On Functions Satisfying a Mean/Median Value Property

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The continuous function $u$ is harmonic in $\Omega \subset \mathbb{R}^N$ if and only if

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for all $x_0 \in \Omega$ and all $r > 0$ such that $B_r(x_0) \subseteq \Omega$. 

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**Question:** Can solutions of other PDEs be characterized in statistical way?

Continuous functions of two variables satisfying local median value property are 1-harmonic (Rudd-Van Dyke).

$p$-harmonic functions ($p > 1$) have asymptotic statistical characterizations (Manfredi, Parviainen, Rossi, Proc AMS, 2010).

$p$-harmonic functions of two variables have additional asymptotic statistical characterizations, including a mean/median value property (H-Rudd, RMJ, 2011).

$p$-harmonic functions are approximated by functions satisfying statistical functional equations ($p$-harmonious functions) (M-P-R, H-Rudd).
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\textbf{Definition} A function $u$ is $p$-harmonic, for $p \geq 1$, if
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Solutions understood in weak (Sobolev) sense or viscosity sense.

For $p > 1$, weak solutions and viscosity solutions are equivalent (Juutinen, Lindqvist, Manfredi, 2001). $u$ is $p$-harmonic if and only if $u$ is a viscosity solution of

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Theorem Suppose $1 < p < \infty$ and $\Omega \subset \mathbb{R}^2$ is open. $u \in C(\Omega)$ is $p$-harmonic if and only if at each $x \in \Omega$,

$$u(x) = \left(\frac{2}{p} - 1\right) \text{median} u + \left(2 - \frac{2}{p}\right) \int_{\partial B_\varepsilon(x)} u(s) \, ds + o(\varepsilon^2)$$

holds in the viscosity sense as $\varepsilon \to 0$. 

Question: What happens if we fix $\varepsilon$ and drop error term?
Asymptotic Characterization of $p$-harmonic Functions

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Viscosity Solution of Asymptotic Formula

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u \text{ is a viscosity supersolution of }
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\varphi(x) \geq \left( \frac{2}{p} - 1 \right) \text{median}_{\partial B_\varepsilon(x)} \varphi + \left( 2 - \frac{2}{p} \right) \int_{\partial B_\varepsilon(x)} \varphi(s) \, ds + o(\varepsilon^2)
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as \( \varepsilon \to 0 \) holds at \( x \in \Omega \) for any smooth \( \varphi \) for which \( |D\varphi(x)| \neq 0 \) and \( u - \varphi \) has a strict minimum at \( x \).
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**Viscosity subsolutions** defined analogously. Formula holds in **viscosity sense** for $u$ if $u$ is both a viscosity supersolution and subsolution.
Let \( q \in (0, 1], h > 0, \Omega \subset \mathbb{R}^N \) bounded and open, and \( g \in C(\partial \Omega) \).

Find \( u \in C(\overline{\Omega}) \) such that

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\begin{cases}
  u(x) = q \text{mean}_{\partial B_x} u + (1 - q) \text{median}_{\partial B_x} u, & x \in \Omega \\
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When a solution $u_h$ exists for all $h > 0$ sufficiently small, $u_h$ converge to solution of $\Delta_p u = 0$, where $p = 2/(2 - q)$, when $N = 2$. 

David Hartenstine  Mean/Median Value Property
For integrable $u : E \rightarrow \mathbb{R}$, a number $m$ is a median for $u$ over $E$, where $0 < |E| < \infty$ if

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Definition of Median

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$\text{median}_E \{u\} =$ set of all medians of $u$ over $E$. 

Also known as the Lévy mean.

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**Lemma** Suppose that $u \in LSC(E)$, $U \in USC(E)$, $E$ is compact and connected and that $u \geq U$. If $m \in \text{median}_E u$ and $M \in \text{median}_E U$, then $m \geq M$. 
Suppose $\Omega \subset \mathbb{R}^n$ is bounded domain. Let $h > 0$. 

Let $r_h : \Omega \to [0, \sqrt{2}h]$ be defined by:

$$r_h(x) = \begin{cases} \sqrt{2}h & \text{if } \text{dist}(x, \partial \Omega) \geq \sqrt{2}h, \\ \text{dist}(x, \partial \Omega) & \text{otherwise}. \end{cases}$$

Let $B_x$ denote the ball $B(x, r_h(x))$.

Define the median operator $\text{median}_h : C(\Omega) \to C(\Omega)$ by:

$$\text{median}_h(u)(x) = \begin{cases} \text{median}_{\partial B_x}u & \text{if } x \in \Omega, \\ u(x) & \text{if } x \in \partial \Omega. \end{cases}$$

This follows from uniform continuity of $u$, continuity of $r_h$, and monotonicity of the median operator.
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Follows from uniform continuity of $u$, continuity of $r_h$ and monotonicity of median.
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and let $B_x$ denote the ball $B(x, r_h(x))$. Define $\text{median}_h : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

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Follows from uniform continuity of $u$, continuity of $r_h$ and monotonicity of median.
Convergence Properties of Medians

1. Suppose that \( \{u_n\} \subset C(\overline{\Omega}) \) is a nondecreasing sequence that converges pointwise to \( u \in LSC(\overline{\Omega}) \cap L^\infty(\overline{\Omega}) \). At each \( x \in \Omega \),

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\text{median } u_n \rightarrow \min_{\partial B_x} \left( \text{median } u \right).
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2. Suppose that \( \{u_n\} \subset C(\overline{\Omega}) \) is a nonincreasing sequence that converges pointwise to \( u \in USC(\overline{\Omega}) \cap L^\infty(\overline{\Omega}) \). At each \( x \in \Omega \),

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Convergence Properties of Medians

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Proof follows from Lemma, basic properties of medians.
Operators $M_{q,h}$ and Dirichlet Problem

Let $q \in (0,1]$ and $h > 0$. For $u : \Omega \rightarrow \mathbb{R}$, define

$$(M_{q,h}u)(x) := \{ q \text{mean} \partial B_x u + (1 - q) \text{median} \partial B_x u, x \in \Omega \}$$

$u(x) \quad x \in \partial \Omega$

Given $g \in C(\partial \Omega)$, find $u \in C(\Omega)$ such that

$$M_{q,h}u(x) = u(x), x \in \Omega$$

$$u = g, \text{on } \partial \Omega$$

$q = 1$: solution given by harmonic function with boundary values $g$. 

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$q = 1$: solution given by harmonic function with boundary values $g$. 
Sub and Supersolutions, Max/Min and Comparison Principles

$v \in C(\bar{\Omega})$ is a **subsolution** to (1) if $v = g$ on $\partial\Omega$ and $M_{q,h}v \geq v$ in $\Omega$.

Similarly, $w \in C(\bar{\Omega})$ is a **supersolution** to (1) if $w = g$ on $\partial\Omega$ and $M_{q,h}w \leq w$ in $\Omega$.
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**Comparison Principle** Suppose that \( v, w \in C(\bar{\Omega}) \) are, respectively, a sub and supersolution to problem (1). Then \( v \leq w \).
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Immediate consequence is \( C(\bar{\Omega}) \) solutions to (1) are unique.
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**Comparison Principle** Suppose that $v, w \in C(\bar{\Omega})$ are, respectively, a sub and supersolution to problem (1). Then $v \leq w$.

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**Max/Min Principle** If $v$ is a subsolution, then $\max_{\partial \Omega} v = \max_{\bar{\Omega}} v$. If $w$ is a supersolution, $\min_{\partial \Omega} w = \min_{\bar{\Omega}} w$. 
Theorem Suppose \( v, w \in C(\overline{\Omega}) \) are respectively a sub- and supersolution of (1). Then there exists \( u \in C(\overline{\Omega}) \) that is a solution of (1).

Main Ideas of Proof

▶ Iteratively apply \( M_q, h \) to \( v \) to produce a non-decreasing sequence of subsolutions \( u_n \), bounded above by \( w \).

▶ This sequence converges pointwise to \( u \in LSC(\Omega) \).

▶ Similarly, produce \( U \in USC(\Omega) \) as limit of iterates of \( w \).

▶ \( u \leq U \) by Comparison Principle.

▶ \( u \) and \( U \) are continuous at \( \partial \Omega \), by continuity of \( v, w \).

▶ Want to show \( u = U \).
**Theorem** Suppose \( v, w \in C(\overline{\Omega}) \) are respectively a sub- and supersolution of (1). Then there exists \( u \in C(\overline{\Omega}) \) that is a solution of (1).

**Main Ideas of Proof**

- Iteratively apply \( M_{q,h} \) to \( v \) to produce an non-decreasing sequence of subsolutions \( u_n \), bounded above by \( w \).
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Mean/Median Value Property
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- Similarly, produce $U \in USC(\overline{\Omega})$ as limit of iterates of $w$. 

David Hartenstine  
Mean/Median Value Property
Sub/Supersolution Theorem

**Theorem** Suppose $v, w \in C(\overline{\Omega})$ are respectively a sub- and supersolution of (1). Then there exists $u \in C(\overline{\Omega})$ that is a solution of (1).

Main Ideas of Proof

- Iteratively apply $M_{q,h}$ to $v$ to produce an non-decreasing sequence of subsolutions $u_n$, bounded above by $w$.
- This sequence converges pointwise to $u \in LSC(\overline{\Omega})$.
- Similarly, produce $U \in USC(\overline{\Omega})$ as limit of iterates of $w$.
- $u \leq U$ by Comparison Principle.
**Theorem** Suppose \( v, w \in C(\overline{\Omega}) \) are respectively a sub- and supersolution of (1). Then there exists \( u \in C(\overline{\Omega}) \) that is a solution of (1).

**Main Ideas of Proof**

- ▶ Iteratively apply \( M_{q,h} \) to \( v \) to produce an non-decreasing sequence of subsolutions \( u_n \), bounded above by \( w \).
- ▶ This sequence converges pointwise to \( u \in LSC(\overline{\Omega}) \)
- ▶ Similarly, produce \( U \in USC(\overline{\Omega}) \) as limit of iterates of \( w \).
- ▶ \( u \leq U \) by Comparison Principle
- ▶ \( u \) and \( U \) are continuous at \( \partial \Omega \), by continuity of \( v, w \).
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- Iteratively apply \( M_{q,h} \) to \( v \) to produce a non-decreasing sequence of subsolutions \( u_n \), bounded above by \( w \).
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Proof of Sub/supersolution Theorem

By Convergence Lemma

\[ u(x) = \lim_{n \to \infty} M_{q,h} u_n(x) = q \text{ mean } \{u\} + (1 - q) \min_{\partial B_x} \left( \text{median } u \right) \]

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Suppose \( U - u \) has max at \( x^* \). Then

\[ u(x) - u(x^*) \geq U(x) - U(x^*) \]

and

\[ 0 = q \text{ mean } \{u - u(x^*)\} + (1 - q) \min_{\partial B_x} \left( \text{median } u - u(x^*) \right) \]

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If \( u - u(x^*) > U - U(x^*) \) on set of pos measure in \( \partial B_{x^*} \), get contradiction to Lemma.

So \( u - u(x^*) = U - U(x^*) \) ae in \( \partial B_{x^*} \).

Semicontinuity implies \( u - u(x^*) = U - U(x^*) \).

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Semicontinuity implies \( u - u(x^*) = U - U(x^*) \). Conclude \( u = U \).
Theorem

Let $\Omega \subset \mathbb{R}^N$ be strictly convex and $g \in C(\partial \Omega)$. Then problem (1) has a $C(\Omega)$ solution.

Proof: Sufficient to find a subsolution and supersolution. Let $v \in C(\Omega)$ be the convex solution of
\[
\begin{cases}
det D^2 u = 1 &\text{in } \Omega \\
u = g &\text{on } \partial \Omega
\end{cases}
\]
Because $v$ is convex, $v(x) \leq \text{median } \partial B_x v$. $v$ is also subharmonic, so $v(x) \leq \text{mean } \partial B_x v$. Thus, $v$ is a subsolution.

If $W$ is the convex $C(\Omega)$ solution of
\[
\begin{cases}
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Theorem Let $\Omega \subset \mathbb{R}^N$ be strictly convex and $g \in C(\partial \Omega)$. Then problem (1) has a $C(\overline{\Omega})$ solution.
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Theorem

Suppose that for all $h > 0$ sufficiently small, problem (1) has a $C(\Omega)$ solution $u_h$, where $\Omega \subset \mathbb{R}^2$. As $h \to 0$, $u_h$ converges locally uniformly to the $p$-harmonic function with boundary values $g$, where $p = \frac{2}{2 - q}$.

Solutions of (1) approximate $p$-harmonic functions in two dimensions when $1 < p \leq 2$.

Following M-P-R, call solution $u_h$ $p$-harmonious.

Proof via Barles-Souganidis framework.

David Hartenstine
Mean/Median Value Property
Connections with $p$-harmonic functions

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Application of Barles & Souganidis Approach

In order to prove theorem, need that $M_q, h$ satisfies:

1. $M_q, 0 v = v$ for any $v \in C(\Omega)$,
2. $M_q, h(v + c) = M_q, h v + c$ for any $v \in C(\Omega)$ and constant $c$,
3. $M_q, h v \leq M_q, h w$ whenever $v, w \in C(\Omega)$ satisfy $v \leq w$, and
4. for any smooth function $\phi$ and any $x \in \mathbb{R}^2$ where $D_x \phi(x)$,
   
   \[ \lim_{h \to 0} \left( \phi(x) - (M_q, h \phi)(x) h \right) = A \phi(x) \]
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4. for any smooth function $\phi$ and any $x \in \mathbb{R}^2$ where $D\phi(x), 0$, $\lim_{h \to 0} (\phi(x) - (M_{q,h}\phi)(x))h = A\phi(x)$.
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David Hartenstine
Mean/Median Value Property
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4. for any smooth function $\varphi$ and any $x \in \mathbb{R}^2$ where $D\varphi(x) \neq 0$, 
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In order to prove theorem, need that $M_{q,h}$ satisfies:

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M-P-R (2010): $u$ is $p$-harmonic if and only if

$$u(x) = \frac{1 - q}{2} \left\{ \max_{B(x,\varepsilon)} u + \min_{B(x,\varepsilon)} u \right\} + q \int_{B(x,\varepsilon)} u(y) \, dy + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0$$

holds in the viscosity sense as $\varepsilon \to 0$, and $q = (2 + N)/(p + N)$. 
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M-P-R: For $\varepsilon > 0$, let $\Gamma_\varepsilon = \{ x \in \mathbb{R}^N \setminus \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon \}$.
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M-P-R: For $\varepsilon > 0$, let $\Gamma_\varepsilon = \{ x \in \mathbb{R}^N \setminus \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon \}$

Given $F : \Gamma_\varepsilon \to \mathbb{R}$, there exists unique $u_\varepsilon$ with $u = F$ in $\Gamma_\varepsilon$ and

$$u_\varepsilon(x) = \frac{1 - q}{2} \left\{ \sup_{B(x, \varepsilon)} \{u_\varepsilon\} + \inf_{B(x, \varepsilon)} \{u_\varepsilon\} \right\} + q \int_{B(x, \varepsilon)} u_\varepsilon(y) \, dy$$
If $p \geq 2$ and $\partial \Omega$ is sufficiently regular (exterior cone condition), then solutions $u_\varepsilon$ converge uniformly to unique solution of $\Delta_p u = 0$ in $\Omega$ with boundary values $F$. 

Call solutions $u_\varepsilon$ $p$-harmonious. $u$ is $p$-harmonic if and only if $u$ is visc solution of 

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where \(\Delta_\infty u = \frac{1}{|Du|^2} \sum_{i,j=1}^{N} \partial_x u \partial_x x_i \partial_x^2 u \partial_x x_j \).
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Method uses connection of functional equation and probabilistic tug-of-war games.
Our methods lead to

**Theorem** If there exist a subsolution $v_h$ and supersolution $w_h$ to following problem, there exists a $C(\overline{\Omega})$ solution.

\[
\begin{cases}
  u(x) = q \text{mean}_{B_x} \{u\} + \frac{1-q}{2} \left\{ \max_{B_x} u_h + \min_{B_x} u_h \right\} & x \in \Omega \\
  u = g, & \text{on } \partial \Omega
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(2)
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(2)

**Theorem** Let $p \geq 2$, and let $q = (2 + N)/(p + N)$. Suppose that for all $h$ sufficiently small, (2) has a solution $u_h \in C(\overline{\Omega})$. As $h \to 0$, $u_h$ converges locally uniformly to $p$-harmonic function with boundary values $g$. 

Let $\Omega \subset \mathbb{R}^N$ be open and $x \in \Omega$. If $\varphi \in C^2(\Omega)$ and $|D\varphi(x)| \neq 0$, then:
Approximation Formulas

Let $\Omega \subset \mathbb{R}^N$ be open and $x \in \Omega$. If $\varphi \in C^2(\Omega)$ and $|D\varphi(x)| \neq 0$, then:

$$\varphi(x) - \left. \text{median}_{\partial B_\varepsilon(x)} \varphi \right| = -\frac{\varepsilon^2}{2} \Delta_1 \varphi(x) + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0. \quad N = 2$$
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$$\varphi(x) - \int_{\partial B_\varepsilon(x)} \varphi(s) \, ds = -\frac{\varepsilon^2}{4} \Delta \varphi(x) + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$
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$$\varphi(x) - \int_{\partial B_\varepsilon(x)} \varphi(s) \, ds = -\frac{\varepsilon^2}{4} \Delta \varphi(x) + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$

$$\varphi(x) - \frac{1}{2} \left\{ \max_{B_\varepsilon(x)} \varphi + \min_{B_\varepsilon(x)} \varphi \right\} = -\frac{\varepsilon^2}{2} \Delta_\infty \varphi(x) + o(\varepsilon^2)$$
What Happens When $q = 0$?

Proof of Comparison Principle fails.
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Problem related to local median value property (Rudd-Van Dyke)
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In general, have nonexistence or nonuniqueness of solutions for Dirichlet problem for LMVP
Open Questions and Future Work

1. Can two-dimensional results be extended to higher dimensions?
2. Do solutions of functional equation with mean and median on $B$ approximate $p$-harmonic functions?
3. More general results concerning the existence of sub and supersolutions.
4. Explore $q = 0$ problem and connections with LMVP and GMVP.
5. Use of results here to establish properties of $p$-harmonic functions.
6. Statistical functional equations make sense in metric measure space context. Approach here could possibly be used to define $p$-harmonic functions in this setting. Comparison to previous work in this area.
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Thanks!