# Zernike Polynomials and Optical Aberrations

## Introduction

The Zernike Polynomials are an infinite set of orthogonal polynomials that are defined on the unit disk. Much like the Legengre Polynomials are formed from orthogonalization of the basis set { $x^n$ }, n=0, 1, 2,…, the Zernike Polynomials are formed from the orthogonalization of the basis set { $r^p e^{iq\theta}$ } such that p = 0, 1, 2,…, and q = 0, 1, 2,…, with the stipulation that [ $p-|q|$ ] is even and $|q| \le p$ .

To begin my comprehension of the Zernike Polynomials' role in optical aberrations, I decided it would be best to start from the very beginning and derive the beasts themselves. Using the Gram-Schmidt orthogonalization process as outlined in Linear Algebra and its Applications, 2nd Edition, by David C. Lay, Addison-Wesely, I was able to derive the first 15 Zernike Polynomials with the help of Mathematica 4.0.2. The Gram-Schmidt orthogonalization process is a simple algorithm for producing an orthogonal basis for any non-zero subspace of $\Re^n$ ; for our purposes the case of the unit disk, $\Re^2$. Thus, we can transform a non-orthogonal basis {$\mathbf{x}_1$, $\mathbf{x}_2$,…, $\mathbf{x}_p$} for a subspace W of $\Re^n$ into an orthogonal basis {$\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_p$} of W. The Gram-Schmidt algorithm is as follows:

$$v_p = x_p - \frac{\langle x_p | v_1 \rangle}{\langle v_1 | v_1 \rangle} v_1 - \frac{\langle x_p | v_2 \rangle}{\langle v_2 | v_2 \rangle} v_2 - \text{K} - \frac{\langle x_p | v_{p-1} \rangle}{\langle v_{p-1} | v_{p-1} \rangle} v_{p-1}$$

Where, $\langle x | v \rangle$ is defined in *general* as the scalar product of the vectors **x** and **v**, such that $\langle x | v \rangle = \int x^* v dt$ , here x = x(t), v = v(t), and $x^*$ is the complex conjugate of x. The resulting orthogonal basis can easily be made orthonormal by dividing each element $\mathbf{v}_n$ by its norm $\sqrt{\langle v_n | v_n \rangle}$ .

Due to the fact that the sequence of basis functions to be orthogonalised are defined on the unit disk and are functions of both r and θ, the integral representing the scalar product takes a slightly different form than the *general* case. The scalar product, in our case, takes the form of a double integral:

$$\langle x | v \rangle = \int_0^{2\pi} \int_0^1 x^* v \, r \, dr \, d\theta$$

Here, the extra multiplication by r comes from the ΔA term in the Riemann sum, describing the area of a portion in polar coordinates:

$$\sum_{i,j} f(r_i, \theta_j)\Delta A$$

Since the integral is formed from the limit of the ΔA term approaching zero, and $\Delta A \cong r\Delta r\Delta\theta$, for small ΔA, we see where the form of the double integral for our scalar product comes from.

Now that the Gram-Schmidt orthogonalization process has been outlined and the scalar products have been defined in our unit-disk-space. I will share the results of Mathematica's power and display the first 15 orthonormal Zernike Polynomials:

| Zernike Moment | p | q | $Z^q_p(r,\theta)$ |
|---|---|---|---|
| 1 | 0 | 0 | $\frac{1}{\sqrt{\pi}}$ |
| 2 | 1 | 1 | $\frac{4\,r\,\mathrm{Sin}[\theta]}{\sqrt{\pi}}$ |
| 3 | 1 | -1 | $\frac{4\,r\,\mathrm{Cos}[\theta]}{\sqrt{\pi}}$ |
| 4 | 2 | 2 | $2\sqrt{\frac{6}{\pi}}\;r^2\,\mathrm{Sin}[2\,\theta]$ |
| 5 | 2 | 0 | $4\sqrt{\frac{3}{\pi}}\;\left(-\frac{1}{2}+r^2\right)$ |
| 6 | 2 | -2 | $2\sqrt{\frac{6}{\pi}}\;r^2\,\mathrm{Cos}[2\,\theta]$ |
| 7 | 3 | 3 | $4\sqrt{\frac{2}{\pi}}\;r^3\,\mathrm{Sin}[3\,\theta]$ |
| 8 | 3 | 1 | $12\sqrt{\frac{2}{\pi}}\;\left(-\frac{2\,r}{3}+r^3\right)\,\mathrm{Sin}[\theta]$ |
| 9 | 3 | -1 | $12\sqrt{\frac{2}{\pi}}\;\left(-\frac{2\,r}{3}+r^3\right)\,\mathrm{Cos}[\theta]$ |
| 10 | 3 | -3 | $4\sqrt{\frac{2}{\pi}}\;r^3\,\mathrm{Cos}[3\,\theta]$ |

| | | | |
|---|---|---|---|
| 11 | 4 | 4 | $2\sqrt{\frac{10}{\pi}}\ r^4\,\text{Sin}[4\theta]$ |
| 12 | 4 | 2 | $8\sqrt{\frac{10}{\pi}}\left(-\frac{3r^2}{4}+r^4\right)\text{Sin}[2\theta]$ |
| 13 | 4 | 0 | $12\sqrt{\frac{5}{\pi}}\left(\frac{1}{6}-r^2+r^4\right)$ |
| 14 | 4 | -2 | $8\sqrt{\frac{10}{\pi}}\left(-\frac{3r^2}{4}+r^4\right)\text{Cos}[2\theta]$ |
| 15 | 4 | -4 | $2\sqrt{\frac{10}{\pi}}\ r^4\,\text{Cos}[4\theta]$ |

Keep in mind that there are an infinite number of Zernike Polynomials. For the $n^{th}$ Zernike Polynomial it requires (n-1) terms involving scalar products, each one building on all of the previous Zernike Polynomials before it. In other words, using the Gram-Schmidt orthogonalization process one cannot get the $n^{th}$ Zernike Polynomial without first obtaining all the Zernike Polynomials of {n-1, n-2, …, 1}. Thus the Gram-Schmidt orthogonalization process can become cumbersome in the pursuit of high order Zernike Polynomials.

Another way of constructing the orthonormal set of Zernike Polynomials is by means of a summation process. This in achieved by creating individual Zernike Polynomials, and then dividing by their respected norms. The individual Zernike Terms are formed by the following process:

$$Z^q_p(r,\theta) = R^q_p(r,\theta) \times W^q(\theta)$$

In which

$$R^q_p(r,\theta) = \sum_{s=0}^{\frac{1}{2}(p-\text{Abs}[q])} \frac{\{(-1)^s (p-s)!\, r^{p-2s}\}}{\{s!\left(\frac{1}{2}(p+\text{Abs}[q])-s\right)!\left(\frac{1}{2}(p-\text{Abs}[q])-s\right)!\}}$$ , and

$W^q(\theta)$ = Cos[q θ], Sin[q θ], and 1 for $q < 1$, $q > 1$, and $q = 1$, respectively.

This development is outlined in *Principles of Optics*, M. Born, E. Wolf (Pergammon Press, Oxford, 1987). The orthonormal Zernike Set can then be achieved by dividing each element by its respected norm. The summation process brings much power to the development of the Zernike Polynomials as well as to their applications. For instance, using the summation process, we can find the Zernike Polynomial corresponding to the index of p = 43, q = -33 without resorting to the cumbersome process of finding all the previous terms.

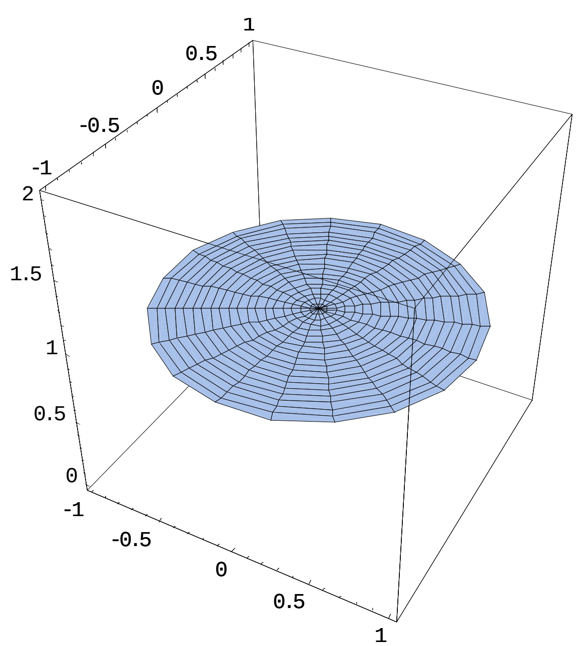
$Z^{43}_{-33}(r,\theta) =$

$$2\sqrt{\frac{22}{\pi}}\left(-501942\, r^{33} + 2878785\, r^{35} - 6580080\, r^{37} + 7493980\, r^{39} - 4253340\, r^{41} + 962598\, r^{43}\right) \mathrm{Cos}[33\,\theta]^2$$

Which is, of course, normalized.

Let's now examine some of the physical manifestations that the Zernike Polynomials describe.

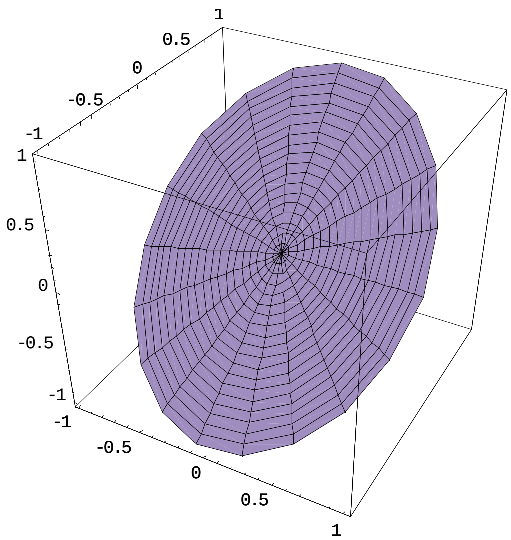
$Z^0_0(r,\theta)$ simply describes the unit disk itself:

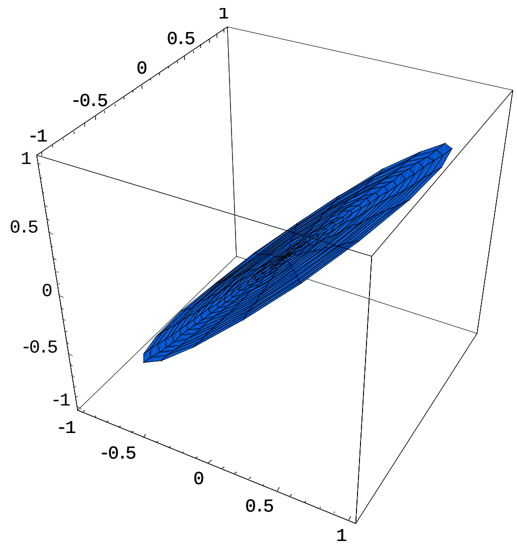



$Z^0_0(r,\theta) = \frac{1}{\sqrt{\pi}}$

$Z^1_1(r,\theta)$ and $Z^{-1}_1(r,\theta)$

tilt the unit disk on the y and x axis's, respectfully:

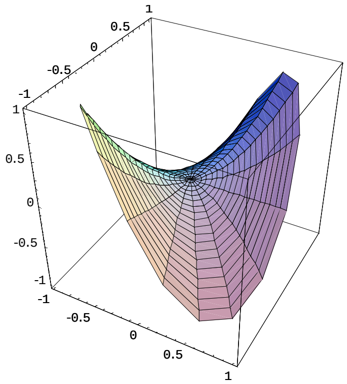


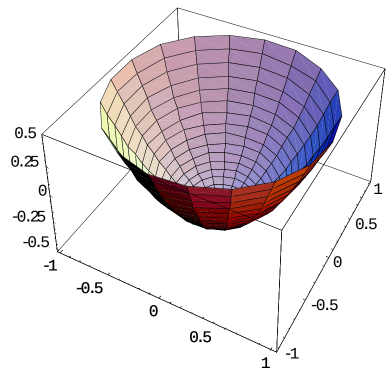



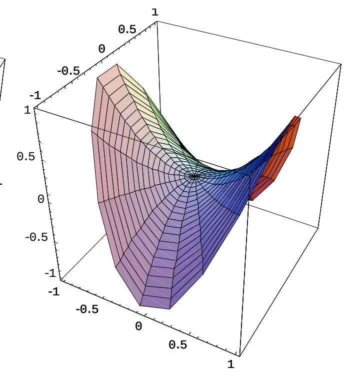

$Z^{1}_{1}(r,\theta) = \frac{4\, r \, \mathrm{Sin}[\theta]}{\sqrt{\pi}}$

$Z^{-1}_{1}(r,\theta) = \frac{4\, r \, \mathrm{Cos}[\theta]}{\sqrt{\pi}}$

$Z^{2}_{2}(r,\theta)$, $Z^{2}_{0}(r,\theta)$, and $Z^{-2}_{2}(r,\theta)$ demonstrate other manipulations of the unit disk:

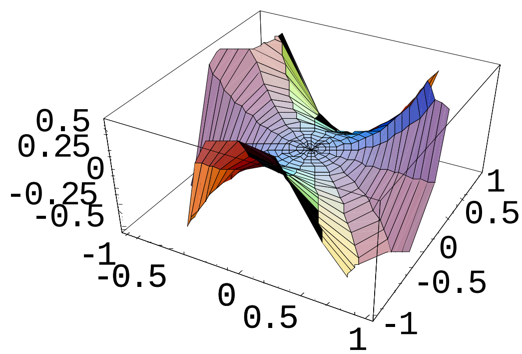


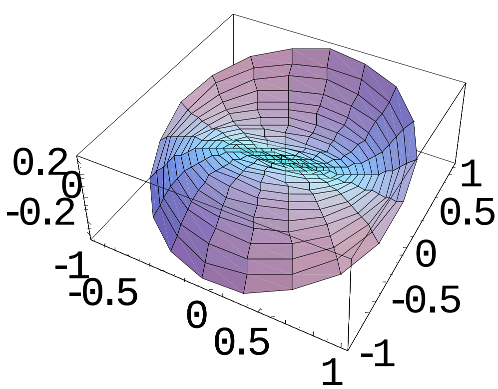



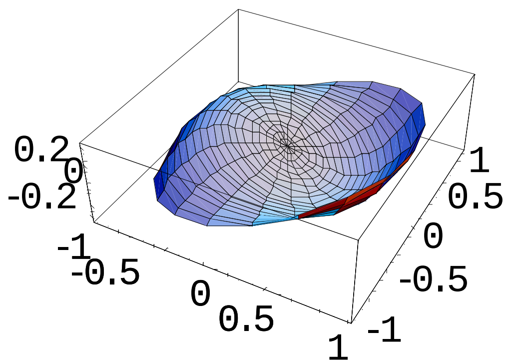

$Z^2_2(r,\theta) = 2\sqrt{\frac{6}{\pi}}\, r^2 \,\mathrm{Sin}[2\theta]$ $\quad Z^2_0(r,\theta) = 4\sqrt{\frac{3}{\pi}}\left(-\frac{1}{2}+r^2\right)$ $\quad Z^{-2}_2(r,\theta) = 2\sqrt{\frac{6}{\pi}}\, r^2 \,\mathrm{Cos}[2\theta]$

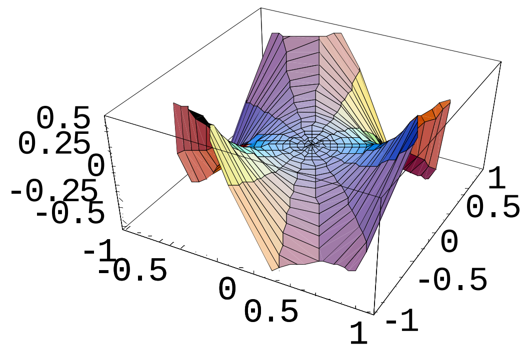
$Z^3_3(r,\theta)$, $Z^1_3(r,\theta)$, $Z^{-1}_3(r,\theta)$, and $Z^{-3}_3(r,\theta)$ manipulate the unit disk in other ways:




$Z^3_3(r,\theta) = 4\sqrt{\frac{2}{\pi}}\, r^3 \,\mathrm{Sin}[3\theta]$ $\quad Z^1_3(r,\theta) = 12\sqrt{\frac{2}{\pi}}\left(-\frac{2r}{3}+r^3\right)\mathrm{Sin}[\theta]$




$Z^{-1}_{3}(r,\theta) = 12\sqrt{\frac{2}{\pi}}\left(-\frac{2r}{3}+r^{3}\right)\mathrm{Cos}[\theta]$ $Z^{-3}_{3}(r,\theta) = 4\sqrt{\frac{2}{\pi}}\, r^{3}\,\mathrm{Cos}[3\theta]$

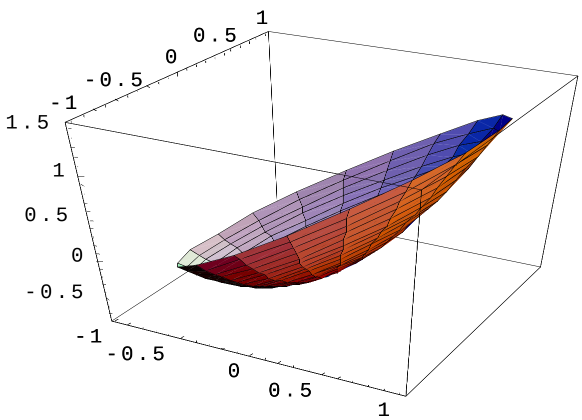
We see that many of the Zernike Polynomials come in “couples”—that is the couples are identical except for the sine and cosine terms. For instance, the $Z^{3}_{3}(r,\theta)$ and $Z^{-3}_{3}(r,\theta)$, corresponding to the 7th and 10th Zernike moments, only differ by the sine and cosine terms. The only difference in the indices themselves are that for a fixed p =3 , q = 3 for the 7th term and q = -3 for the 10th term. In fact, all Zernike polynomials that share common indices p and $|q|$ will be different only by the sine and cosine terms. Graphically, it comes as no surprise that the difference takes the form of a $\frac{\pi}{2}$ phase shift—as demonstrated by the graphs of $Z^{3}_{3}(r,\theta)$ and $Z^{-3}_{3}(r,\theta)$.

**The Power of the Zernike Polynomials**

The infinite set of Zernike Polynomials spans functions defined on the unit disk. In other words, *any distortion or manipulation of the unit disk can be described completely by a linear combination of Zernike Polynomials*. However, in most cases an infinite number of Zernike Terms is not needed to describe, to a very good approximation, the physical state of a warped unit disk. To illustrate this point let's look at a simple linear combination of the 3rd and 5th Zernike Terms.

The linear combination of $Z^{-1}{}_{1}(r,\theta) + Z^{2}{}_{0}(r,\theta)$ is mathematically written as:

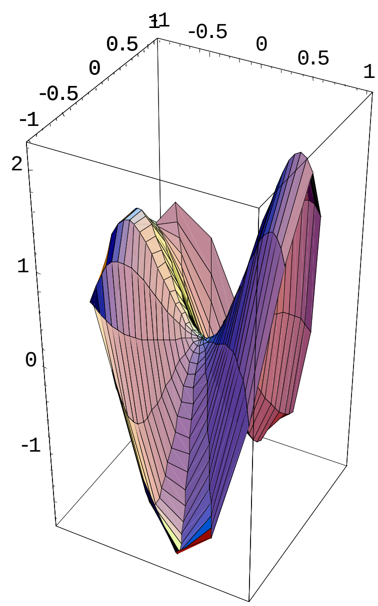
$\frac{4\, r\, \mathrm{Cos}[\theta]}{\sqrt{\pi}} + 4\sqrt{\frac{3}{\pi}}\left(-\frac{1}{2} + r^2\right)$, and graphically described by the following figure:




$$Z^{-1}{}_{1}(r,\theta) + Z^{2}{}_{1}(r,\theta) = \frac{4\, r\, \mathrm{Cos}[\theta]}{\sqrt{\pi}} + 4\sqrt{\frac{3}{\pi}}\left(-\frac{1}{2} + r^2\right)$$

This really comes as no surprise when we look at the two individual graphs of $Z^{-1}{}_{1}(r,\theta)$ and $Z^{2}{}_{0}(r,\theta)$. We are able to achieve more intricate distortions of the unit disk by examining more complicated linear combinations of Zernike Polynomials. For example,

$(2)Z^{2}{}_{2}(r,\theta) + (0.5)Z^{1}{}_{3}(r,\theta) + Z^{-3}{}_{3}(r,\theta) - (4)Z^{0}{}_{4}(r,\theta)$ yields the following figure:




$(2)Z^{2}{}_{2}(r,\theta) + (0.5)Z^{1}{}_{3}(r,\theta) + Z^{-3}{}_{3}(r,\theta) - (4)Z^{0}{}_{4}(r,\theta)$