
Mónica Clapp

Universidad Nacional Autónoma de México

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Introduction

The problem

We consider the problem

\[
\begin{cases}
-\Delta u = |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where
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(\phi_p) \quad \begin{cases} 
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where

- \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( N \geq 3 \),
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where

- \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( N \geq 3 \),
- \( p = 2^* := \frac{2N}{N-2} \) is the critical Sobolev exponent, or
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where

\begin{itemize}
  \item $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 3,$
  \item $p = 2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, or
  \item $p > 2^*$ is supercritical.
\end{itemize}
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- \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( N \geq 3 \),
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- \( p > 2^* \) is supercritical.

Throughout this talk \( p = 2^* \).
Introduction

Why study the critical problem?

Some reasons for studying \((\mathcal{Q}_2^*)\):

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
Introduction

Why study the critical problem?

Some reasons for studying $\mathcal{P}_{2^*}$:

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
   - the Yamabe problem,
Introduction

Why study the critical problem?

Some reasons for studying $f_{2^*}$:

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
   - the Yamabe problem,
   - the prescribed curvature problem.
Introduction

Why study the critical problem?

Some reasons for studying $(\mathcal{R}_{2^*})$:

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
   - the Yamabe problem,
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2. It gives rise to an interesting and challenging variational problem:
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Why study the critical problem?

Some reasons for studying $\left(\mathcal{Q}_{2^*}\right)$:

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
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   - the prescribed curvature problem.

2. It gives rise to an interesting and challenging variational problem:
   - Usual variational methods cannot be applied due to the lack of compactness.
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Some reasons for studying ($\varphi_{2^*}$):

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
   - the Yamabe problem,
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2. It gives rise to an interesting and challenging variational problem:
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3. It has a rich geometric structure.
Introduction

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Some reasons for studying \( (\varphi_2^*) \):

1. It is a simplified model for fundamental problems in Differential Geometry, e.g.
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   - the prescribed curvature problem.

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   - Usual variational methods cannot be applied due to the lack of compactness.

3. It has a rich geometric structure.

4. It has been an amazing source of open problems and new ideas.
Introduction

The Yamabe problem

- Let $M$ be a compact Riemannian manifold.
Introduction
The Yamabe problem

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- Two metrics $g$ and $\bar{g}$ on $M$ are \textit{conformally equivalent} if there exists a smooth function $\rho > 0$ such that $\bar{g} = \rho g$. 

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The Yamabe problem
The Yamabe problem

In dimension 2 one has the classical result:
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**Theorem (Klein-Poincaré uniformization theorem)**

_Every surface admits a metric of constant curvature._
Introduction

The Yamabe problem

- In dimension 2 one has the classical result:

**Theorem (Klein-Poincaré uniformization theorem)**

*Every surface admits a metric of constant curvature.*

**Problem (Yamabe)**

*If* $(M, g)$, $\text{dim} M \geq 3$, *does there exist a metric* $\bar{g}$ *conformally equivalent to* $g$ *such that* $(M, \bar{g})$ *has constant scalar curvature?*
Problem (Yamabe)

If \((M, g)\), \(\text{dim} M \geq 3\), does there exist a metric \(\bar{g}\) conformally equivalent to \(g\) such that \((M, \bar{g})\) has constant scalar curvature?
Yamabe's problem

The variational problem

The classical results

The geometric structure

Multiple solutions

Punctured domains

Symmetries

Thin holes

The end

Introduction

The Yamabe problem

Problem (Yamabe)

If \((M, g)\), \(\text{dim} M \geq 3\), does there exist a metric \(\bar{g}\) conformally equivalent to \(g\) such that \((M, \bar{g})\) has constant scalar curvature?

- Yamabe (1960) claimed there exists such a metric, but ...
**Introduction**

**The Yamabe problem**

**Problem (Yamabe)**

*If $\mathcal{M} = (M, g)$, $\dim M \geq 3$, does there exist a metric $\bar{g}$ conformally equivalent to $g$ such that $(\mathcal{M}, \bar{g})$ has constant scalar curvature?*

- Yamabe (1960) claimed there exists such a metric, but . . .
- Trudinger (1968) found a fundamental mistake in Yamabe’s proof.
Introduction
The Yamabe problem

Problem (Yamabe)

If \((M, g)\), \(\text{dim} M \geq 3\), does there exist a metric \(\bar{g}\) conformally equivalent to \(g\) such that \((M, \bar{g})\) has constant scalar curvature?

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Theorem (Yamabe, Trudinger, Aubin 1976, Schoen 1984)

The answer to Yamabe’s problem is YES.
Problem (Yamabe)

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Theorem (Yamabe, Trudinger, Aubin 1976, Schoen 1984)

The answer to Yamabe’s problem is YES.

If we write \(\rho := u^{2^*-2}\) and \(\bar{g} := \rho g\), then the scalar curvatures \(R_g\) of \((M, g)\) and \(R_{\bar{g}}\) of \((M, \bar{g})\) satisfy

\[-c_N \Delta_g u + R_g u = R_{\bar{g}} u^{2^*-1}\]
on \(M\).
The variational problem

The functional

The solutions to problem

\[
\begin{cases}
-\Delta u = |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

\( p \in (2, 2^*] \),

\( \Omega \subset \mathbb{R}^N \) bounded smooth domain, \( N \geq 3 \),

\( 2^* := \frac{2N}{N-2} \),
The variational problem

The functional

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(\varphi_p) \quad \begin{cases}
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\(\Omega \subset \mathbb{R}^N\) bounded smooth domain, \(N \geq 3\), \(2^* := \frac{2N}{N-2}\),

- are the critical points of

\[
J_p(u) = \frac{1}{2} \|u\|_{H^1_0}^2 - \frac{1}{p} \|u\|_{L^p}^p, \quad u \in H^1_0(\Omega),
\]
The variational problem

The functional

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$$J_p(u) = \frac{1}{2} \|u\|_{H^1_0}^2 - \frac{1}{p} \|u\|_{L^p}^p, \quad u \in H^1_0(\Omega),$$

• where \( \|u\|_{H^1_0}^2 := \int_\Omega |
\nabla u|^2 \).
The variational problem

The graph of the functional

\[ J_p(u) = \frac{1}{2} \| u \|^2_{H^1_0} - \frac{1}{p} \| u \|^p_{L^p}, \quad p \in (2, 2^*]. \]
The variational problem

Variational methods

- Variational methods: Follow the negative gradient flow to obtain critical points.
The variational problem

Variational methods

- **Variational methods**: Follow the negative gradient flow to obtain critical points.

- The problem is: The flow lines do not necessarily take us to a critical point!
The variational problem

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  - They do if $p < 2^*$. 
The variational problem

Variational methods

- **Variational methods**: Follow the negative gradient flow to obtain critical points.

- The problem is: The flow lines do not necessarily take us to a critical point!
  - They do if $p < 2^*$.
  - But not necessarily when $p = 2^*$. 
The variational problem
supercritical vs subcritical

In fact,
- if $p \in (2, 2^*)$ variational methods give infinitely many solutions to problem

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(\mathcal{P}_p) \quad \begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}
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whereas,
The variational problem
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In fact,

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The variational problem
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  - e.g. \( \Omega = \text{ball} \).
The variational problem
supercritical vs subcritical

In fact,

- if $p \in (2, 2^*)$ variational methods give infinitely many solutions to problem

$$\begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

whereas,

- if $p \geq 2^*$ there are domains $\Omega$ for which the problem has no solution,
  - e.g. $\Omega = \text{ball}$.
  - the existence of solutions depends on $\Omega$. 

\[ \text{critical & supercritical} \]
\[ \text{Mónica Clapp} \]

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The classical results

Introduction

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Theorem (Pohozhaev 1965)

Problem

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\begin{cases}
-\Delta u = |u|^{p-2} u & \text{in } \Omega, \\
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\end{cases}
\]

with \( p \geq 2^* \) does not have a nontrivial solution if \( \Omega \) is strictly starshaped.

Nonexistence
The classical results

Existence

Theorem (Kazdan-Warner 1975)

If $\Omega$ is an annulus, i.e.

$$\Omega = \{ x \in \mathbb{R}^N : 0 < a < |x| < b \},$$

then $(\varphi_p)$ has infinitely many radial solutions for every $p > 2$. 
The classical results

Existence in punctured domains

Theorem (Coron 1984)

Let \( \Omega \) be a bounded smooth domain, \( \xi \in \Omega \) and \( \varepsilon > 0 \). Then

\[
\begin{cases}
-\Delta u = |u|^{2^* - 2} u & \text{in } \Omega_\varepsilon := \Omega \setminus B_\varepsilon(\xi), \\
u = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}
\]

has a positive solution for \( \varepsilon \) small enough.
The classical results

Existence

**Theorem (Bahri-Coron 1988)**

If $\tilde{H}_*(\Omega; \mathbb{Z}/2) \neq 0$, then $(\varphi_{2^*})$ has a positive solution.

- The proof relies on the fact that one knows all positive solutions to the problem in $\mathbb{R}^N$. 
The classical results

Existence

Theorem (Bahri-Coron 1988)

If \( \tilde{H}_*(\Omega; \mathbb{Z}/2) \neq 0 \), then \((\varphi_{2^*})\) has a positive solution.

- The proof relies on the fact that one knows all positive solutions to the problem in \( \mathbb{R}^N \).
- It uses delicate estimates and
The classical results

Existence

Theorem (Bahri-Coron 1988)

If $\tilde{H}_*(\Omega; \mathbb{Z}/2) \neq 0$, then $(\varphi_{2*})$ has a positive solution.

- The proof relies on the fact that one knows all positive solutions to the problem in $\mathbb{R}^N$.
- It uses delicate estimates and sophisticated tools from algebraic topology.
The classical results

Existence in contractible domains

• Is it true that there is no solution if $\Omega$ is contractible?
The classical results
Existence in contractible domains

- Is it true that there is no solution if $\Omega$ is contractible?

Examples (Dancer 1988, Ding 1989, Passaseo 1989)
There are nontrivial solutions in some contractible domains, e.g.

Annulus with a very thin tunnel
The geometric structure
Möbius invariance

- If \( u \) is a solution to

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The geometric structure

Möbius invariance

• If $u$ is a solution to

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• then, for any Möbius transformation

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\phi : \mathbb{R}^N \cup \{\infty\} \to \mathbb{R}^N \cup \{\infty\},
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The geometric structure
Möbius invariance

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- then, for any Möbius transformation

\[
\phi : \mathbb{R}^N \cup \{\infty\} \to \mathbb{R}^N \cup \{\infty\},
\]

- the function

\[
u_\phi := |\det D\phi|^{\frac{1}{2^*}} (u \circ \phi)
\]

is a solution to

\[
\begin{cases}
-\Delta u_\phi = |u_\phi|^{2^*-2} u_\phi & \text{in } \phi^{-1}(\Omega), \\
u_\phi = 0 & \text{on } \partial (\phi^{-1}(\Omega)) .
\end{cases}
\]
The geometric structure

Möbius transformations

- A **Möbius transformation** is a finite composition of reflections on planes and inversions on spheres.
The geometric structure
Möbius transformations

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- Examples:
The geometric structure

Möbius transformations

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Examples:
- euclidean isometries, i.e. translations and linear isometries,
The geometric structure

Möbius transformations

- A Möbius transformation is a finite composition of reflections on planes and inversions on spheres.

Examples:
- euclidean isometries, i.e. translations and linear isometries,
- dilations: $x \mapsto \lambda x$, $\lambda > 0$. 
The geometric structure

Multiplicity in domains with spherical boundaries

Example (C.-Pacella 2008)

If \( \partial \Omega = \) union of two disjoint spheres, then problem \( (\varphi_2^*) \) has infinitely many solutions.
The geometric structure

Multiplicity in domains with spherical boundaries

Examples (C.-Pacella 2008)

If \( \partial \Omega = \) union of two disjoint spheres, then problem \((\varphi_{2*})\) has infinitely many solutions.
Examples (C.-Pacella 2008)
If $\partial \Omega = \text{union of two disjoint spheres}$, then problem $(\varphi_{2^*})$ has infinitely many solutions.

Proof.
There exists an inversion which maps $\Omega$ onto an annulus:
The geometric structure

Positive entire solutions

- Consider the problem in \( \mathbb{R}^N \)

\[
\begin{cases}
- \Delta u = |u|^{2^* - 2} u & \text{in } \mathbb{R}^N, \\
u(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]
The geometric structure

Standard bubbles

Theorem (Aubin, Talenti 1976, Gidas-Ni-Nirenberg 1979, Lions 1985)

The standard bubble

\[ U(x) = a_N \left( \frac{1}{1 + |x|^2} \right)^{\frac{N-2}{2}} \]

is the only positive solution to \((\mathcal{O}_{\mathbb{R}^N})\), up to translations and dilations.
The geometric structure causes blow-up in bounded domains

- As for
The geometric structure causes blow-up in bounded domains

- As for

\[ J(u_t) \to c, \quad \nabla J(u_t) \to 0, \quad \text{as } t \to \infty, \]

- if \( \Omega \) is a bounded domain, there are trajectories \( t \mapsto u_t \) in \( H_0^1(\Omega) \), such that
The geometric structure
causes blow-up in bounded domains

- As for

- if $\Omega$ is a bounded domain, there are trajectories $t \mapsto u_t$ in $H^1_0(\Omega)$, such that

$$J(u_t) \to c, \quad \nabla J(u_t) \to 0, \quad \text{as } t \to \infty,$$

- but $(u_t)$ does not converge to a critical point as $t \to \infty$!
The geometry of the problem causes blow-up in bounded domains

- They look like this:
The geometry of the problem causes blow-up in bounded domains

- They look like this:
The geometry of the problem causes blow-up in bounded domains

- They look like this:

- Struwe (1984) showed that the lack of compactness is solely due to this phenomenon.
Multiple solutions
Our program

- We now go back to the problem

\[
\begin{align*}
\phi_{2^*} \quad \begin{cases} 
- \Delta u &= |u|^{2^*-2} u \quad \text{in } \Omega, \\
            u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

in a bounded smooth domain \( \Omega \subset \mathbb{R}^N, \ N \geq 3, \)
\[
2^* := \frac{2N}{N-2}.
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Multiple solutions

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- QUESTIONS:
We now go back to the problem

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**QUESTIONS:**

- In those cases where existence is known, are there other solutions?
Multiple solutions
Our program

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in a bounded smooth domain \( \Omega \subset \mathbb{R}^N, \, N \geq 3, \)

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- **QUESTIONS:**
  - In those cases where existence is known, are there other solutions?
  - How do they look like?
Multiple solutions

Answers & methods

• SOME ANSWERS:

1. In punctured domains (like those of Coron) much progress has been made.
2. In other slightly perturbed domains &
3. In more general domains with nontrivial topology there are a few recent results.

THE METHODS:
1. Lyapunov-Schmidt reduction, which works very well for punctured domains.
2. Variational methods + symmetries, the symmetries help us deal with the lack of compactness.
Multiple solutions

Answers & methods

- **SOME ANSWERS:**

1. In punctured domains (like those of Coron)
Multiple solutions

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The classical results

The geometric structure

Multiple solutions

Punctured domains

Symmetries

Thin holes

The end

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Multiple solutions
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Multiple solutions
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2. Variational methods + symmetries,
   • the symmetries help us deal with the lack of compactness.
Multiple solutions
In punctured domains

- The data:
Multiple solutions
In punctured domains

- The data:
  - $\Omega$ a bounded smooth domain in $\mathbb{R}^N$,
Multiple solutions

In punctured domains

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  - $\xi \in \Omega$, 

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Multiple solutions
In punctured domains

- The data:
  - $\Omega$ a bounded smooth domain in $\mathbb{R}^N$,
  - $\xi \in \Omega$,
  - $\varepsilon > 0$ small enough,
Multiple solutions
In punctured domains

- The data:
  - $\Omega$ a bounded smooth domain in $\mathbb{R}^N$,
  - $\xi \in \Omega$,
  - $\epsilon > 0$ small enough,
  - $\Omega_\epsilon := \Omega \setminus B_\epsilon(\xi)$. 

The data: $\Omega$ a bounded smooth domain in $\mathbb{R}^N$, $\xi \in \Omega$, $\epsilon > 0$ small enough, $\Omega_\epsilon := \Omega \setminus B_\epsilon(\xi)$. 

The end
Multiple solutions
In punctured domains

- The data:
  - \( \Omega \) a bounded smooth domain in \( \mathbb{R}^N \),
  - \( \xi \in \Omega \),
  - \( \varepsilon > 0 \) small enough,
  - \( \Omega_\varepsilon := \Omega \setminus B_\varepsilon(\xi) \).

Problem

Does problem

\[
\left\{
\begin{array}{ll}
-\Delta u = |u|^{2^*-2}u & \text{in } \Omega_\varepsilon, \\
u = 0 & \text{on } \partial \Omega_\varepsilon,
\end{array}
\right.
\]

have more than one solution for \( \varepsilon > 0 \) small enough?
Multiple solutions
In punctured domains

Theorem (Ge-Musso-Pistoia 2010)

$$\# \text{ of solutions to } (\phi_{2^*},\varepsilon) \xrightarrow{\varepsilon \to 0} \infty.$$  

- The solutions look like superpositions of standard bubbles with alternating signs (bubble towers):

![Diagram of bubble towers]
Symmetries
The energy functional

- Recall that the solutions to

\[
\begin{aligned}
\tag{8^*}
-\Delta u &= \left|u\right|^{2^*-2} u & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

\[\]

The classical results
The geometric structure
Multiple solutions
Punctured domains
Symmetries
Thin holes
The end
• Recall that the solutions to

\[
\begin{cases}
  -\Delta u = |u|^{2^*-2} u & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

are the critical points of the functional

\[
J(u) = \frac{1}{2} \| u \|_{H_0^1}^2 - \frac{1}{2^*} \| u \|_{L^{2^*}}^{2^*} , \quad u \in H_0^1(\Omega).
\]
Symmetries
The energy functional

- $J$ has the mountain pass geometry:
Symmetries
The energy functional

\[ J \] has the mountain pass geometry:

- but the first mountain pass is never attained!!!
Let $G \subset O(N)$ be a group of linear isometries of $\mathbb{R}^N$. 
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We assume that $\Omega$ is $G$-invariant, i.e.
Let $G \subset O(N)$ be a group of linear isometries of $\mathbb{R}^N$.

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- $Gx \subset \Omega$ for all $x \in \Omega$. 

Symmetries and variational methods
Symmetries
and variational methods

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Symmetries
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Symmetries
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  - $Gx \subset \Omega$ for all $x \in \Omega$,
- and look for $G$-invariant solutions $u$, i.e.
  - $u$ is constant on each $Gx$.
- They are the critical points of the restriction of

$$J(u) = \frac{1}{2} \| u \|_{H_0^1}^2 - \frac{1}{2^*} \| u \|_{L^{2^*}}$$

to the subspace

$$H_0^1(\Omega)^G := \{ u \in H_0^1(\Omega) : u \text{ is } G\text{-invariant} \}.$$
Symmetries produce compactness

**Theorem (compactness)**

*If*

\[
c < \min_{x \in \Omega} (\# Gx) c_{\infty}, \quad c_{\infty} := \frac{1}{N} S^{N/2},
\]

*then J satisfies the Palais-Smale \((PS)_c^G\), i.e.*

- **every sequence s.t.**

  \[
u_n \in H^1_0(\Omega)^G, \quad J(u_n) \to c, \quad \nabla J(u_n) \to 0,
  \]

  contains a convergent subsequence.*
Symmetries
produce compactness

Theorem (compactness)
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contains a convergent subsequence.

Corollary
If \( \# Gx = \infty \) for every \( x \in \overline{\Omega} \) then problem \((\varphi_{2^*})\) has infinitely many solutions.
Example (Kazdan-Warner)
If $G = O(N)$ and $\Omega = \text{annulus}$, then problem ($\varphi_{2^*}$) has infinitely many radial solutions.

Example
If $G = SO(2)$ and $\Omega = \text{torus}$, then ($\varphi_{2^*}$) has infinitely many solutions which are invariant under rotations.
Example (Pohožhaev)
If $G = O(N)$ and $\Omega = \text{ball}$, problem $(\mathcal{E}_{2^*})$ does not have a nontrivial solution!!!
Domains with a thin hole

The setting

- The data:
Domains with a thin hole

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  - $\Omega$ a bounded smooth domain in $\mathbb{R}^N$, 

The end
Domains with a thin hole

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The end
Domains with a thin hole

The setting

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- $\Omega$ a bounded smooth domain in $\mathbb{R}^N$,
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- $\varepsilon > 0$ small enough,
Domains with a thin hole

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Domains with a thin hole

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Problem

Does problem

$$
\begin{cases}
-\Delta u = |u|^{2^*-2} u & \text{in } \Omega_\varepsilon, \\
u = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}
$$

have more than one solution for $\varepsilon$ small enough?
Domains with a thin hole
Symmetries & sign changing solutions

• To obtain sign changing solutions we consider domains with symmetries. We assume
Domains with a thin hole
Symmetries & sign changing solutions

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- \( G \subset O(N) \) is a finite group of linear isometries of \( \mathbb{R}^N \),
Domains with a thin hole
Symmetries & sign changing solutions

- To obtain sign changing solutions we consider domains with symmetries. We assume
  - $G \subset O(N)$ is a finite group of linear isometries of $\mathbb{R}^N$,
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Domains with a thin hole
Symmetries & sign changing solutions

To obtain sign changing solutions we consider domains with symmetries. We assume

- $G \subset O(N)$ is a finite group of linear isometries of $\mathbb{R}^N$,
- $\Omega$ and $M$ are $G$-invariant,
- For simplicity, $G$ acts freely on $\Omega$. 
Domains with a thin hole
Symmetries & sign changing solutions

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  • $G \subset O(N)$ is a finite group of linear isometries of $\mathbb{R}^N$,
  • $\Omega$ and $M$ are $G$-invariant,
  • For simplicity, $G$ acts freely on $\Omega$,
    • i.e. $gx \neq x$ for all $g \in G$, $x \in \Omega$. 

Introduction
Yamabe's problem
The variational problem
The classical results
The geometric structure
Multiple solutions
Punctured domains
Symmetries
Thin holes
The end
Domains with a thin hole
Symmetries & sign changing solutions

Example

\[ G_n \defeq \text{group generated by the rotation of angle } \frac{2\pi}{n} \text{ about the } \]
\[ z\text{-axis in } \mathbb{R}^3, \]
Domains with a thin hole
Symmetries & sign changing solutions

Example

\(G_n := \text{group generated by the rotation of angle } \frac{2\pi}{n} \text{ about the } z\text{-axis in } \mathbb{R}^3,\)

- \(\Omega\) is a torus of revolution about the \(z\)-axis,
Domains with a thin hole
Symmetries & sign changing solutions

Example

$G_n := \text{group generated by the rotation of angle } \frac{2\pi}{n} \text{ about the z-axis in } \mathbb{R}^3$,

- $\Omega$ is a torus of revolution about the z-axis,
- $M \subset \Omega$ is a toroidal knot:

$G_5$-inv.  $G_8$-inv.  $G_{10}$-inv.
Domains with a thin hole
Symmetries & sign changing solutions

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$$G_5\text{-inv.}, \quad G_8\text{-inv.}, \quad G_{10}\text{-inv.}$$

- Then $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, M) > \varepsilon\}$ is $G_n$-invariant, but $\#G_n x = n < \infty.$
Domains with a thin hole

A trick to produce sign changing solutions

- Let $\tau : G \to \mathbb{Z}/2 := \{1, -1\}$ be a group homomorphism.
Domains with a thin hole

A trick to produce sign changing solutions

- Let $\tau : G \to \mathbb{Z}/2 := \{1, -1\}$ be a group homomorphism.
- We look for solutions $u$ which satisfy

$$u(gx) = \tau(g)u(x) \quad \forall g \in G, \forall x \in \Omega_\varepsilon.$$
Domains with a thin hole
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- Let $\tau : G \to \mathbb{Z}/2 := \{1, -1\}$ be a group homomorphism.
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Domains with a thin hole
A trick to produce sign changing solutions

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    \]
- Therefore, if \( \tau \) is surjective, \( u \) changes sign.
Domains with a thin hole
Symmetries & sign changing solutions

Example

$G_{2n} := \text{group generated by the rotation } \varrho_{2n} \text{ by } \frac{\pi}{n} \text{ about the } z\text{-axis in } \mathbb{R}^3,$
Domains with a thin hole
Symmetries & sign changing solutions

Example

$G_{2n} :=$ group generated by the rotation $\varrho_{2n}$ by $\frac{\pi}{n}$ about the $z$-axis in $\mathbb{R}^3$,

- $\tau(\varrho_{2n}^k) := (-1)^k, \quad k = 0, 1, \ldots, 2n - 1.$
Domains with a thin hole
Symmetries & sign changing solutions

Example

$G_{2n} :=$ group generated by the rotation $\rho_{2n}$ by $\dfrac{\pi}{n}$ about the $z$-axis in $\mathbb{R}^3$,

- $\tau(\rho_{2n}^k) := (-1)^k$, $k = 0, 1, \ldots, 2n - 1$.
- A function satisfying $u(gx) = \tau(g)u(x)$ for $G_4$ is:
Domains with a thin hole
Symmetries & sign changing solutions

We look for solutions to

\[(\mathcal{D}_2^*, \varepsilon) \quad \left\{ \begin{array}{l}
-\Delta u = |u|^{2^*-2} u \quad \text{in } \Omega_\varepsilon, \\
u = 0 \quad \partial \Omega_\varepsilon,
\end{array} \right.\]

where \(\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, M) > \varepsilon\}\),
Domains with a thin hole
Symmetries & sign changing solutions

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where \( \Omega_\epsilon := \{ x \in \Omega : \text{dist}(x, M) > \epsilon \} \),

- which satisfy

\[
u(gx) = \tau(g)u(x) \quad \forall g \in G, \; x \in \Omega_\epsilon.
\]
Domains with a thin hole

Existence

Theorem (C.-Grossi-Pistoia ~2010)

For each $\varepsilon > 0$ sufficiently small problem $(\mathcal{P}_{\varepsilon}^*, \varepsilon)$ has at least one nontrivial solution $u$ which satisfies

$$u(gx) = \tau(g)u(x) \quad \forall g \in G, \ x \in \Omega_\varepsilon.$$ 

- $u$ is positive if $\tau$ is the trivial homomorphism,
Domains with a thin hole

Existence

Theorem (C.-Grossi-Pistoia ~2010)

For each \( \varepsilon > 0 \) sufficiently small problem \((\delta \Omega_{2^*}, \varepsilon)\) has at least one nontrivial solution \(u\) which satisfies

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\]

- \( u \) is positive if \( \tau \) is the trivial homomorphism,
- \( u \) changes sign if \( \tau \) is surjective.
Example

$\Omega \subset \mathbb{R}^3$ a solid of revolution about the $z$-axis, $M := S^1 \times \{0\}$ such that

$$M \subset \Omega \quad \text{and} \quad \Omega \cap (z\text{-axis}) = \emptyset.$$
Domains with a thin hole
Multiplicty in highly symmetric domains

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- \( \Omega_\varepsilon \) is invariant under rotations about the z-axis, hence
Domains with a thin hole
Multiplicity in highly symmetric domains

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• $\Omega_\varepsilon$ is invariant under rotations about the $z$-axis, hence
• $(\varphi_\varepsilon)$ has infinitely many rotationally invariant solutions.
Domains with a thin hole
Multiplicity in highly symmetric domains

Corollary

Let \( m \in \mathbb{N} \), \( \Omega \) and \( M \) as above. Then, for \( \varepsilon \) small enough, \((\varphi_\varepsilon)\) has \( m \) pairs of solutions \( \pm u_1, \ldots, \pm u_m \) such that

\[
 u_n(\varphi_{2^n}^k x) = (-1)^k u_n(x), \quad k = 0, 1, \ldots, 2^n - 1.
\]

Proof. We apply the previous theorem to the group \( G_2 \).
Corollary

Let \( m \in \mathbb{N} \), \( \Omega \) and \( M \) as above. Then, for \( \varepsilon \) small enough, \((\varphi_{\varepsilon})\) has \( m \) pairs of solutions \( \pm u_1, \ldots, \pm u_m \) such that

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Proof.

We apply the previous theorem to the group \( G_{2^n} \):

\[G_2\] \hspace{1cm} \[G_4\] \hspace{1cm} \[G_8\]
Domains with a thin hole
Open problems

Problem

In general domains of the form (without symmetries)

\[ \Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, M) > \varepsilon \}, \]

- work in progress by Juan Carlos Fernández shows there are at least two solutions.
Domains with a thin hole

Open problems

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- Is it true that, as for punctured domains, the number of solutions increases arbitrarily as \( \varepsilon \to 0 \)?
Domains with a thin hole

Open problems

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Domains with a thin hole
Open problems

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- Are there bubble towers?
- Are there multibump solutions?
Domains with a thin hole

Open problems

Problem

*In general domains of the form (without symmetries)*

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- *work in progress by Juan Carlos Fernández shows there are at least two solutions.*
- *Is it true that, as for punctured domains, the number of solutions increases arbitrarily as \( \varepsilon \to 0 \)?*
- *Are there bubble towers?*
- *Are there multibump solutions?*
- *Are there solutions with layers concentrating along \( M \) as \( \varepsilon \to 0 \)?
Thank you very much for your attention!