Multiplicity and symmetry of positive solutions to semi-linear elliptic problems with Neumann boundary conditions

Christophe Troestler
(Joint work with D. Bonheure & C. Grumiau)

Institut de Mathématique
Université de Mons

Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, June 6–9, 2012
The Lane-Emden problem

Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, $N \geq 2$, and $2 < p < 2^* := \frac{2N}{N-2}$. We consider

$$\begin{cases}
-\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\
\partial_\nu u = 0, & \text{on } \partial \Omega.
\end{cases}$$

Solutions are critical points of the functional

$$E_p : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p} \int_\Omega |u|^p$$

$$\partial E_p(u) : H^1(\Omega) \to \mathbb{R} : v \mapsto \int_\Omega \nabla u \nabla v + uv - \int_\Omega |u|^{p-2}uv$$

Notation: $1 = \lambda_1 < \lambda_2 < \cdots$ denote the eigenvalues of $-\Delta + 1$

$E_i$ denote the corresponding eigenspaces

Remark: 0 is always a (trivial) solution.
Outline

1. $p \approx 2$: ground state solutions
2. $p \approx 2$: positive solutions
3. $p$ large: symmetry breaking of the ground state
4. $p$ large: bifurcations from 1
5. $p$ large: multiplicity results (radial domains)
$p \approx 2$: ground state solutions  

$\begin{align*}
-\Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{align*}$

- The ground state solution is positive and is even w.r.t. any hyperplane leaving $\Omega$ invariant (when $\Omega$ is convex). In particular, it is radially symmetric on a ball.

- Uniqueness of the positive solution when $\Omega$ is a ball.

- If $\Omega$ is strictly starshaped and $p \geq 2^*$, no solution exists.
Existence of ground state solutions \((p < 2^*)\)

**Theorem (Z. Nehari, A. Ambrosetti, P.H. Rabinowitz)**

*For any \(p \in ]2, 2^*[, there exists a ground state solution to \((\mathcal{P}_p)\). It is a one-signed function.*

**Sketch of the proof.**

- The functional \(E_p\) possesses a mountain pass structure.
- \(\exists u_0 \neq 0, E_p(u_0) = \inf_{u \neq 0} \max_{\lambda > 0} E_p(\lambda u)\)
  \[= \inf_{u \in \mathcal{N}_p} E_p(u)\]
  where \(\mathcal{N}_p\) is the Nehari manifold of \(E_p\).
- For any sign-changing solution \(u\): if \(u^\pm \neq 0, u^\pm \in \mathcal{N}_p\) and \(E_p(u^\pm) < E_p(u)\), where \(u^\pm := \pm \max\{\pm u, 0\} \).
Theorem (D. Bonheure, V. Bouchez, C. Grumiau, C. T., J. Van Schaftingen, ’08)

For $p$ close to 2 and any $R \in O(N)$ s.t. $R(\Omega) = \Omega$, ground state solutions to $(\mathcal{P}_p)$ are symmetric w.r.t. $R$.

E.g. if $\Omega$ is radially symmetric, so must the ground state solution be.

Remark that the seminal method of moving planes is not applicable.
Uniqueness of the positive solution

**Theorem**

1 is the unique positive solution for $p$ small.
Uniqueness of the positive solution

**Theorem**

1 is the unique positive solution for $p$ small.

Let $v := P_{E_1} u_p$ (constant function) and $w := P_{E_1^\perp} u_p$.

\[ \lambda_2 \int_{\Omega} w^2 \leq \int_{\Omega} |\nabla w|^2 + w^2 \]

\[ = \int_{\Omega} |u_p|^{p-1} w = \int_{\Omega} ((v + w)^{p-1} - v^{p-1}) w \]

\[ = \int_{\Omega} (p-1)(v + \vartheta_p w)^{p-2} w^2 \quad (\vartheta_p \in ]0, 1[) \]

\[ \leq (p-1)(|v| + \|w\|_{\infty})^{p-2} \int_{\Omega} w^2 \leq (p-1)K^{p-2} \int_{\Omega} w^2. \]

As $\lambda_1 = 1 < \lambda_2$, for $p \approx 2$, $w = 0$ and then $u_p = v = 1$. 
A priori bounds for positive solutions

**Lemma**

*Positive solutions* $(u_p)$ *are bounded in* $L^\infty$ *as* $p \approx 2$.

- **Integration & Hölder:** $\int_\Omega u_p^{p-1} = \int_\Omega u_p \leq |\Omega|$ (recall $u_p > 0$).
- **Brezis-Strauss:** from the bound on $\int_\Omega u_p^{p-1}$, we deduce a bound on $\|u_p\|_{W^{1,q}(\Omega)}$, $1 \leq q < N/(N-1)$.
- **Sobolev embedding:** $(u_p)$ bounded in $L^r(\Omega)$, $1 < r < N/(N-2)$.
- **Bootstrap:** $\|u_p\|_{W^{2,r}(\Omega)}$ is bounded for some $r > N/2$ when $p \approx 2$. 
A priori bounds for positive solutions

Proposition

Let $2 < \bar{p} < 2^*$. There exists $C_{\bar{p}} > 0$ such that any positive solution to $(P_p)$ with $2 < p \leq \bar{p}$ satisfies $\max\{||u||_{H^1}, ||u||_{L^\infty}\} \leq C_{\bar{p}}$. 
A priori bounds for positive solutions

Proposition

Let \(2 < \bar{p} < 2^*\). There exists \(C_{\bar{p}} > 0\) such that any positive solution to \((P_p)\) with \(2 < p \leq \bar{p}\) satisfies \(\max\{\|u\|_{H^1}, \|u\|_{L^\infty}\} \leq C_{\bar{p}}\).

It remains to obtain a bound for \(2 < p < \bar{p} < 2^*\) in \(L^\infty\). Blow up argument (Gidas-Spruck). Suppose on the contrary that there is a sequence \((p_n) \subseteq [p, \bar{p}]\) and \((u_{p_n})\) s.t.

\[
u_{p_n}(x_{p_n}) := \|u_{p_n}\|_{L^\infty} \to +\infty \quad \text{and} \quad p_n \to p^* \in [p, \bar{p}].
\]

(Drop index \(n\).) Define

\[
u_p(y) := \mu_p u_p(\mu_p^{(p-2)/2}y + x_p)
\]

where \(\mu_p := 1/\|u_p\|_{L^\infty} \to 0\).

Note: \(\nu_p(0) = \|\nu_p\|_{L^\infty} = 1\).
A priori bounds for positive solutions

The rescaled function $v_p$ satisfies

$$-\Delta v_p + \mu_p^{p-2} v_p = v_p^{p-1} \quad \text{on } \Omega_p := (\Omega - x_p) / \mu_p^{(p-2)/2}$$

with NBC. By elliptic regularity, $(v_p)$ is bounded in $W^{2,r}$ and $C^{1,\alpha}$, $0 < \alpha < 1$ on any compact set. Thus, taking if necessary a subsequence,

$$v_n \to v^* \quad \text{in } W^{2,r} \text{ and } C^{1,\alpha} \quad \text{on compact sets of } \Omega^* = \mathbb{R}^N \text{ or } \mathbb{R}^{N-1} \times \mathbb{R}_{>a}.$$

One has $v^* \geq 0$, $v^*(0) = 1 = \|v\|_{L^\infty}$ and $v^*$ satisfies

$$-\Delta v^* = (v^*)^{p^*-1} \quad \text{in } \mathbb{R}^N \quad \text{or} \quad \begin{cases} -\Delta v^* = (v^*)^{p^*-1} & \text{in } \mathbb{R}^{N-1} \times \mathbb{R}_{>a} \\ \partial_N v^* = 0 & \text{when } x_N = a \end{cases}$$

Liouville theorems imply $v^* = 0$. 

\[\square\]
Theorem

As \( p \to 2^* \), least energy solutions go to 0 everywhere except around a single peak located at a point \( Q^* \in \partial \Omega \) where the boundary is most curved.
Corollary

1 cannot remain the ground state for all \( p \).
Corollary

1 cannot remain the ground state for all $p$.

Lemma

1 cannot remain the ground state solution for $p > 1 + \lambda_2$.

Proof. The Morse index of 1 is the sum of the dimension of the eigenspaces corresponding to negative eigenvalues $\lambda$ of

$$\begin{cases}
-\Delta v + v = (p - 1)v + \lambda v, & \text{in } \Omega, \\
\partial_{\nu} v = 0, & \text{on } \partial\Omega.
\end{cases}$$

i.e. eigenvalues of $-\Delta + 1$ less than $p - 1$. When $p > 1 + \lambda_2$, the Morse index of the solution 1 is $> 1$. 
$p$ large: symmetry breaking of the ground state

Proposition (Lopez, ’96)

On radial domains, the ground state is either constant or (e.g. when $p > 1 + \lambda_2$) not radially symmetric.
\( p \) large: symmetry breaking of the ground state

**Proposition (Lopez, ’96)**

On radial domains, the ground state is either constant or (e.g. when \( p > 1 + \lambda_2 \)) not radially symmetric.

**Proposition**

When \( \Omega \) is a ball or an annulus, the Morse index of a non-constant positive radial solution is at least \( N + 1 \).

Based on: A. Aftalion, F. Pacella, Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, CRAS, 339(5), ’04.

Let \( u \) be non-constant positive radial solution of \((P_p)\). We have to show that

\[
L v := -\Delta v + v - (p - 1)|u|^{p-2} v
\]

with NBC possesses \( N + 1 \) negative eigenvalues.
$p$ large: symmetry breaking of the ground state

$u$ radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial \Omega$ and on $\Omega_i$. 

\[ \Omega_i^- \quad \Omega_i^+ \]
\[ x_i \]
\( p \) large: symmetry breaking of the ground state

\[ u \text{ radial} \Rightarrow \partial_{x_i} u = 0 \text{ on } \partial \Omega \text{ and on } \Omega_i. \]

Let \( \bar{x} \in \Omega_i^+ \text{ s.t. } \partial_{x_i} u(\bar{x}) \neq 0. \text{ Let } D \text{ be the connected component of } \{\partial_{x_i} u(\bar{x}) \neq 0\} \text{ containing } \bar{x}. \ D \subseteq \Omega_i^+. \]
$p$ large: symmetry breaking of the ground state

$u$ radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial \Omega$ and on $\Omega_i$.

Let $\bar{x} \in \Omega_i^+$ s.t. $\partial_{x_i} u(\bar{x}) \neq 0$. Let $D$ be the connected component of $\{\partial_{x_i} u(\bar{x}) \neq 0\}$ containing $\bar{x}$. $D \subseteq \Omega_i^+$.

$L(\partial_{x_i} u) = 0$, on $D$; \quad $\partial_{x_i} u = 0$, on $\partial D$. 

\[
\begin{align*}
\Omega_i^- & \quad \Omega_i^+ \\
D & \quad x_i
\end{align*}
\]
**p large: symmetry breaking of the ground state**

\( u \) radial \( \Rightarrow \partial_{x_i} u = 0 \) on \( \partial \Omega \) and on \( \Omega_i \).

Let \( \bar{x} \in \Omega_i^+ \) s.t. \( \partial_{x_i} u(\bar{x}) \neq 0 \). Let \( D \) be the connected component of \( \{ \partial_{x_i} u(\bar{x}) \neq 0 \} \) containing \( \bar{x} \). \( D \subseteq \Omega_i^+ \).

\[
L(\partial_{x_i} u) = 0, \quad \text{on} \ D; \quad \partial_{x_i} u = 0, \quad \text{on} \ \partial D.
\]

\[\Rightarrow \lambda_1(L, D, \text{DBC}) = 0\]

\[\Rightarrow \lambda_1(L, \Omega_i^+, \text{DBC}) \leq 0\]
\( p \) large: symmetry breaking of the ground state

\( u \) radial \( \Rightarrow \partial_{x_i} u = 0 \) on \( \partial \Omega \) and on \( \Omega_i \).
Let \( \bar{x} \in \Omega_i^+ \) s.t. \( \partial_{x_i} u(\bar{x}) \neq 0 \). Let \( D \) be the connected component of \( \{ \partial_{x_i} u(\bar{x}) \neq 0 \} \) containing \( \bar{x} \). \( D \subseteq \Omega_i^+ \).

\[
L(\partial_{x_i} u) = 0, \quad \text{on } D; \quad \partial_{x_i} u = 0, \quad \text{on } \partial D.
\]

\( \Rightarrow \lambda_1(L, D, \text{DBC}) = 0 \)

\( \Rightarrow \lambda_1(L, \Omega_i^+, \text{DBC}) \leq 0 \)

\( \Rightarrow \mu_i := \lambda_1(L, \Omega_i^+, \text{DBC on } \Omega_i \text{ and NBC on } \partial \Omega_i^+ \setminus \Omega_i) < 0 \)
$p$ large: symmetry breaking of the ground state

$u$ radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial \Omega$ and on $\Omega_i$.

Let $\bar{x} \in \Omega_i^+$ s.t. $\partial_{x_i} u(\bar{x}) \neq 0$. Let $D$ be the connected component of $\{\partial_{x_i} u(\bar{x}) \neq 0\}$ containing $\bar{x}$. $D \subseteq \Omega_i^+$.

$$L(\partial_{x_i} u) = 0, \quad \text{on } D; \quad \partial_{x_i} u = 0, \quad \text{on } \partial D.$$ 

$\Rightarrow \lambda_1(L, D, \text{DBC}) = 0$

$\Rightarrow \lambda_1(L, \Omega_i^+, \text{DBC}) \leq 0$

$\Rightarrow \mu_i := \lambda_1(L, \Omega_i^+, \text{DBC on } \Omega_i \text{ and NBC on } \partial \Omega_i^+ \setminus \Omega_i) < 0$

If $\psi_i > 0$ is the first eigenfunction of $L$ on $\Omega_i^+$ with DBC on $\Omega_i$ and NBC on $\partial \Omega_i^+ \setminus \Omega_i$, its odd extension $\psi_i^*$ to $\Omega$ satisfies

$$L(\psi_i^*) = \mu_i \psi_i^*, \quad \text{on } \Omega, \quad \partial_{\nu} \psi_i^* = 0, \quad \text{on } \partial \Omega.$$
$p$ large: symmetry breaking of the ground state

$u$ radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial \Omega$ and on $\Omega_i$.

Let $\bar{x} \in \Omega_i^+$ s.t. $\partial_{x_i} u(\bar{x}) \neq 0$. Let $D$ be the connected component of $\{\partial_{x_i} u(\bar{x}) \neq 0\}$ containing $\bar{x}$. $D \subseteq \Omega_i^+$.

$$L(\partial_{x_i} u) = 0, \quad \text{on } D; \quad \partial_{x_i} u = 0, \quad \text{on } \partial D.$$ 

$\Rightarrow \lambda_1(L, D, \text{DBC}) = 0$

$\Rightarrow \lambda_1(L, \Omega_i^+, \text{DBC}) \leq 0$

$\Rightarrow \mu_i := \lambda_1(L, \Omega_i^+, \text{DBC on } \Omega_i \text{ and NBC on } \partial \Omega_i^+ \setminus \Omega_i) < 0$

If $\psi_i > 0$ is the first eigenfunction of $L$ on $\Omega_i^+$ with DBC on $\Omega_i$ and NBC on $\partial \Omega_i^+ \setminus \Omega_i$, its odd extension $\psi_i^*$ to $\Omega$ satisfies

$$L(\psi_i^*) = \mu_i \psi_i^*, \quad \text{on } \Omega, \quad \partial \nu \psi_i^* = 0, \quad \text{on } \partial \Omega.$$ 

All $\psi_j^*, j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.
\( p \) large: symmetry breaking of the ground state

\( u \) radial \( \Rightarrow \partial_{x_i} u = 0 \) on \( \partial \Omega \) and on \( \Omega_i \).

Let \( \bar{x} \in \Omega_i^+ \) s.t. \( \partial_{x_i} u(\bar{x}) \neq 0 \). Let \( D \) be the connected component of \( \{ \partial_{x_i} u(\bar{x}) \neq 0 \} \) containing \( \bar{x} \). \( D \subseteq \Omega_i^+ \).

\[
L(\partial_{x_i} u) = 0, \quad \text{on } D; \quad \partial_{x_i} u = 0, \quad \text{on } \partial D.
\]

\( \Rightarrow \lambda_1(L, D, \text{DBC}) = 0 \)

\( \Rightarrow \lambda_1(L, \Omega_i^+, \text{DBC}) \leq 0 \)

\( \Rightarrow \mu_i := \lambda_1(L, \Omega_i^+, \text{DBC on } \Omega_i \text{ and NBC on } \partial \Omega_i^+ \setminus \Omega_i) < 0 \)

If \( \psi_i > 0 \) is the first eigenfunction of \( L \) on \( \Omega_i^+ \) with DBC on \( \Omega_i \) and NBC on \( \partial \Omega_i^+ \setminus \Omega_i \), its odd extension \( \psi_i^* \) to \( \Omega \) satisfies

\[
L(\psi_i^*) = \mu_i \psi_i^*, \quad \text{on } \Omega, \quad \partial_v \psi_i^* = 0, \quad \text{on } \partial \Omega.
\]

All \( \psi_j^*, j \neq i \) vanish on the axis \( x_i \) \( \Rightarrow \) the family \( (\psi_j^*)_{j=1}^N \) is lin. indep.

None of the \( (\psi_j^*)_{j=1}^N \) is a first eigenfunction.
$p$ large: symmetry breaking of the ground state

**Theorem (Lopes, ’96)**

*On radial domains, ground state solutions are symmetric w.r.t. any hyperplane containing a line $L$ passing through the origin.*

**Theorem (J. Van Schaftingen, ’04)**

*On radial domains, ground state solutions are foliated Schwarz symmetric.*

There exists a unit vector $d$ s.t. $u$ depends only on $r = |x|$ and $\vartheta = \arccos(\frac{x}{|x|} \cdot d)$ and is non-increasing in $\vartheta$. 
$p \approx 2$: ground state solutions  
$p \approx 2$: positive solutions  
Symmetry breaking  
Bifurcations  
Multiplicity

$p$ large: non radially symmetric ground state

For $p = 5.5$, $p = 6.5$, and $p = 8$, the non-radially symmetric ground states are illustrated. Each figure shows a 3D plot of the ground state solution for different values of $p$. The accompanying 2D plots demonstrate the non-radial symmetry of these solutions.
Symmetry breaking at exactly $p = 1 + \lambda_2$?

The linearisation of the equation around $u = 1$,

$$Lv := -\Delta v + v - (p - 1)v$$

is not invertible iff $p = 1 + \lambda_i, \ i \geq 2$. 
Symmetry breaking at exactly $p = 1 + \lambda_2$?

The linearisation of the equation around $u = 1$,

$$Lv := -\Delta v + v - (p - 1)v$$

is not invertible iff $p = 1 + \lambda_i, \ i \geq 2$.

Eigenfunctions of $-\Delta + 1$ with NBC have the form:

$$u(x) = r^{-\frac{N-2}{2}} J_{\nu}(\sqrt{\mu}r) P_k\left(\frac{x}{|x|}\right), \quad \text{where} \ \nu = k + \frac{N-2}{2},$$

$r = |x|$, and $P_k : \mathbb{R}^N \to \mathbb{R}$ is an harmonic homogenous polynomial of degree $k$ for some $k \in \mathbb{N}$. To satisfy the boundary conditions:

$$\sqrt{\mu}R \text{ is a root of } z \mapsto (k - \nu)J_{\nu}(z) + z\partial J_{\nu}(z) = kJ_{\nu}(z) - zJ_{\nu+1}(z).$$

$$\Rightarrow \lambda_i = 1 + \mu$$
Symmetry breaking at exactly $p = 1 + \lambda_2$?

In particular, a basis of $E_2$ is

$$x \mapsto r^{-\frac{N-2}{2}} J_{N/2}(\sqrt{\mu}r) \frac{x_j}{|x|}, \quad j = 1, \ldots, N.$$ 

There is single function (up to a multiple) that is invariant under rotation in $(x_2, \ldots, x_N)$. 
Symmetry breaking at exactly $p = 1 + \lambda_2$?

In particular, a basis of $E_2$ is

$$x \mapsto r^{-\frac{N-2}{2}} J_{N/2}(\sqrt{\mu} r) \frac{x_j}{|x|}, \quad j = 1, \ldots, N.$$ 

There is single function (up to a multiple) that is invariant under rotation in $(x_2, \ldots, x_N)$.

**Theorem (Ambrosetti-Prodi)**

Let $X$ and $Y$ two Banach spaces, $u^* \in X$, and a function $F : \mathbb{R} \times X \rightarrow Y : (p, u) \mapsto F(p, u)$ such that $\forall p \in \mathbb{R}, F(p, u^*) = 0$. Let $p^* \in \mathbb{R}$ be such that $\ker(\partial_u F(p^*, u^*)) = \text{span}\{\varphi^*\}$ has a dimension 1 and $\text{codim}(\text{Im}(\partial_u F(p^*, u^*))) = 1$. Let $\psi : Y \rightarrow \mathbb{R}$ be a continuous linear map such that $\text{Im}(\partial_u F(p^*, u^*)) = \{y \in Y : \langle \psi, y \rangle = 0\}$. 

---

Christophe Troestler (UMONS)  
Symmetries and symmetry breaking of solutions with NBC  
June 6–9, 2012  
18 / 36
Symmetry breaking at exactly \( p = 1 + \lambda_2 \)?

**Theorem (Ambrosetti-Prodi (cont’d))**

If \( a := \langle \psi, \partial_{p,u} F(p^*, u^*)[\varphi^*] \rangle \neq 0 \), then \((p^*, u^*)\) is a bifurcation point for \( F \). In addition, the set of non-trivial solutions of \( F = 0 \) around \((p^*, u^*)\) is given by a unique \( C^1 \) curve \( p \mapsto u_p \). The local behavior of the branch \((p, u_p)\) for \( p \) close to \( p^* \) is as follows.

- If \( b := -\frac{1}{2a} \langle \psi, \partial_{u}^2 F(p^*, u^*)[\varphi^*, \varphi^*] \rangle \neq 0 \) then the branch is transcritical and

\[
  u_p = u^* + \frac{p - p^*}{b} \varphi^* + o(p - p^*).
\]

\[
  u \uparrow
\]

\[
  p^* \quad p
\]
Symmetry breaking at exactly $p = 1 + \lambda_2$?

**Theorem (Ambrosetti-Prodi (cont’d))**

If $a := \langle \psi, \partial_{p,u} F(p^*, u^*)[\varphi^*] \rangle \neq 0$, then $(p^*, u^*)$ is a bifurcation point for $F$. In addition, the set of non-trivial solutions of $F = 0$ around $(p^*, u^*)$ is given by a unique $C^1$ curve $p \mapsto u_p$. The local behavior of the branch $(p, u_p)$ for $p$ close to $p^*$ is as follows.

- If $b := -\frac{1}{2a} \langle \psi, \partial_{u}^2 F(p^*, u^*)[\varphi^*, \varphi^*] \rangle \neq 0$ then the branch is transcritical and

  $$u_p = u^* + \frac{p - p^*}{b} \varphi^* + o(p - p^*).$$

In our case,

$$a = -\int_\Omega \varphi_2^2 = -1 \quad \text{and} \quad b = -\frac{1}{2} \lambda_2 (\lambda_2 - 1) \int_\Omega \varphi_2^3 = 0.$$
Symmetry breaking at exactly \( p = 1 + \lambda_2 \)?

**Theorem (Ambrosetti-Prodi (cont’d))**

- If \( b = 0 \), let us define

\[
c := -\frac{1}{6a} \left( \langle \psi, \partial_u F(p^*, u^*)[\varphi^*, \varphi^*, \varphi^*] \rangle 
+ 3 \langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, w] \rangle \right)
\]

where \( w \in X \) is any solution of the equation

\[
\partial_u F(p^*, u^*)[w] = -\partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*].
\]

If \( c \neq 0 \) then

\[
u_p = u^* \pm \left( \frac{p - p^*}{c} \right)^{1/2} \varphi^* + o(|p - p^*|^{1/2}).
\]

In particular, the branch is supercritical if \( c > 0 \) and subcritical if \( c < 0 \).
Symmetry breaking at exactly $p = 1 + \lambda_2$?

In our case,

$$c = \frac{1}{6} \lambda_2(\lambda_2 - 1) \left( - (\lambda_2 - 2) \int_{B_R} \varphi_2^4 - 3 \lambda_2 (\lambda_2 - 1) \int_{B_R} \varphi_2^2 w \right)$$

where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on $B_R$. 

Symmetry breaking at exactly $p = 1 + \lambda_2$?

In our case,

$$c = \frac{1}{6} \lambda_2 (\lambda_2 - 1) (- (\lambda_2 - 2) \int_{B_R} \varphi_2^4 - 3 \lambda_2 (\lambda_2 - 1) \int_{B_R} \varphi_2^2 w)$$

where $(-\Delta + 1 - \lambda_2) w = \varphi_2^2$ with NBC on $B_R$.

$$= \frac{1}{6} \bar{\mu}_2 R^{-(N+2)} (1 + \bar{\mu}_2 \frac{R^2}{\bar{\mu}_2}) \left( (\beta - \alpha) \frac{\bar{\mu}_2}{R^2} + \beta + \alpha \right)$$

where $\alpha := \int_{B_1} \bar{\varphi}_2^4$, $\beta := -3 \bar{\mu}_2 \int_{B_1} \bar{\varphi}_2^2 \bar{w}$,

$$(-\Delta - \bar{\mu}_2) \bar{w} = \bar{\varphi}_2^2$$

with NBC on $B_1$,

$\bar{\varphi}_2$ and $\bar{\mu}_2 > 0$ are the second eigenfunction and eigenvalue of $-\Delta$ with NBC on $B_1$ s.t. $|\bar{\varphi}_2|_{L^2} = 1$. 
Symmetry breaking at exactly $p = 1 + \lambda_2$?

We numerically have

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\beta - \alpha$</th>
<th>$\beta + \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5577</td>
<td>0.5884</td>
<td>0.0306</td>
<td>1.1461</td>
</tr>
<tr>
<td>3</td>
<td>0.4632</td>
<td>0.3096</td>
<td>-0.1536</td>
<td>0.7728</td>
</tr>
<tr>
<td>4</td>
<td>0.4222</td>
<td>0.1694</td>
<td>-0.2528</td>
<td>0.5916</td>
</tr>
<tr>
<td>5</td>
<td>0.4171</td>
<td>0.0858</td>
<td>-0.3313</td>
<td>0.5029</td>
</tr>
<tr>
<td>6</td>
<td>0.4421</td>
<td>0.0250</td>
<td>-0.4171</td>
<td>0.4671</td>
</tr>
</tbody>
</table>
Symmetry breaking at exactly \( p = 1 + \lambda_2 \) ?

**Conjecture**

*When \( R \) is large enough or \( N = 2, 1 \) is the ground state of*

\[
(P_p) \begin{cases} 
-\Delta u + u = |u|^{p-2}u, & \text{in } B_R \\
\partial_\nu u = 0, & \text{on } \partial B_R.
\end{cases}
\]

*iff \( p \leq 1 + \lambda_2 \).*
Lemma

When \( p > 2 \) is increasing,

1. a bifurcation sequence start from 1 \( \text{iff} \) \( p \) crosses \( 1 + \lambda_i \);
2. this is actually a continuum if \( \lambda_i \) has odd multiplicity.
Krasnoselskii-Boehme-Marino theorem (1/2)

Theorem (Krasnoselskii-Boehme-Marino)

Let $F : I \times H \to K : (t, u) \mapsto F(t, u)$ be a continuous function, where $I \subseteq \mathbb{R}$ is an interval, and $H$ and $K$ are Banach spaces, such that $F(\lambda, 0) = 0$ for any $\lambda \in I$.

- If $F$ is of class $C^1$ in a neighborhood of $(\lambda, 0)$ and $(\lambda, 0)$ is a bifurcation point of $F$ then $\partial_u F(\lambda, 0)$ is not invertible.
- Let assume that for each $(\lambda, u) \in I \times H$,

$$F(\lambda, u) = L(\lambda, u) - N(\lambda, u), \quad L(\lambda, \cdot) = \lambda 1 - T \quad \text{and} \quad N(\lambda, u) = o(\|u\|),$$

with $T$ linear, $T$ and $N$ compact, and the last equality being uniform on each compact set of $\lambda$.

If $\lambda_*$ is an eigenvalue of $T$ with odd multiplicity, then $(\lambda_*, 0)$ is a global bifurcation point for $F(t, u) = 0$. 

Krasnoselskii-Boehme-Marino theorem (2/2)

Theorem (Krasnoselskii-Boehme-Marino (cont’d))

Let assume that $H$ is a Hilbert space and that for each $(\lambda, u) \in I \times \mathbb{R}$, $F(\lambda, u) = \nabla_u h(\lambda, u)$ where

$$h(\lambda, u) = \frac{1}{2} \langle L(\lambda, u), u \rangle - g(\lambda, u),$$

$$L(\lambda, \cdot) = \lambda I - T,$$

and

$$\nabla g(\lambda, u) = o(\|u\|),$$

with $T$ linear and symmetric, $g(\lambda, \cdot) \in C^2$ for all $\lambda$, and the last equality being uniform on each compact set of $\lambda$.

If $\lambda_*$ is an eigenvalue of $T$ with finite multiplicity and $h(\lambda, \cdot)$ verifies the Palais-Smale condition for each $\lambda$, then $(\lambda_*, 0)$ is a bifurcation point for $F(t, u) = 0$. 
p large: transcritical radial bifurcations

$\lambda_{i,\text{rad}}$ eigenvalues that possess a radial eigenfunction (simple in $H^1_{\text{rad}}$).

**Proposition**

On balls, two branches radial solutions in $C^{2,\alpha}(\Omega)$ of

$$
(P_p) \begin{cases}
-\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\
\partial_{\nu} u = 0, & \text{on } \partial \Omega.
\end{cases}
$$

start from each $(p, u) = (1 + \lambda_{i,\text{rad}}, 1)$, $i > 1$. Locally, these branches form a unique $C^1$-curve. Moreover, for $i$ large enough independent of the measure of $\Omega$, the bifurcation is **transcritical**.
Proof. \( \Omega = B_R \). Using Ambrosetti-Prodi theorem, one has to show

\[
b = -\frac{1}{2} \lambda_i (\lambda_i - 1) \int_{B_R} \varphi_{i,\text{rad}}^3 \neq 0.
\]

Given that radial eigenfunctions are given by constant spherical harmonics \( (k = 0, \nu = (N - 2)/2) \), this amounts to

\[
\int_0^R \left( r^{-\frac{N-2}{2}} J_\nu(r \sqrt{\bar{\mu}_{i,\text{rad}}}/R) \right)^3 r^{N-1} \, dr \neq 0 \quad \text{i.e.} \quad \int_0^\infty t^{1-\nu} J_\nu^3(t) \, dt \neq 0
\]

where \( \lambda_{i,\text{rad}} = 1 + \bar{\mu}_{i,\text{rad}}/R^2 \). This is true for large \( i \) because

\[
\int_0^\infty t^{1-\nu} J_\nu^3(t) \, dt = \frac{2^{\nu-1} (3/16)^{\nu-1/2}}{\pi^{1/2} \Gamma(\nu + 1/2)} > 0.
\]
Numerical computations indicate that

\[ \forall z \in ]0, +\infty[, \quad \int_0^z t^{1-\nu} J_\nu^3(t) \, dt > 0, \quad \nu = (N-2)/2, \]

and therefore that radial bifurcations are transcritical for all \( i \).
Corollary

The branches consist of positive functions.

Sketch: If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0.
$p$ large: positive transcritical radial bifurcations

**Corollary**

*The branches consist of positive functions.*

**Sketch:** If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0. □

**Theorem**

*Radial bifurcations obtained for the $C^{2,\alpha}(\Omega)$-norm are unbounded and do not intersect each other. Moreover, along bifurcations starting from $(1 + \lambda_{i,\text{rad}}, 1)$, the solutions always possess the same number of intersections with 1.*

**Sketch:** The number of crossings with 1 stays constant because otherwise a non-constant radial solution $u$ s.t. $u - 1$ has a double root would exists. Since the branches do not intersect each other, Rabinowitz’s principle says they must be undounded.
$p$ large: multiplicity results (radial domains)

**Theorem**

Assume $\Omega$ is a ball.

- In dimension $2$, for any $n \in \mathbb{N}_0$, there exists $p_n > 2$ such that, for any $p > p_n$, at least $n$ positive solutions exist.
- In dimension $\geq 3$, for any $2 < p < 2^*$ and $n \in \mathbb{N}_0$, at least $n$ different positive solutions exist if the measure of the ball $\Omega$ is large enough.
$p$ large: multiplicity results (radial domains)

**Theorem**

Assume $\Omega$ is a ball.

- In dimension $2$, for any $n \in \mathbb{N}_0$, there exists $p_n > 2$ such that, for any $p > p_n$, at least $n$ positive solutions exist.

- In dimension $\geq 3$, for any $2 < p < 2^*$ and $n \in \mathbb{N}_0$, at least $n$ different positive solutions exist if the measure of the ball $\Omega$ is large enough.

**Theorem**

On balls, there exists a degenerate positive radial solution for some $p$ provided that the measure of $\Omega$ is large enough.
\[ p \geq 2^* \]

**Theorem (Serra & Tilli, ’11)**

Assume \( a \in L^1(]0, R[) \) is increasing, not constant and satisfies \( a > 0 \) in \( ]0, R[ \), then for any \( p \in ]2, +\infty[ \), \(-\Delta u + u = a(|x|)|u|^{p-2}u \) with NBC possesses a positive radially increasing solution.

Trick: work on the space of radially increasing functions.
Proposition

Assume $\Omega$ is a ball of radius $R$. If $u$ is a radial solution of $(P_p)$ such that $u(0) < 1$, then $\|u\|_{L^\infty} \leq \exp(1/2)$.
Proposition

Assume \( \Omega \) is a ball of radius \( R \). If \( u \) is a radial solution of \((P_p)\) such that \( u(0) < 1 \), then \( \|u\|_{L^\infty} \leq \exp(1/2) \).

Proof. In radial coordinates, the equation writes

\[
-u'' - \frac{N - 1}{r} u' + u = u^{p-1}.
\]

Multiplying by \( u' \), we get

\[
\frac{d}{dr} h(r) = - \frac{N - 1}{r} u'^2(r) \leq 0,
\]

where

\[
h(r) := \frac{u'^2(r)}{2} + \frac{u^p(r)}{p} - \frac{u^2(r)}{2}.
\]

In particular, this means that \( h(r) \leq h(0) \) for any \( r \).
\( p \geq 2^* \)

**Proof (cont’d).** The assumption \( u(0) < 1 \) implies

\[
h(0) = \frac{u^p(0)}{p} - \frac{u^2(0)}{2} = u^2(0)\left( \frac{u^{p-2}(0)}{p} - \frac{1}{2} \right) \leq 0.
\]

Thus

\[
\|u\|_{L^\infty} \leq \left( \frac{p}{2} \right)^{1/(p-2)} \leq \exp(1/2).
\]

\( \square \)
\( p \geq 2^* \)

**Theorem**

Assume \( \Omega \) is a ball. Then, for any \( n \in \mathbb{N}_0 \), there exists \( p_n \) s.t., for any \( p \in [p_n, +\infty[ \), \(( \mathcal{P}_p )\) has at least \( n \) positive radially symmetric solutions.
$p \geq 2^*$

**Theorem**

Assume $\Omega$ is a ball. Then, for any $n \in \mathbb{N}_0$, there exists $p_n$ s.t., for any $p \in [p_n, +\infty[$, $(\mathcal{P}_p)$ has at least $n$ positive radially symmetric solutions.

**Sketch:** Radial bifurcations are transcritical, thus, as $p \approx 1 + \lambda_{i,\text{rad}},$

$$u_p = 1 + \frac{p - 1 - \lambda_{i,\text{rad}}}{b} \varphi_{i,\text{rad}} + o(p - 1 - \lambda_{i,\text{rad}}).$$

Along the left or right branch $u_p(0) < 1$. This latter property persists along the whole branch. Thus all $u$ belonging to that branch must satisfy $\|u\|_{L^\infty} \leq \exp(1/2)$. Since 1 is the only solution for $p \approx 2$, the branch must exist for all $p$ large. □
| $p \approx 2$: ground state solutions | $p \approx 2$: positive solutions | Symmetry breaking | Bifurcations | Multiplicity |

Thank you for your attention.