. The algorithm found in Problems 0.9 and 0.10 is called *Newton's method* for finding roots.

Problem 0.12. (Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be C^2, $f(p) = 0$ and $f'(p) \neq 0$. Then there is a $\delta > 0$ such that Newton's method converges to p for all initial point $p_0 \in (p - \delta, p + \delta)$.