Elliptic boundary value problems with critical and supercritical nonlinearities. Part 2.

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Universidad Nacional Autónoma de México

Flagstaff, June 2012

Introduction

The problem

- We consider the problem

$$(\wp_p) \quad \begin{cases} -\Delta u = |u|^{p-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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- $p > 2^*$ is supercritical.

Coauthors

Jorge Faya (Universidad Nacional Autónoma de México)

Angela Pistoia (Università di Roma "La Sapienza")

The critical case

The Bahri-Coron theorem

- **Bahri-Coron, 1988**: If $\tilde{H}_*(\Omega;\mathbb{Z}/2) \neq 0$, then

$$(\wp_{2^*,\Omega}) \quad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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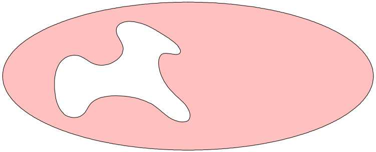

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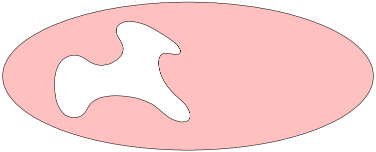


Problem

Are there multiple solutions in general domains (which are not small perturbations of a given one)?

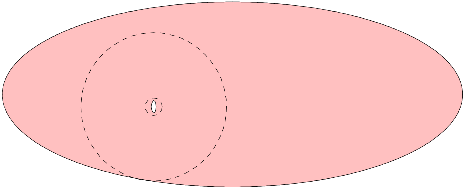
The critical case

Another look at Coron's theorem

- **Coron, 1984**: If Ω is annular-shaped, i.e.

$$\Omega \supset \{x : 0 < a \leq |x| \leq b\}, \qquad 0 \notin \Omega,$$

and $\frac{b}{a}$ *is large enough*, then $(\wp_{2^*,\Omega})$ has a positive solution.



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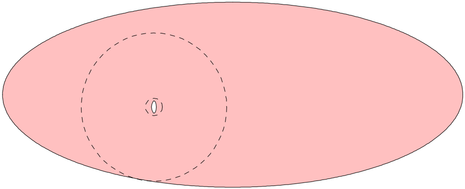

- The solution, as well as those of Ge-Musso-Pistoia, look like radial solutions in the annulus.

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Highly symmetric domains

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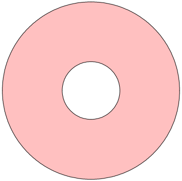

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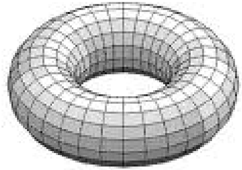

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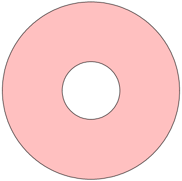

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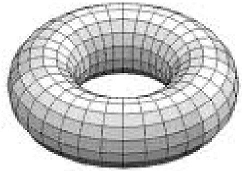

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- **QUESTION:** Is it true that, if Ω contains a domain of this type, problem $(\wp_{2^*,\Omega})$ has multiple solutions?

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Multiplicity in domains with finite symmetries

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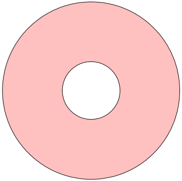

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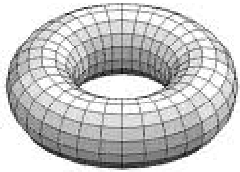

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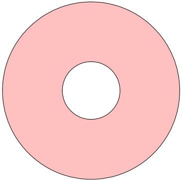

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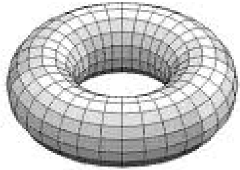

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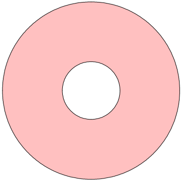

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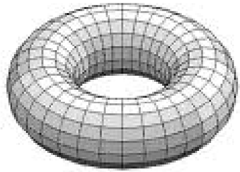

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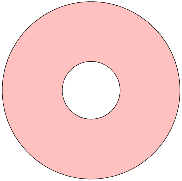

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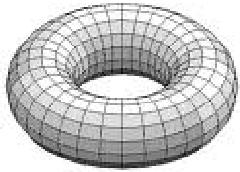

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- We consider domains Ω such that
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  - Ω is G-invariant under some closed subgroup G of Γ.

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Multiplicity in domains with nontrivial topology

Theorem (C.-Faya, preprint 2012)

There exists (ℓ_m) *nondecreasing, depending only on* Γ *and* D*, s.t.: If* $\Omega \supset D$*,* Ω *is* G*-invariant under a closed subgroup* $G \subset \Gamma$ *and*

$$\min_{x \in \Omega} \# Gx > \ell_m,$$

- *then* $(\wp_{2^*,\Omega})$ *has at least* m *pairs* $\pm u_1, \ldots, \pm u_m$ *of* G*-invariant solutions such that*

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- $$\int_\Omega |\nabla u_k|^2 \leq \ell_k S^{N/2} \qquad \text{for each } k = 1, \ldots, m,$$
- *u_1 is positive and $u_2, \ldots, u_m$ change sign.*

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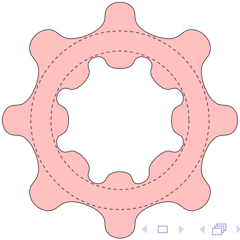
  - Hence, if $n > \ell_m$, problem $(\wp_{2^*,\Omega})$ has m pairs of solutions.



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    if $G \neq O(3), SO(3)$ and $\Omega \supset D$.
    - But the numbers ℓ_m become larger as the annulus becomes thinner.

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Multiplicity in toroidal domains

Example

If $\Gamma = SO(2)$ and $D =$ torus in $\mathbb{R}^3$ then, for each m, there are domains Ω in which $(\wp_{2^*,\Omega})$ has m pairs of solutions:

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- Similarly, in other dimensions.

The proof

Recall the statement

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- We look for critical points of the energy functional J on

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Let W be a linear subspace of $H_0^1(\Omega)^G$. If J satisfies $(PS)_c^G$ for all $c \leq \sup_W J$, then J has at least $\dim(W)$ pairs of critical points $\pm u$ in $H_0^1(\Omega)^G$.

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- for ε small enough, so that $\ell_m c_\infty + \varepsilon < (\min_{x \in \Omega} \#Gx)\, c_\infty$.

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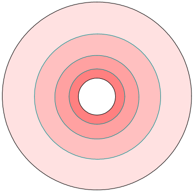

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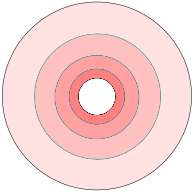

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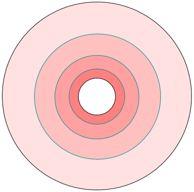

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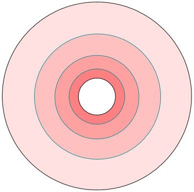

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  - $\ell_m := c_\infty^{-1} c_m$.

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  - Since ω_{D_i} and ω_{D_j} have a.e. disjoint supports for $i \neq j$,

$$\dim W_m = m,$$

and, since ω_{D_i} lies on the Nehari manifold,

$$\sup_{W_m} J \leq \sum_{i=1}^{m} J(\omega_{D_i}) \leq \ell_m c_\infty + \varepsilon. \qquad \square$$

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Conclusions

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Problem (open)

Does $(\wp_{2^*,\Omega})$ *have multiple solutions in every domain with nontrivial topology?*

The supercritical problem

- Next we look at the supercritical problem

$$(\wp_{p,\Omega}) \quad \begin{cases} -\Delta u = |u|^{p-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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- $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 3$,
- $p > 2^*$ is supercritical.

The supercritical problem

Rabinowitz's question

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Problem (Rabinowitz)

Is it true that, If $\tilde{H}_*(\Omega; \mathbb{Z}/2) \neq 0$, then $(\wp_{p,\Omega})$ has a nontrivial solution?

The supercritical problem

Passaseo's answer

Theorem (Passaseo 1995)

For each $1 \leq k \leq N-3$ *there exists* Ω *such that*

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- $$2^*_{N,k} := \frac{2(N-k)}{N-k-2} = (k+1)\text{-st critical exponent.}$$

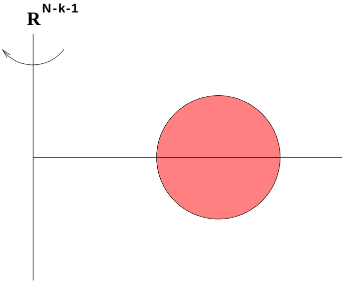
The supercritical problem

Passaseo's example

Passaseo's domains are

$$\Omega := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in B\}$$

where B is a ball contained in $(0, \infty) \times \mathbb{R}^{N-k-1}$ with center in $(0, \infty) \times \{0\}$.



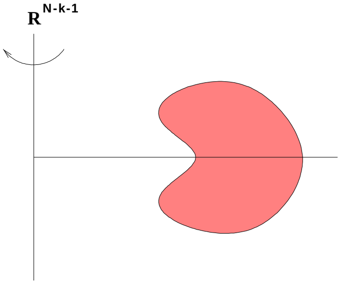
The supercritical problem

Existence at higher critical exponents

- **Wei-Yan 2011**: Constructed infinitely many positive solutions for $p = 2^*_{N,k}$, $N \geq 5$, in domains Ω of the form

$$\Omega := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\},$$

where $\overline{\Theta} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$ has a particular shape:



The supercritical problem

A geometric nonexistence condition

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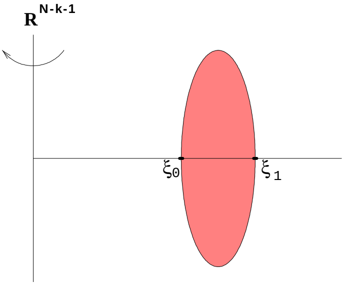

The supercritical problem

A nonexistence result

Theorem (C.-Faya-Pistoia, preprint 2012)

If $\Theta \subset (0,\infty) \times \mathbb{R}^{N-k-1}$ *is doubly starshaped,* $0 \leq k \leq N-3$ *and*

$$\Omega := \{(y,z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\},$$

then problem $(\wp_{p,\Omega})$ *does not have a nontrivial solution for* $p \geq 2^*_{N,k}$ *and has infinitely many solutions for* $p \in (2, 2^*_{N,k})$.

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Given $k = k_1 + \cdots + k_m \leq N-3$ and $\varepsilon > 0$ there exists a domain $\Omega \simeq \mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_m}$, in which problem $(\wp_{p,\Omega})$ does not have a nontrivial solution for $p \geq 2^*_{N,k} + \varepsilon$ and has infinitely many solutions for $p \in (2, 2^*_{N,k})$.

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  - i.e. there are k cohomology classes in $H^1(\Omega;\mathbb{Z})$ whose cup-product is not zero.

The supercritical problem

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- Our domains are of the form

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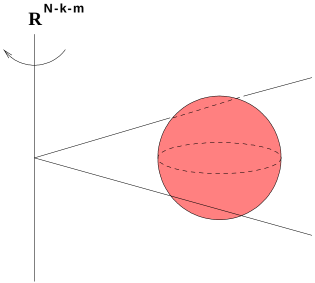

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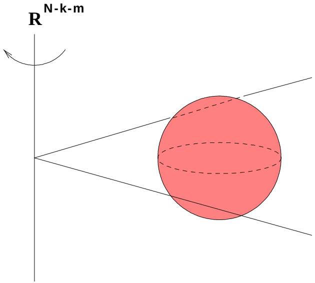

- whose radius becomes smaller as $\varepsilon \to 0$.

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  - $+$ other convenient properties.

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Problem (open)

If $\Theta \subset (0,\infty) \times \mathbb{R}^{N-k-1}$ *is starshaped w.r. to the point in* $(0,\infty) \times \{0\} \cap \overline{\Theta}$ *which is closest to the origin, is it true that problem* $(\wp_{p,\Omega})$ *does not have a nontrivial solution in*

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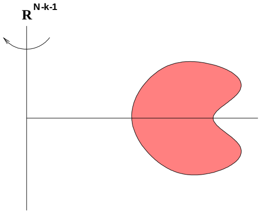

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Existence at higher critical exponents

Some recent perturbative existence results:

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- **Wei-Yan, 2011:** Positive multipeak solutions for $p = 2^*_{N,k}$.

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Existence via the Hopf fibrations

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- The *Hopf map* $\pi : \mathbb{R}^N = \mathbb{K} \times \mathbb{K} \to \mathbb{K} \times \mathbb{R} = \mathbb{R}^{(N/2)+1}$,

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- Is there a simple relationship between

$$\Delta_M u \qquad \text{and} \qquad \Delta_N v?$$

The supercritical problem

Harmonic morphisms

- A smooth map $\pi : M \to N$ is a *harmonic morphism* with *dilation* $\lambda : M \to [0, \infty)$ if
$$\Delta_M (v \circ \pi)(x) = \lambda^2(x) \left[(\Delta_N v)(\pi(x)) \right]$$
for each function $v : U \to \mathbb{R}$ defined on an open subset U of N s.t. $\pi^{-1}(U) \neq \emptyset$.

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Example

A Riemannian submersion $\pi : M \to N$ s.t. the mean curvature of each fiber $\pi^{-1}(y)$ in M is zero is a harmonic morphism.

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The Hopf maps $\pi : \mathbb{R}^N = \mathbb{K} \times \mathbb{K} \to \mathbb{K} \times \mathbb{R} = \mathbb{R}^{(N/2)+1}$ are harmonic morphisms with dilation

$$\lambda(x, y) = \sqrt{2(|x|^2 + |y|^2)}.$$

The supercritical problem

A comparison result

- **Proposition.** Let $N = 4, 8, 16$, U be a domain in $\mathbb{R}^{(N/2)+1}$ s.t. $0 \notin U$. If v solves

$$(\wp^*_{p,\Omega}) \quad \begin{cases} -\Delta v = \frac{1}{2|x|} \left|v\right|^{p-2} v & \text{in } U, \\ v = 0 & \text{on } U, \end{cases}$$

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- then $u := v \circ \pi$ is a solution of

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where $\pi : \mathbb{R}^N \to \mathbb{R}^{(N/2)+1}$ is the Hopf map.

The supercritical problem

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- Let $N = 4, 8, 16$.
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Theorem (C.-Faya-Pistoia, preprint 2012)

There exists (ℓ_m) nondecreasing, depending only on Γ and D, s.t.: If $U \supset D$ is invariant under a subgroup G of Γ and

$$\min_{x \in U} (\# Gx) \, |x|^{(N/2)-1} > \ell_m,$$

then, for $p = 2^*_{N,(N/2)-1}$, problem $(\wp_{p,\Omega})$ has m pairs of solutions $\pm u_1, \ldots, \pm u_m$ in $\Omega := \pi^{-1}(U)$, u_1 is positive and $u_2, \ldots, u_m$ change sign.

The supercritical problem

An example

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Fix $D :=$ torus in $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, $\quad \Gamma := SO(2)$.

- $U :=$ "punk" torus in $\mathbb{R}^3$:

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- $U :=$ "punk" torus in $\mathbb{R}^3$:

- Then $\Omega \cong U \times \mathbb{S}^1 \subset \mathbb{R}^4$,

The supercritical problem

An example

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- $U :=$ "punk" torus in $\mathbb{R}^3$:

- Then $\Omega \cong U \times \mathbb{S}^1 \subset \mathbb{R}^4$,
- and we obtain m pairs of solutions to problem $(\wp_{p,\Omega})$ in Ω for $p = 6$.

Thanks

Thank you very much for your attention!