Q2 661

Work 2 of the 5 problems, clearly marking which problems are to be graded. No notes, books, or calculators.

(1) Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Perform 2 iterations of Gauss-Sidel and 2 iterations of Jacobi to approximate a solution to $Ax = b$, using the zero vector as an initial guess. Compare your approximations to the actual answer using an appropriate norm.

(2) Approximate solutions to $x^2 - 2 = 0$ and $x^2 = 0$ using 2 iterations of Newton's method with an initial guess of $x = 1$. Compare your approximations to the actual roots, and discuss the applicability of the Newton convergence theorem (Problem 0.12) in each case.

(3) Let $h_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $h_1(x, y) = \frac{1}{2}(x^2 + y^2)$ and $h_2(x, y) = \frac{1}{2}(x^2 - y^2)$. Perform 1 iteration of Newton's method to approximate a critical point of each function, using an initial guess of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compare your approximations to the actual zeroes of the corresponding gradients and use the eigenvalues of the corresponding Jacobians of the gradients (Hessians of the original functions) to classify the critical points as minima, maxima, or saddle points. Can you explain why the algorithm worked as well as it did in each case?

(4) Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Write down the domain and co-domain of the functions f', $f'(x)$, f'', and $f''(x)$ and discuss whether the functions are linear, bilinear, or nonlinear in each case. Where appropriate, relate the functions to the gradient and/or the Hessian of f.

(5) Let $g : [a, b] \to [a, b]$ be a continuous function. Prove that g has a fixed point p in $[a, b]$. If in addition g is differentiable on (a, b) with $|g'(x)| \leq k < 1$ for all $x \in (a, b)$, prove that the fixed point is unique, and that the iteration $p_0 \in (a, b)$, $p_k = g(p_{k-1})$ converges to p.