Elliptic boundary value problems with critical and supercritical nonlinearities. Part 1.

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Introduction

The problem

- We consider the problem

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- Throughout this talk $p = 2^*$.

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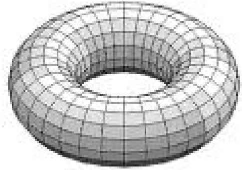
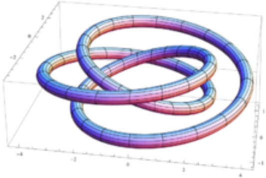
3. It has a rich geometric structure.
4. It has been an amazing source of open problems and new ideas.

Introduction

The Yamabe problem

- Let M be a compact Riemannian manifold.





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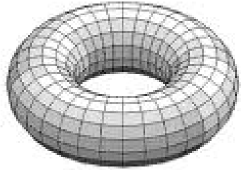
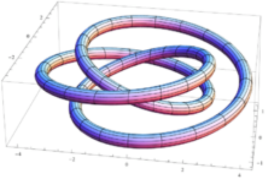

- Two metrics g and $\overline{g}$ on M are *conformally equivalent* if there exists a smooth function $\rho > 0$ such that $\overline{g} = \rho g$.

Introduction

The Yamabe problem

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Every surface admits a metric of constant curvature.

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Problem (Yamabe)

If (M, g), $dimM \geq 3$, does there exist a metric $\overline{g}$ conformally equivalent to g such that $(M, \overline{g})$ has constant scalar curvature?

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- Trudinger (1968) found a fundamental mistake in Yamabe's proof.

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Theorem (Yamabe, Trudinger, Aubin 1976, Schoen 1984)

The answer to Yamabe's problem is YES.

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Theorem (Yamabe, Trudinger, Aubin 1976, Schoen 1984)

The answer to Yamabe's problem is YES.

- If we write $\rho := u^{2^*-2}$ and $\overline{g} := \rho g$, then the scalar curvatures R_g of (M, g) and $R_{\overline{g}}$ of $(M, \overline{g})$ satisfy

$$-c_N \Delta_g u + R_g u = R_{\overline{g}} u^{2^*-1} \quad \text{on } M.$$

The variational problem

The functional

- The solutions to problem

$$(\wp_p) \quad \begin{cases} -\Delta u = |u|^{p-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad p \in (2, 2^*],$$

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$$J_p(u) = \frac{1}{2}\, \|u\|_{H_0^1}^2 - \frac{1}{p}\, \|u\|_{L^p}^p\,, \qquad u \in H_0^1(\Omega),$$

  - where $\|u\|_{H_0^1}^2 := \int_\Omega |\nabla u|^2$.

The variational problem

The graph of the functional

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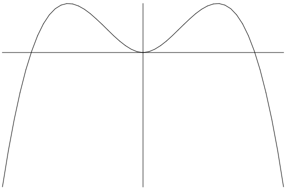
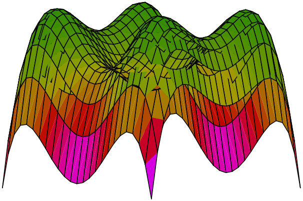
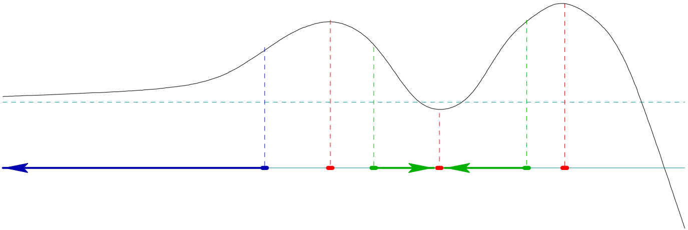




The variational problem

Variational methods

- Variational methods: Follow the negative gradient flow to obtain critical points.



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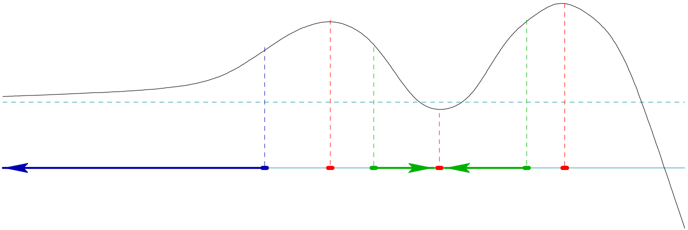

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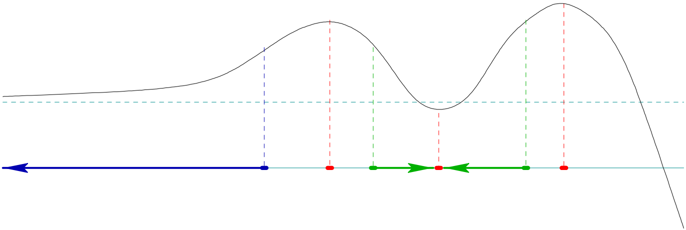

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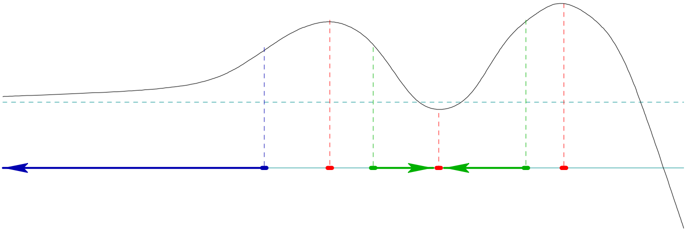

- The problem is: The flow lines do not necessarily take us to a critical point!
  - They do if $p < 2^*$.
  - But not necessarily when $p = 2^*$.

The variational problem

supercritical vs subcritical

In fact,

- if $p \in (2, 2^*)$ variational methods give infinitely many solutions to problem

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  - e.g. $\Omega =$ ball.
  - the existence of solutions depends on Ω.

The classical results

Nonexistence

Theorem (Pohozhaev 1965)

Problem

$$(\wp_p)\quad \begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $p \geq 2^*$ *does not have a nontrivial solution if* Ω *is strictly starshaped.*

The classical results

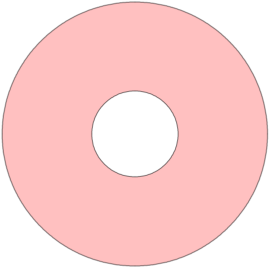
Existence

Theorem (Kazdan-Warner 1975)

If Ω is an annulus, i.e.

$$\Omega = \{x \in \mathbb{R}^N : 0 < a < |x| < b\},$$

then $(\wp_p)$ has infinitely many radial solutions for every $p > 2$.



The classical results

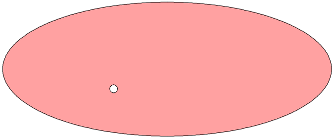
Existence in punctured domains

Theorem (Coron 1984)

Let Ω be a bounded smooth domain, $\xi \in \Omega$ and $\varepsilon > 0$. Then

$$(\wp_{2^*,\varepsilon}) \qquad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega_\varepsilon := \Omega \smallsetminus B_\varepsilon(\xi), \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

has a positive solution for ε small enough.

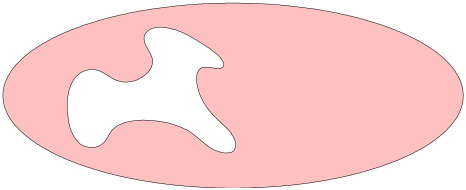


The classical results

Existence

Theorem (Bahri-Coron 1988)

If $\tilde{H}_*(\Omega;\mathbb{Z}/2) \neq 0$, *then* $(\wp_{2^*})$ *has a positive solution.*



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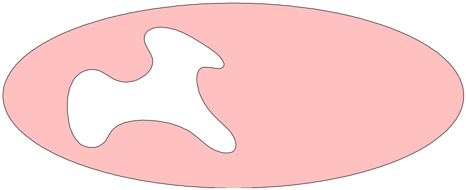

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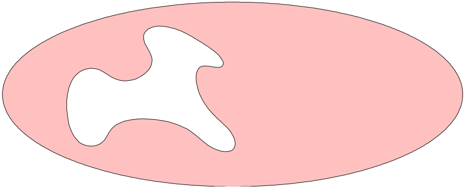


- *The proof relies on the fact that one knows all positive solutions to the problem in $\mathbb{R}^N$.*
- *It uses delicate estimates and*
- *sofisticated tools from algebraic topology.*

The classical results

Existence in contractible domains

- Is it true that there is no solution if Ω is contractible?

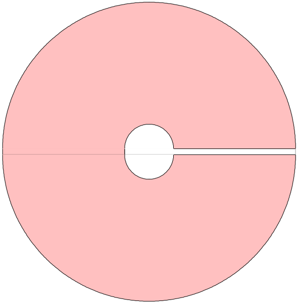
The classical results

Existence in contractible domains

- Is it true that there is no solution if Ω is contractible?

Examples (Dancer 1988, Ding 1989, Passaseo 1989)

There are nontrivial solutions in some contractible domains, e.g.



Annulus with a very thin tunnel

The geometric structure

Möbius invariance

- If u is a solution to

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$$\phi : \mathbb{R}^N \cup \{\infty\} \to \mathbb{R}^N \cup \{\infty\},$$

- the function

$$u_\phi := |\det D\phi|^{\frac{1}{2^*}} (u \circ \phi)$$

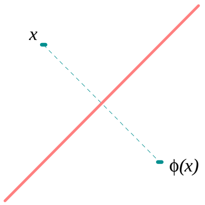
  is a solution to

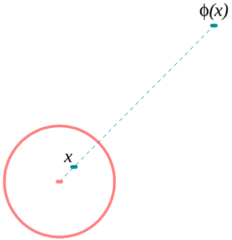
$$\begin{cases} -\Delta u_\phi = |u_\phi|^{2^*-2} u_\phi & \text{in } \phi^{-1}(\Omega), \\ u = 0 & \text{on } \partial\left(\phi^{-1}(\Omega)\right). \end{cases}$$

The geometric structure

Möbius transformations

- A **Möbius transformation** is a finite composition of reflections on planes and inversions on spheres.





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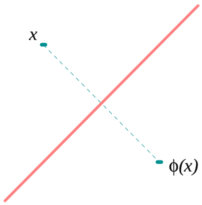

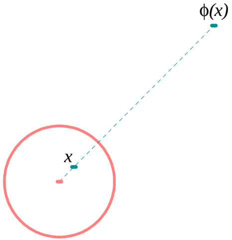

- Examples:

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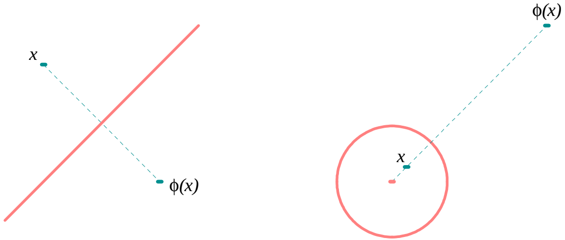

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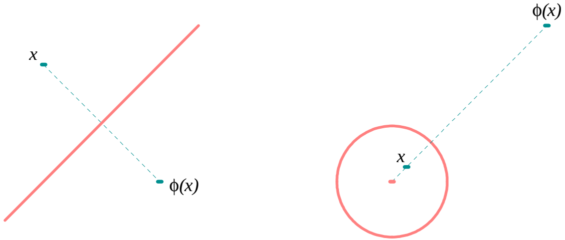


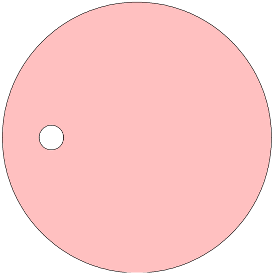
- Examples:
  - euclidean isometries, i.e. translations and linear isometries,
  - dilations: $x \mapsto \lambda x, \;\; \lambda > 0$.

The geometric structure

Multiplicity in domains with spherical boundaries

Example (C.-Pacella 2008)

If $\partial\Omega =$ union of two disjoint spheres, then problem $(\wp_{2^*})$ has infinitely many solutions.



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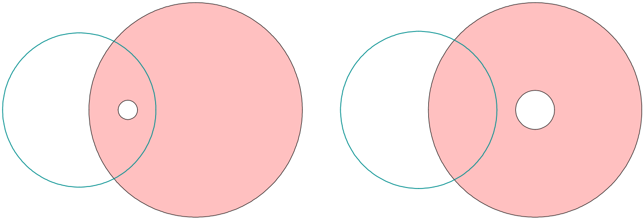
Proof.

There exists an inversion which maps Ω onto an annulus:



$\square$

The geometric structure

Positive entire solutions

- Consider the problem in $\mathbb{R}^N$

$$(\wp_{\mathbb{R}^N}) \quad \begin{cases} -\Delta u = |\, u \,|^{2^*-2} u & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

The geometric structure

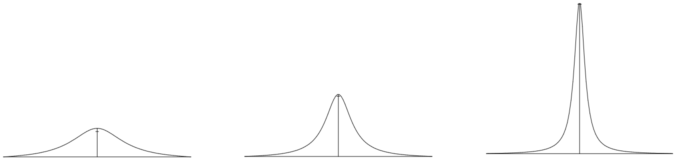
Standard bubbles

Theorem (Aubin, Talenti 1976, Gidas-Ni-Nirenberg 1979, Lions 1985)

The standard bubble

$$U(x) = a_N \left(\frac{1}{1 + |x|^2} \right)^{\frac{N-2}{2}}$$

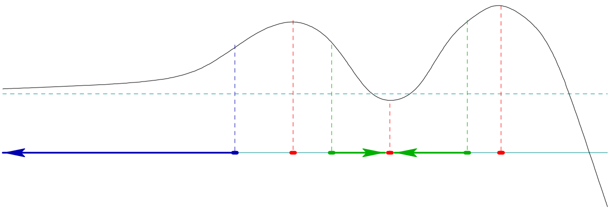
is the only positive solution to $(\wp_{\mathbb{R}^N})$, *up to translations and dilations.*



The geometric structure

causes blow-up in bounded domains

- As for



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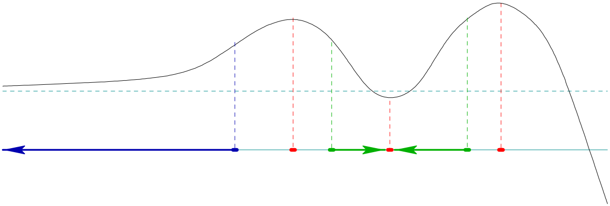

- if Ω is a bounded domain, there are trajectories $t \mapsto u_t$ in $H_0^1(\Omega)$, such that

$$J(u_t) \to c, \quad \nabla J(u_t) \to 0, \quad \text{as } t \to \infty,$$

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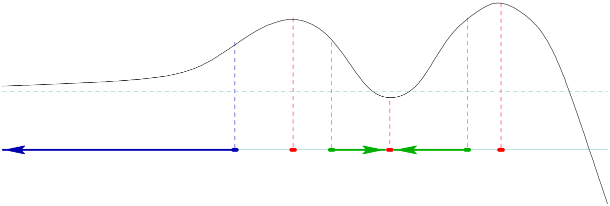

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$$J(u_t) \longrightarrow c, \qquad \nabla J(u_t) \longrightarrow 0, \qquad \text{as } t \longrightarrow \infty,$$

- but (u_t) does not converge to a critical point as $t \longrightarrow \infty$!

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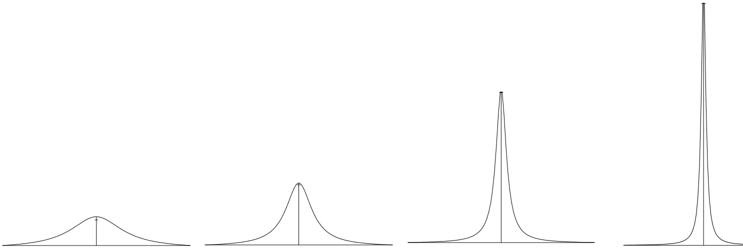

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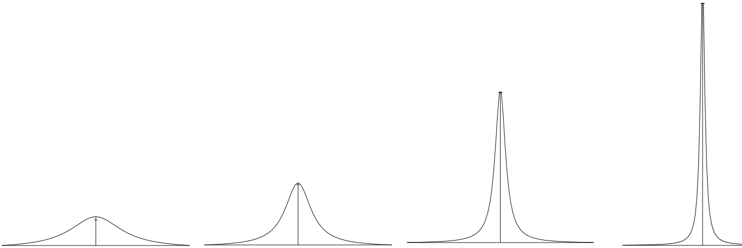

- **Struwe (1984)** showed that the lack of compactness is solely due to this phenomenon.

Multiple solutions

Our program

- We now go back to the problem

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Answers & methods

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have more than one solution for $\varepsilon > 0$ small enough?

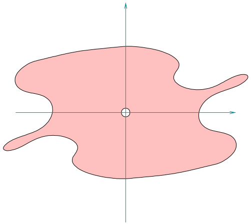
Multiple solutions

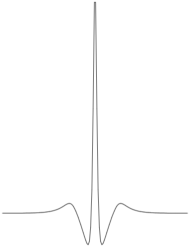
In punctured domains

Theorem (Ge-Musso-Pistoia 2010)

$$\#\ \textit{of solutions to}\ (\wp_{2^*,\varepsilon}) \underset{\varepsilon\to 0}{\longrightarrow} \infty.$$

- The solutions look like superpositions of standard bubbles with alternating signs (*bubble towers*):





Symmetries

The energy functional

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$$(\wp_{2^*}) \qquad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

- are the critical points of the functional

$$J(u) = \frac{1}{2} \|u\|^2_{H^1_0} - \frac{1}{2^*} \|u\|^{2^*}_{L^{2^*}}, \qquad u \in H^1_0(\Omega).$$

Symmetries

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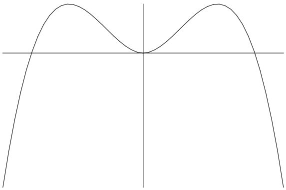
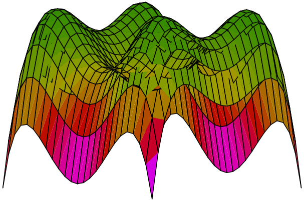
- J has the mountain pass geometry:





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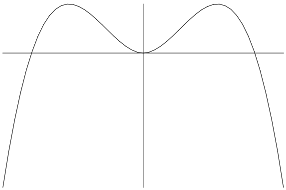

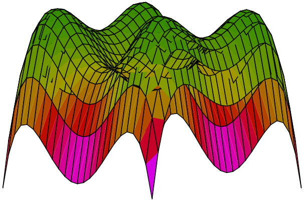

- but the first mountain pass is never attained!!!

Symmetries

and variational methods

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  - u is constant on each Gx.
- They are the critical points of the restriction of

$$J(u) = \frac{1}{2} \|u\|^2_{H^1_0} - \frac{1}{2^*} \|u\|^{2^*}_{L^{2^*}}$$

to the subspace

$$H^1_0(\Omega)^G := \{u \in H^1_0(\Omega) : u \text{ is } G\text{-invariant}\}.$$

Symmetries

produce compactness

Theorem (compactness)

If

$$c < \min_{x \in \overline{\Omega}} (\# Gx) c_\infty, \qquad c_\infty := \frac{1}{N} S^{N/2},$$

then J satisfies the Palais-Smale $(PS)_c^G$, i.e.

- *every sequence s.t.*

$$u_n \in H_0^1(\Omega)^G, \qquad J(u_n) \to c, \qquad \nabla J(u_n) \to 0,$$

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Corollary

If $\# Gx = \infty$ for every $x \in \overline{\Omega}$ then problem $(\wp_{2^*})$ has infinitely many solutions.

Symmetries

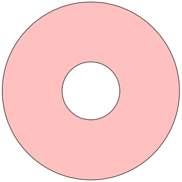
produce compactness

Example (Kazdan-Warner)

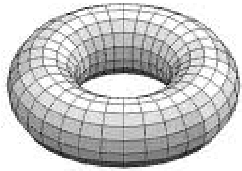
If $G = O(N)$ and $\Omega =$ annulus, then problem $(\wp_{2^*})$ has infinitely many radial solutions.

Example

If $G = SO(2)$ and $\Omega =$ torus, then $(\wp_{2^*})$ has infinitely many solutions which are invariant under rotations.



annulus



torus

Symmetries

The orbits must be infinite

Example (Pohozhaev)

If $G = O(N)$ and $\Omega =$ ball, problem $(\wp_{2^*})$ does not have a nontrivial solution!!!

Domains with a thin hole

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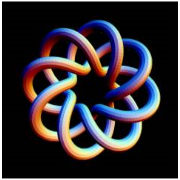

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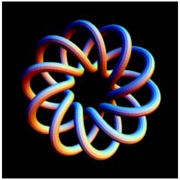

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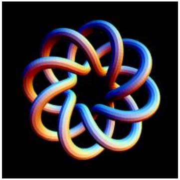

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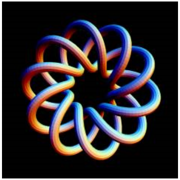

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- Then $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, M) > \varepsilon\}$ is G_n-invariant, but $\#G_n x = n < \infty$.

Domains with a thin hole

A trick to produce sign changing solutions

- Let $\tau : G \to \mathbb{Z}/2 := \{1, -1\}$ be a group homomorphism.

Domains with a thin hole

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  - Therefore, if τ is surjective, u changes sign.

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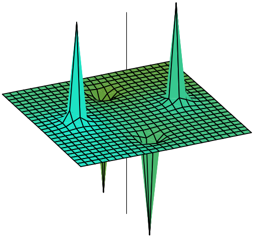
- A function satisfying $u(gx) = \tau(g)u(x)$ for G_4 is:



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Domains with a thin hole

Existence

Theorem (C.-Grossi-Pistoia ∼2010)

For each $\varepsilon > 0$ *suficiently small problem* $(\wp_{2^*,\varepsilon})$ *has at least one nontrivial solution* u *which satisfies*

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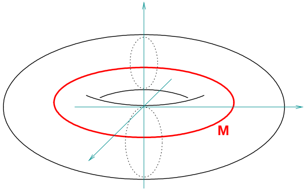
Domains with a thin hole

Multiplicity in highly symmetric domains

Example

$\Omega \subset \mathbb{R}^3$ a solid of revolution about the z-axis, $M := \mathbb{S}^1 \times \{0\}$ such that

$$M \subset \Omega \quad \text{and} \quad \Omega \cap (z\text{-axis}) = \emptyset.$$



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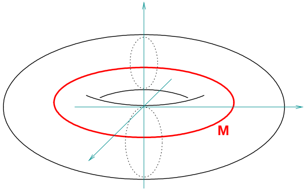

- Ω_ε is invariant under rotations about the z-axis, hence

Domains with a thin hole

Multiplicity in highly symmetric domains

Example

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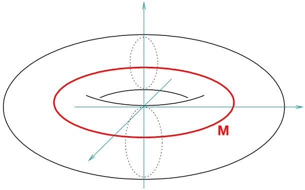

- Ω_ε is invariant under rotations about the z-axis, hence
- $(\wp_\varepsilon)$ has infinitely many rotationaly invariant solutions.

Domains with a thin hole

Multiplicity in highly symmetric domains

Corollary

Let $m \in \mathbb{N}$, Ω *and* M *as above Then, for* ε *small enough,* $(\wp_\varepsilon)$ *has* m *pairs of solutions* $\pm u_1, \ldots, \pm u_m$ *such that*

$$u_n(\varrho_{2^n}^k x) = (-1)^k\, u_n(x), \qquad k = 0, 1, \ldots, 2^n - 1.$$

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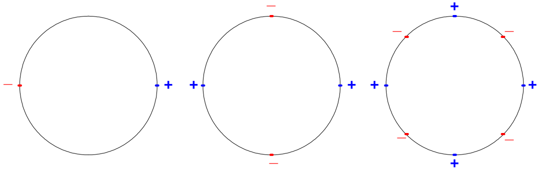
Proof.

We apply the previous theorem to the group G_{2^n} :



G_2 $\quad$ G_4 $\quad$ G_8

Domains with a thin hole

Open problems

Problem

In general domains of the form (without symmetries)

$$\Omega_\varepsilon := \{x \in \Omega : dist(x, M) > \varepsilon\},$$

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- *Are there bubble towers?*
- *Are there multibump solutions?*
- *Are there solutions with layers concentrating along* M *as* $\varepsilon \longrightarrow 0$*?*

Thanks

Thank you very much for your attention !