

# **Numerical Analysis Of Nonlinear Differential Equations**

Hon 485  
Undergraduate Research

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Differential equations are often used in modeling the physical world. Astrophysics uses them to model energy transport, gravitational forces, and many other aspects of stars. One particular type of nonlinear partial differential equation used in modeling gravitational potential in stars is the Lane-Emden equation. This equation can be taken to be a singular ordinary differential equation (ODE) when modeling the gravitational potential throughout a star that is assumed to be radially symmetric. The purpose of this research was to solve this equation using numerical techniques not traditionally used by physicists and to develop an equation that would model the more physically realistic non-radially symmetric stars.

The radially symmetric case of this is what is commonly used to model stars. This means that the symmetric Lane-Emden has only a radial component and has no angular dependences. The partial differential equation (PDE) looks like  $\Delta u + u^n = 0$  and the radially symmetric case looks like  $\frac{d^2 u}{dr^2} + \frac{2}{r} \frac{du}{dr} + u^n = 0$ . The  $u$  in these equations is a unit less function that appears as  $u = \frac{\Phi}{\Phi_c} = \left( \frac{\rho}{\rho_c} \right)^{\frac{1}{n}}$ , where  $\Phi$  and  $\Phi_c$  are the gravitational potentials a distance  $R$  away from the center of the star and at the star's center respectively. The symbols  $\rho$  and  $\rho_c$  represent the density a distance  $r$  away from the center of the star and at the star's center respectively.

Three well-known physics equations are used to derive the Lane-Emden equation (Kippenhahn & Weigert). The first of these is the equation of radially symmetric hydrostatic equilibrium,  $\frac{dP}{dr} = -\frac{d\Phi}{dr} \rho$ . In this equation, the pressure  $P$  is related to the gravitational potential  $\Phi$  and the density  $\rho$ . Poisson's equation is the second equation used in the derivation. Poisson's equation (radially symmetric form),

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho,$$

relates the gravitational potential to the density, where  $G$  is the gravitational constant. Finally the third equation of significance in the derivation of the

Lane-Emden is the Polytropic relation,  $P = K \rho^\gamma = K \rho^{1 + \frac{1}{n}}$ . This equation relates the

pressure to the density, where  $K$  is some polytropic constant and  $n$  is the polytropic index for a particular star.

The derivation of the Lane-Emden begins with the hydrostatic equilibrium (HE),

$$\frac{dP}{dr} = -\frac{d\Phi}{dr} \rho, \text{ equation and the polytropic relation (PR), } P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}. \text{ Taking a}$$

derivative of (PR) with respect to  $r$ , we obtain  $\frac{dP}{dr} = K(\gamma)\rho^{\gamma-1} \frac{d\rho}{dr}$ . Substituting this into

$$\text{(HE), one gets } -\frac{d\Phi}{dr} \rho = K(\gamma)\rho^{\gamma-1} \frac{d\rho}{dr} \text{ which implies that } \frac{d\Phi}{dr} = -K(\gamma)\rho^{\gamma-2} \frac{d\rho}{dr}.$$

Integrating both sides with respect to  $r$ , we obtain  $\Phi = -\gamma K \left( \frac{1}{\gamma-1} \right) \rho^{\gamma-1}$ . But since

$$n = \frac{1}{\gamma-1}, \text{ we have } \Phi = -(n+1)K\rho^{\gamma-1}. \text{ This can be rewritten as } \rho^{\frac{1}{n}} = \frac{-\Phi}{(n+1)K} \text{ or}$$

$$\rho = \left( \frac{-\Phi}{(n+1)K} \right)^n. \text{ Substituting this into Poisson's Eq. yields}$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \left( \frac{-\Phi}{(n+1)K} \right)^n. \text{ Canceling terms, we obtain (1)}$$

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r} \frac{d\Phi}{dr} = 4\pi G \left( \frac{-\Phi}{(n+1)K} \right)^n. \text{ To remove the constants, the following unit}$$

substitutions will be made:  $z = Ar$ ,  $dz = A dr$ ,

$$A^2 = \frac{4\pi G}{(n+1)^n K^n} (-\Phi_c)^{n-1} = \frac{4\pi G}{(n+1)K} \rho^{\frac{n-1}{n}}, \text{ and } u = \frac{\Phi}{\Phi_c} = \left( \frac{\rho}{\rho_c} \right)^{\frac{1}{n}}. \text{ The left side of (1)}$$

$$\text{then becomes } A^2 \frac{d^2\Phi}{dz^2} + \frac{2A}{z} \frac{d\Phi}{dz} A = A^2 \left( \frac{d^2\Phi}{dz^2} + \frac{2}{z} \frac{d\Phi}{dz} \right). \text{ Using the fact}$$

$$\text{that } A^2 = \frac{4\pi G}{(n+1)^n K^n} (-\Phi_c)^{n-1}, \text{ (1) now appears as}$$

$$A^2 \left( \frac{d^2\Phi}{dz^2} + \frac{2}{z} \frac{d\Phi}{dz} \right) = \frac{4\pi G}{(n+1)^n K^n} (-\Phi)^n \text{ or}$$

$\frac{4\pi G}{(n+1)^n K^n} (-\Phi_c)^{n-1} \left( \frac{d^2\Phi}{dz^2} + \frac{2}{z} \frac{d\Phi}{dz} \right) = \frac{4\pi G}{(n+1)^n K^n} (-\Phi)^n$ . This reduces to

$(-\Phi_c)^{n-1} \left( \frac{d^2\Phi}{dz^2} + \frac{2}{z} \frac{d\Phi}{dz} \right) = (-\Phi)^n$  and after substituting in  $u = \frac{\Phi}{\Phi_c} = \left( \frac{\rho}{\rho_c} \right)^{\frac{1}{n}}$ , we obtain

$\frac{d^2u}{dz^2} + \frac{2}{z} \frac{du}{dz} = \frac{(-1)^n}{(-1)^{n-1}} \left( \frac{\Phi}{\Phi_c} \right)^n$  or  $\frac{d^2u}{dz^2} + \frac{2}{z} \frac{du}{dz} = -u^n$ . We now have the Lane-Emden

equation for the radially symmetric case:

$$(2) \frac{d^2u}{dz^2} + \frac{2}{z} \frac{du}{dz} + u^n = 0,$$

where  $z$  is the radial distance from the center.

This derivation yields a nonlinear, singular, second order ordinary differential equation (ODE). In the past, this equation was solved using a power series expansion of  $u(z)$ , i.e.  $u(z) = 1 + a_1z + a_2z^2 + a_3z^3 + \dots$  (Kippenhahn & Weigert). We used a different numerical approach to solve this equation. We use Bessel's Equation and the Euler Method to approximate solutions to the ODE. It is not possible to use Euler's Method alone to look for solutions starting at  $z = 0$ . Instead the nonlinear ODE must be approximated near zero via a linearized equation, and then using that result to generate initial data for Euler's Method.

To find these initial points to be used in the Euler Method, we used Bessel's equation. We first linearized (2) by linearizing  $u^n$  at a point  $(d, d^n)$ , where  $d$  is whatever peak value that is to be used for  $u(0)$ . To do this, we take the derivative of  $u$  to find the slope of the tangent line at that point. Next, we use the equation for a line,  $y = mx + b$ , where in this case,  $y$  is  $f(u)$ ,  $x$  is  $u$ ,  $m$  is the derivative of  $u^n$  at the point  $(d, d^n)$ , and  $b$  is just  $d$ . Thus  $u^n \rightarrow d + nd^{n-1}u$  when linearized. Our new linear equation is

$\frac{d^2u}{dz^2} + \frac{2}{z} \frac{du}{dz} + nd^{n-1}u + d = 0$ . Comparing this to the general Bessel's equation,

$$\frac{d^2u}{dz^2} - \left( \frac{2a-1}{z} \right) \frac{du}{dz} + \left( b^2c^2z^{2c-2} + \frac{a^2 - v^2c^2}{z^2} \right) u = 0,$$

one sees that  $a = -\frac{1}{2}$ ,  $b = \sqrt{nd^{n-1}}$ ,  $c=1$ , and  $v = \frac{1}{2}$ . This yields a homogenous solution to the linearized equation of the form  $\frac{c_1}{\sqrt{z}} J_{\frac{1}{2}}(\sqrt{nd^{n-1}}z)$ . We need, however, the particular solution to the linearized equation of the form  $\frac{c_1}{\sqrt{z}} J_{\frac{1}{2}}(\sqrt{nd^{n-1}}z) + c_2$ . We see that for our problem,  $c_2 = \frac{n-1}{n}d$ . The general solution is  $w(r) = \frac{c_1}{\sqrt{z}} J_{\frac{1}{2}}(\sqrt{nd^{n-1}}z) + \frac{n-1}{n}d$ , where  $c_1$  can be found by evaluating  $\frac{1}{\sqrt{z}} J_{\frac{1}{2}}(\sqrt{nd^{n-1}}z)$  at a point very close to zero using the initial conditions  $u(0) = d$  and  $u'(0) = 0$ . Figure 1 below shows over what region this solution was used to find the initial points of the Euler Method.

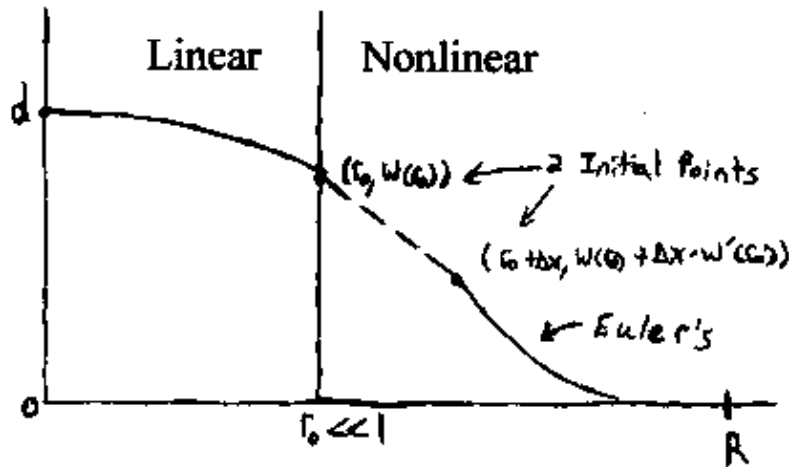


Figure 1

The first of the two initial points used in the Euler Method were found by first defining the general solution to the linearized equation  $w(r)$ . The first point is found at a point  $r_0$ , where  $r_0$  is a point much less than 1,  $r_0 \ll 1$ . The second point is found by following the tangent line through the point  $w(r_0)$  to a point that is some interval  $\Delta z$  away from the point  $z = r_0$ . Keep in mind that  $r$  and  $z$  are interchangeable since both represent

the radial distance from the center of the star. The two initial points for the Euler approximation are then

$$(r_0, w(r_0)) \text{ and } (r_0 + \Delta z, w(r_0) + \Delta z * w'(r_0)).$$

Now that two initial points have been found, the Euler Method can be applied. It is actually a double Euler approximation since the equation to be solved is of second

order. In our case we used a second difference to evaluate  $\frac{d^2u}{dz^2}$  in (2). We wrote a

program in Mathematica to accomplish the task of finding a solution to the radially symmetric Lane-Emden. Below is the portion of this code that actually uses the Euler Method, given the above to initial points found via the linearized equation at  $r_0$ .

```
y={{0, d},{r0, w[r0]},{r0 + dx, w[r0] + dx * (d[w[s], s]/.s->r0)}};
For[i = 3, i ≤ n*R, i ++,
  x = (i-1) * dx + r0;
  x1 = x-dx;
  yim2 = y[[i-1, 2]];
  yim1 = y[[i, 2]];
  yi = (x1/(x1 + dx)) * (2yim1-yim2-dx^2f[yim1] + (dx/x1) * yim2);
  y = Append[y, {x,yi}];
];
```

The line, “ $y_i = (x_1/(x_1 + dx)) * (2y_{i-1} - y_{i-2} - dx^2 f[y_{i-1}] + (dx/x_1) * y_{i-2})$ ,” is the actual Euler step used in the approximation. The Appendix contains the entire code that was used as well as a description of what each symbol represents.

We plotted several solutions to the Lane-Emden ODE for different values of the polytropic index,  $n$ . Some of these solutions can be seen below for the  $n = 3$  and  $n = 5$  cases. These match up precisely with solutions that were found before by others using a power series expansion. The curve represents the ratio of the gravitational potential in the star a distance  $r$  away from its center to the potential at the center. For actual modeling, the gravitational potential at the center is known or the density at the center is known. With that known, it is no problem to find the potential or the density a distance  $r$  away from the star’s center. In the  $n = 3$  case (Figure 2), the point where the potential line crosses the zero axis marks the boundary of the star. In the  $n = 5$  case (Figure 3), the boundary is assumed to be where the potential line is very nearly zero. Note also that in

the  $n = 5$  case, it appears on that scale that the potential line goes to infinity at zero. This is not the case. It actually has a value of one at zero.

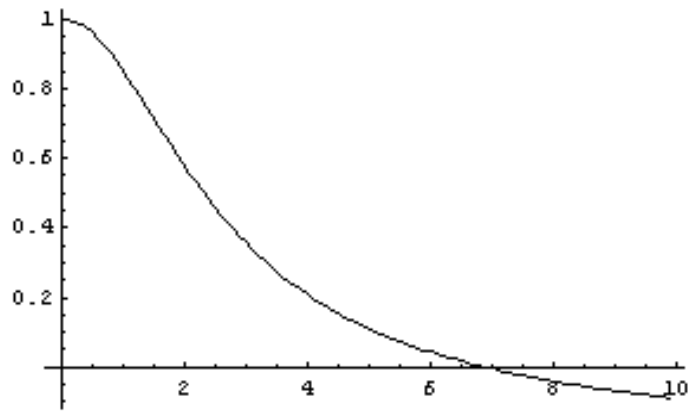


Figure 2

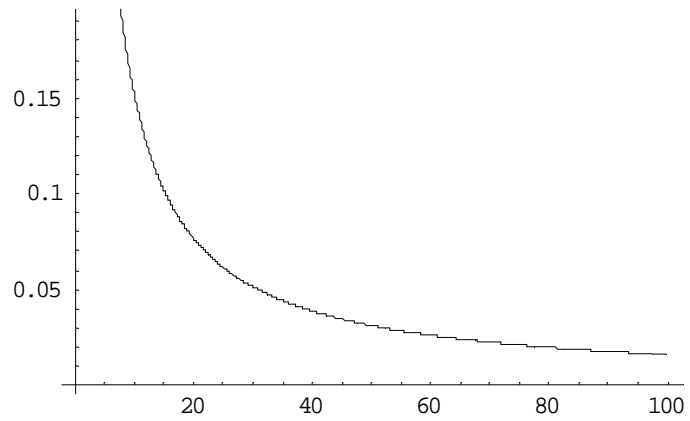


Figure 3

We also decided to graph solutions in three dimensions to look at the drop in potential across real space. The following are graphs of the  $n = 3$  (Figure 4) and  $n = 5$  (Figure 5) cases that have been plotted parametrically in three dimensions.

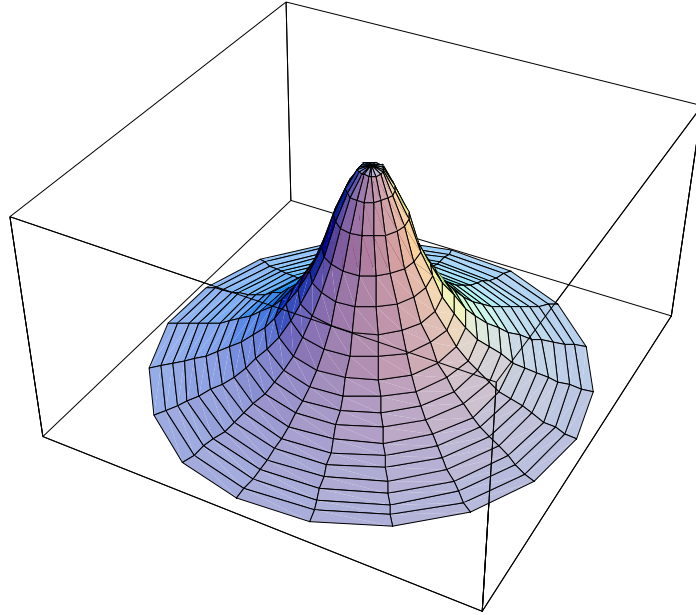


Figure 4

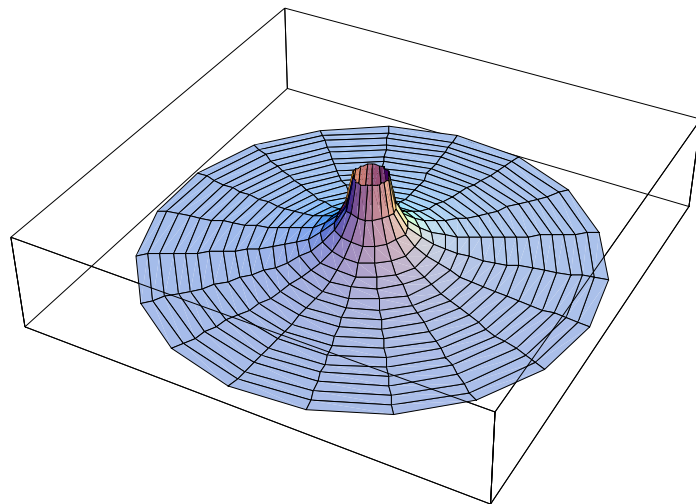


Figure 5

Our next goal was to form an equation similar to the Lane-Emden that would model the gravitational potential in stars that are not radially symmetric, specifically those that are squashed in one direction. This case is much more difficult to derive and solve since the original Lane-Emden was derived from two equations, hydrostatic equilibrium and Poisson's Eq., that were themselves radially symmetric. We therefore had to use the non-radially symmetric versions of these to form our equation. The non-spherically symmetric version of hydrostatic equilibrium (NHE) is  $\nabla^{\mu} P = -\rho \nabla^{\mu} \Phi + \rho \dot{D}^{\mu}$  and of Poisson's Eq. (NPE) is  $\nabla^2 \Phi = 4\pi G \rho$  (Collins II). In this case,  $\nabla$  is the gradient

vector and  $D$  is some perturbing force vector. The perturbing force is what would cause the squashing or loss of symmetry in the star. Examples of different perturbing forces would be a gravitational distortion from an external point mass, distortion from a toroidal magnetic field, or just from rigid rotation of the star. Our new equation for non-spherically symmetric stars was formed by the following derivation.

First using (PR) and the (NHE), substitute one into the other to obtain  $\nabla K\rho^\gamma = -\rho\nabla\Phi + \rho D$  or  $K\nabla\rho^\gamma = \rho(D - \nabla\Phi)$ . We want  $\nabla$  in spherical coordinates. This can be rewritten as  $\frac{\nabla\rho^\gamma}{\rho} = \frac{D - \nabla\Phi}{K}$ . Taking the gradients of each side yields  $\gamma\rho^{\gamma-2}\left(\rho_r\hat{r} + \frac{1}{r}\rho_\theta\hat{\theta} + \frac{1}{r\sin\theta}\rho_\phi\hat{\phi}\right) = \frac{D}{K} - \left(\Phi_r\hat{r} + \frac{1}{r}\Phi_\theta\hat{\theta} + \frac{1}{r\sin\theta}\Phi_\phi\hat{\phi}\right)$ . Now take the absolute magnitude of both vectors. We obtain  $\gamma K\rho^{\gamma-2}\rho_r = D_{r_c} - \Phi_r$ , where  $D_{r_c}$  is the  $r$  component of  $D$ , but  $\Phi_r, \rho_r$  are derivatives with respect to  $r$ . Integrating both sides with respect to  $r$  yields  $\frac{\gamma}{\gamma-1}K\rho^{\gamma-1} = \int D_{r_c} dr - \Phi$ . Now let  $N = \int D_{r_c} dr$  and  $\frac{1}{\gamma-1} = n$ . We obtain  $(n+1)K\rho^n = N - \Phi$  where  $\rho = \left[\frac{N - \Phi}{K(n+1)}\right]^{\frac{1}{n}}$ . Substituting into (NPE), one gets

$\nabla^2\Phi = 4\Pi G \left[\frac{N - \Phi}{K(n+1)}\right]^n$ . In spherical coordinates, this becomes

$\frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 4\Pi G \left[\frac{N - \Phi}{K(n+1)}\right]^n$ . However, for our case we assume that the star is radially symmetric in the  $\phi$  direction. Our final equation is then:  $\frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 4\Pi G \left[\frac{N - \Phi}{K(n+1)}\right]^n$ .

Note that our new equation incorporates the elements from the spherically symmetric case. Also note now that our new equation is a nonlinear elliptic partial differential equation. Dr. Neuberger has an algorithm that we can use to solve this equation given a predetermined boundary. Unfortunately, that is not enough. Our next step would be to find a way to solve it over all of  $\mathfrak{R}^2$  or as a “free boundary problem”, for different perturbing forces.

## Literature Cited

1. Collins II George W., 1989, *The Fundamentals of Stellar Astrophysics*. W. H. Freeman And Company, New York.
2. Kippenhahn R., Weigert A., 1990, *Stellar Structure and Evolution*. Springer-Verlag, New York, Berlin.

For modern research regarding the Lane-Emden equation, see the following articles:

3. Liu F. K., 1996, "Polytropic gas spheres: an approximate analytic solution of the Lane-Emden equation". *Monthly Notices of the Royal Astronomical Society*, Volume 281, Issue 4, pp. 1197-1205.
4. Roxburgh Ian W., Stockman Lynne M., 1999, "Power series solutions of the polytropic equations". *Monthly Notices of the Royal Astronomical Society*, Volume 303, Issue 3, pp. 466-470.

## The Appendix

### Program to approximate solutions to the Lane-Emden equation using the shooting method.

#### Symbols:

**n** is the number of divisions/unit,

**r0** is a point very close to **d** used as one of the initial points

(linearize on (0,r0) and nonlinear on (r0,R)),

**b[r\_]** is the bessel function  $J_{1/2}$ ,

**d** is the peak value (at r0),

**R** is the length of the interval,

**nn** is the specified polytropic index (power of the nonlinear function),

**aa** is the bessel function solved at a point very close to zero (undefined at zero),

**c1** is the coefficient associated with the bessel function in the inhomogenous bessel solution,

**f[s\_]** is the nonlinear function,

**h[r\_]** is used to find the coefficient **c1**, a solution to the homogenous linear problem,

**w[r\_]** is the inhomogenous solution to the linearized problem.

#### Initialization Cells:

```
n = 10;
```

```
r0 = .000001;
```

```
d = 1;
```

```
R = 10;
```

```
nn = 3;
```

```
dx = 1/n/N;
```

```
f[s_] = s^nn;
```

```
b[r_] = BesselJ[1/2,r];
```

```
h[r_] = b[Sqrt[nn * d^(nn-1)] * r]/Sqrt[r];
```

```
w[r_] = c1/Sqrt[r] * b[Sqrt[nn * d^(nn-1)] * r] + ((nn-1)/nn) * d;
```

```
aa = h[.00000000001];
```

```
c1 = d/(3 * aa);
```

The following code computes the approximation of the second order nonlinear ordinary differential Lane-Emden equation. We use (r0,w[r0]) and (r0+dx,w[r0]+dx\*(D[w[s],s]/.s->r0)) as our initial points used in the Euler method.

Euler Loop:

```
y = {{0, d}, {r0, w[r0]}, {r0 + dx, w[r0] + dx * (d[w[s], s]/.s->r0)}};
For[i = 3, i ≤ n*R, i ++,
  x = (i-1) * dx + r0;
  x1 = x-dx;
  yim2 = y[[i-1, 2]];
  yim1 = y[[i, 2]];
  yi = (x1/(x1 + dx)) * (2yim1-yim2-dx^2f[yim1] + (dx/x1) * yim2);
  y = Append[y, {x,yi}];
];
```

```
ListPlot[y, PlotJoined → True];
```

```
Z[r_] := y[[Floor[r * n] + 1]][[2]];
```

```
ParametricPlot3D[{r * Cos[t], r * Sin[t], 10z[r]}, {r, 0, R}, {t, 0, 2 * Pi}, Axes → False];
```