Multiple sign changing solutions of nonlinear elliptic problems in exterior domains

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Coauthors

Joint work with

Mónica Clapp
We consider the problem

\[
(\mathcal{P}) \quad \begin{cases} 
-\Delta u + (V_\infty + V(x))u = |u|^{p-2}u, \\
u \in H^1_0(\Omega),
\end{cases}
\]

where

- \( \Omega \subset \mathbb{R}^N \ (N \geq 3) \) is an unbounded smooth domain whose complement is bounded.
- \( 2 < p < 2^* := \frac{2N}{N-2} \).
- The potential \( V_\infty + V \) satisfies

\[
(V_0) \quad V \in C^0(\mathbb{R}^N); \ V_\infty \in (0, \infty), \ \inf_{\mathbb{R}^N} \{V_\infty + V(x)\} > 0; \\
\lim_{|x| \to \infty} V(x) = 0.
\]

Description of the lack of compactness. Benci-Cerami (1987)
We consider the problem

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  \[(V_0) \quad V \in C^0(\mathbb{R}^N); \ V_{\infty} \in (0, \infty), \ \inf_{\mathbb{R}^N} \{V_{\infty} + V(x)\} > 0; \ \lim_{|x| \to \infty} V(x) = 0.\]

Description of the lack of compactness. Benci-Cerami (1987)
## POSITIVE SOLUTIONS

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<td><img src="image1" alt="Graph" /></td>
<td>$\Omega = \mathbb{R}^N$</td>
<td>A positive solution. Lions (1984)</td>
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## MULTIPLE SOLUTIONS

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<td>$V$ radial, $V(r) \to 0$ in a polynomial way</td>
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## Background

### LOW ENERGY SOLUTIONS

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<td>$\Omega = \mathbb{R}^N$</td>
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Multiple sign changing solutions
## LOW ENERGY SOLUTIONS

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- **Clapp-Weth (2004)**
- **Cerami-Clapp (2007)**
- **Carvalho-Maia-Miyagaki (2011)**
Our goal

The topology of the orbit space of certain subsets of the domain

\[ \Downarrow \]

The number of low energy sign changing solutions to \((\mathcal{O})\)
We consider

- A closed subgroup $\Gamma$ of $O(N)$.
- A continuous homomorphism $\phi : \Gamma \to \mathbb{Z}/2 := \{1, -1\}$.

We denote by

- $G := \ker \phi$.
- $\ell := \min\{\#Gx : x \in \mathbb{S}^{N-1}\}$, where $Gx := \{gx : g \in G\}$.

Recall

- $X \subset \mathbb{R}^N$ is $\Gamma$-invariant if $\Gamma x \subset X$ for every $x \in X$.
- $u : X \to \mathbb{R}$ is $\Gamma$-invariant if it is constant on each $\Gamma$-orbit $\Gamma x$ with $x \in X$. 
We choose \( \gamma \in \Gamma \) such that \( \phi(\gamma) = -1 \).

\[
\mathbb{Z}/2 \times X/G \longrightarrow X/G
\]
\[
(-1, \ Gx) \longmapsto (-1) \cdot Gx := G(\gamma x).
\]

We denote by

\[
\Sigma := \{x \in S^{N-1} : \#Gx = \ell \}
\]
\[
\Sigma_0 := \{x \in \Sigma : Gx = G(\gamma x) \}.
\]

If \( Z \) is a \( \Gamma \)-invariant subset of \( \Sigma \setminus \Sigma_0 \), the action of \( \mathbb{Z}/2 \) on its \( G \)-orbit space \( Z/G \) is free.

**Definition**

If \( Z \neq \emptyset \) the Krasnoselski genus of \( Z/G \), denoted \( \text{genus}(Z/G) \), is defined to be the smallest \( k \in \mathbb{N} \) such that there exists a continuous map \( f : Z/G \rightarrow S^{k-1} \) which is \( \mathbb{Z}/2 \)-equivariant, i.e. \( f((-1) \cdot Gz) = -f(Gz) \) for every \( z \in Z \). We define \( \text{genus}(\emptyset) := 0 \).
We choose $\gamma \in \Gamma$ such that $\phi(\gamma) = -1$.

$$\mathbb{Z}/2 \times X/G \longrightarrow X/G$$

$$( -1 , \ Gx ) \longmapsto ( -1 ) \cdot Gx := G(\gamma x).$$

We denote by

$$\Sigma := \{ x \in \mathbb{S}^{N-1} : \#Gx = \ell \}$$
$$\Sigma_0 := \{ x \in \Sigma : Gx = G(\gamma x) \}.$$

If $Z$ is a $\Gamma$-invariant subset of $\Sigma \setminus \Sigma_0$, the action of $\mathbb{Z}/2$ on its $G$-orbit space $Z/G$ is free.

**Definition**

*If $Z \neq \emptyset$ the Krasnoselski genus of $Z/G$, denoted $\text{genus}(Z/G)$, is defined to be the smallest $k \in \mathbb{N}$ such that there exists a continuous map $f : Z/G \to \mathbb{S}^{k-1}$ which is $\mathbb{Z}/2$-equivariant, i.e. $f((-1) \cdot Gz) = -f(Gz)$ for every $z \in Z$. We define $\text{genus}(\emptyset) := 0.$*
For each subgroup $K$ of $\Gamma$ we set

$$\mu(Kz) := \begin{cases} 
\inf\{|\alpha_1 z - \alpha_2 z| : \alpha_1, \alpha_2 \in K, \alpha_1 z \neq \alpha_2 z\} & \text{if } \#Kz \geq 2, \\
2|z| & \text{if } \#Kz = 1,
\end{cases}$$

$$\mu_K(Z) := \inf_{z \in Z} \mu(Kz)$$ and $$\mu^K(Z) := \sup_{z \in Z} \mu(Kz).$$

From now on, we will assume that:

- $\Omega$ is $\Gamma$-invariant.
- $V$ is a $\Gamma$-invariant function and $(V_0)$ holds.
- $\ell < \infty$.

$c_\infty$ is the energy of the positive solution to the limit problem

$$\begin{cases} 
-\Delta u + V_\infty u = |u|^{p-2}u, \\
u \in H^1(\mathbb{R}^N).
\end{cases}$$
Theorem (1)

If $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, $Z$ is a $\Gamma$-invariant subset of $\Sigma \setminus \Sigma_0$, and $V$ satisfies the following:

\[(V_1)\quad \text{There exist } r_0 > 0, \ c_0 > 0 \text{ and } \lambda \in (0, \mu_{\Gamma}(Z)\sqrt{V_{\infty}}) \text{ such that}
\]

\[V(x) \leq -c_0e^{-\lambda|x|} \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0,
\]

then problem $\varphi$ has at least $\text{genus}(Z/G)$ pairs of sign changing solutions $\pm u$ such that

\[u(\alpha x) = \phi(\alpha)u(x) \quad \text{for all } \alpha \in \Gamma, \ x \in \Omega, \quad (1)
\]

and

\[\int_{\Omega} |u|^p < \frac{4p}{p-2}lc_{\infty}. \quad (2)
\]
Theorem (1)

If \( \phi : \Gamma \to \mathbb{Z}/2 \) is an epimorphism, \( Z \) is a \( \Gamma \)-invariant subset of \( \Sigma \setminus \Sigma_0 \), and \( V \) satisfies the following:

\((V_1)\) There exist \( r_0 > 0 \), \( c_0 > 0 \) and \( \lambda \in (0, \mu_\Gamma(Z)\sqrt{V_\infty}) \) such that

\[
V(x) \leq -c_0 e^{-\lambda|x|} \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0,
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then problem \( (\phi) \) has at least \( \text{genus}(Z/G) \) pairs of sign changing solutions \( \pm u \) such that

\[
u(\alpha x) = \phi(\alpha)u(x) \quad \text{for all } \alpha \in \Gamma, \ x \in \Omega,
\]

and

\[
\int_{\Omega} |u|^p < \frac{4p}{p - 2} \ell c_\infty.
\]
Multiplicity of sign changing solutions with symmetries

\[ \Gamma = \langle \gamma \rangle \]

\[ \phi : \Gamma \rightarrow \mathbb{Z}/2 \]
\[ \gamma \mapsto -1 \]

\[ G = \{ e \} \]

\[ \Sigma = S^{N-1} \]

\[ \Sigma_0 = W \cap S^{N-1} \]

\[ Z = W^\perp \cap S^{N-1} \]

\[ \mu_{\Gamma}(Z) = 2 \]

\[ \text{genus}(Z/G) = N - \dim W \]

\[ 0 \leq \dim W < N \]
Multiplicity of sign changing solutions with symmetries

\[ N = 2n, \quad \mathbb{R}^N = \mathbb{C}^n, \quad \rho(z_1, \ldots, z_n) := (e^{\pi i/m}z_1, \ldots, e^{\pi i/m}z_n) \]

\[ \Gamma = \langle \rho \rangle \quad \phi : \Gamma \longrightarrow \mathbb{Z}/2 \quad G = \ker \phi = \langle \rho^2 \rangle \]

\[ \rho \longmapsto -1 \]

- \( \Sigma = S^{N-1} \quad \Sigma_0 = \emptyset \quad Z = S^{N-1} \)
- If \( m = 2^k \), then \( \text{genus}(S^{N-1}/G) \geq \frac{N-1}{2^k} + 1 \).
- Since \( \mu_\Gamma(S^{N-1}) = |e^{\pi i/m} - 1| \), condition \((V_1)\) becomes more restrictive as \( m \) increases.
- If \((V_1)\) holds for \( m = 2^k \), it will also hold for \( m = 2^j \) with \( 0 \leq j < k \).
- Moreover, if \( u_j \) is a solution provided by Theorem (1) for \( m = 2^j \), then \( u_k \neq u_j \) if \( k > j \).

More solutions!

Therefore, Theorem (1) provides at least

$$\sum_{j=0}^{k} \frac{N - 1}{2^j} + k + 1 = (N - 1) \frac{2^{k+1} - 1}{2^k} + k + 1$$

pairs of solutions in this case.

No solutions!

$$\rho(z, t) := (e^{\pi i/m} z, t), \quad (z, t) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$$

$$\phi(\rho) := -1 \implies \Sigma = \{\pm(0, 0, 1)\} = \Sigma_0.$$  

One pair of solutions!

If $$\Gamma = \langle \rho, \tau \rangle$$, where $$\tau(z, t) := (z, -t)$$ and

$$\phi : \Gamma \longrightarrow \mathbb{Z}/2$$

$$\rho \mapsto 1$$

$$\tau \mapsto -1$$

then $$\Sigma = \{\pm(0, 0, 1)\}$$ and $$\Sigma_0 = \emptyset.$$
Theorem (2)

Let \( \ell \geq 2 \) and \( Z \) be a compact \( \Gamma \)-invariant subset of \( \Sigma \). Assume that the following hold:

\begin{enumerate}[(Z_0)]
\item \( \text{dist}(\gamma z, Gz) > \mu(Gz) \) for all \( z \in Z \) and \( \gamma \in \Gamma \) with \( \phi(\gamma) = -1 \).
\item \( \text{There exist } c_1 > 0 \text{ and } \kappa > \mu^G(Z) \sqrt{V_\infty} \text{ such that} \)
\[ V(x) \leq c_1 e^{-\kappa|x|} \text{ for all } x \in \mathbb{R}^N. \]
\end{enumerate}

Then \( \varphi \) has at least \( \text{genus}(Z/G) \) pairs of sign changing solutions \( \pm u \), which satisfy (1) and (2).
Theorem (2)

Let $\ell \geq 2$ and $Z$ be a compact $\Gamma$-invariant subset of $\Sigma$. Assume that the following hold:

$(Z_0)$ \( \text{dist}(\gamma z, Gz) > \mu(Gz) \) for all $z \in Z$ and $\gamma \in \Gamma$ with $\phi(\gamma) = -1$.

$(V_2)$ There exist $c_1 > 0$ and $\kappa > \mu^G(Z)\sqrt{\mathcal{V}_\infty}$ such that

$$V(x) \leq c_1 e^{-\kappa|x|} \quad \text{for all } x \in \mathbb{R}^N.$$ 

Then ($\varphi$) has at least $\text{genus}(Z/G)$ pairs of sign changing solutions $\pm u$, which satisfy (1) and (2).
Multiplicity of sign changing solutions with symmetries

\[ \mathbb{R}^{4n} = \mathbb{C}^n \times \mathbb{C}^n \quad m \geq 3 \]

For \((y, z) \in \mathbb{C}^n \times \mathbb{C}^n\)

\[ \rho(y, z) := (e^{\pi i/ m}y, e^{\pi i/ m}z) \]

\[ \gamma(y, z) := (-\bar{z}, \bar{y}) \]

\[ \Gamma = \langle \rho, \gamma \rangle \leq O(4n) \]

\[ \phi : \Gamma \longrightarrow \mathbb{Z}/2 \]

\[ \rho \longmapsto 1 \]

\[ \gamma \longmapsto -1 \]

\[ G = \langle \rho \rangle \quad Z := \mathbb{S}^{4n-1} \]

\[ \mu^G(Z) = |e^{\pi i/ m} - 1| \]

The variational setting

- The energy functional associated to $(\phi)$

\[
J_V : H^1_0(\Omega) \to \mathbb{R}
\]

\[
J_V(u) := \frac{1}{2} \|u\|^2_V - \frac{1}{p} |u|^p_p,
\]

where \( \|u\|_V := \left( \int_\Omega \left( \left| \nabla u \right|^2 + (1 + V(x)) u^2 \right) \right)^{1/2} \).

- The action of $\Gamma$ on $H^1_0(\Omega)$ induced by $\phi$

\[
(\gamma u)(x) := \phi(\gamma)u(\gamma^{-1}x) \quad \gamma \in \Gamma \text{ and } u \in H^1_0(\Omega).
\]

- The $\phi-$equivariant function space

\[
H^1_0(\Omega)^\phi := \{ u \in H^1_0(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \ \forall \gamma \in \Gamma, x \in \Omega \}.
\]

- The principle of symmetric criticality (Palais 1979)
The variational setting

- The energy functional associated to $(\phi)$

$$J_V : H^1_0(\Omega) \rightarrow \mathbb{R}$$

$$J_V(u) := \frac{1}{2} \|u\|_V^2 - \frac{1}{p} |u|^p_p,$$

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- The principle of symmetric criticality (Palais 1979)
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$$J_V : H_0^1(\Omega) \to \mathbb{R}$$

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$$(\gamma u)(x) := \phi(\gamma)u(\gamma^{-1}x) \quad \gamma \in \Gamma \text{ and } u \in H_0^1(\Omega).$$

- The $\phi$–equivariant function space

$$H_0^1(\Omega)_{\phi} := \{ u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \ \forall \gamma \in \Gamma, \ x \in \Omega \}.$$

- The principle of symmetric criticality (Palais 1979)
The variational setting

- The energy functional associated to $(\varphi)$
  \[ J_V : H^1_0(\Omega) \to \mathbb{R} \]
  \[ J_V(u) := \frac{1}{2} \|u\|_V^2 - \frac{1}{p} |u|^p_p, \]
  where \( \|u\|_V := \left( \int_{\Omega} \left( |\nabla u|^2 + (1 + V(x)) u^2 \right) \right)^{1/2} \).

- The action of \( \Gamma \) on \( H^1_0(\Omega) \) induced by \( \varphi \)
  \[ (\gamma u)(x) := \varphi(\gamma)u(\gamma^{-1}x) \quad \gamma \in \Gamma \text{ and } u \in H^1_0(\Omega). \]

- The \( \varphi \)-equivariant function space
  \[ H^1_0(\Omega)^\varphi := \{ u \in H^1_0(\Omega) : u(\gamma x) = \varphi(\gamma)u(x) \ \forall \gamma \in \Gamma, \ x \in \Omega \}. \]

- The principle of symmetric criticality (Palais 1979)
The variational setting

- The Nehari manifold

\[ N^\phi := \left\{ u \in H^1_0(\Omega)^\phi : u \neq 0, \|u\|^2_V = |u|^p_p \right\}. \]

- Let \( \omega \) be the ground state of

\[ (\phi_\infty) \left\{ \begin{array}{l}
-\Delta u + u = |u|^{p-2}u, \\
u \in H^1(\mathbb{R}^N).
\end{array} \right. \]

\[ \lim_{|x| \to \infty} \frac{|D^\nu \omega(x)| |x|^\frac{N-1}{2}}{\exp{|x|}} = b_\nu > 0 \quad \text{for} \quad \nu = 0, 1. \]

- The energy functional \( J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R} \) associated to problem \((\phi_\infty)\) is given by

\[ J_\infty(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p} |u|^p_p \quad \text{and} \quad c_\infty := J_\infty(\omega). \]
The variational setting

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- Let \( \omega \) be the ground state of
  \[
  (\rho_\infty) \quad \begin{cases} 
  -\Delta u + u = |u|^{p-2}u, \\
  u \in H^1(\mathbb{R}^N). 
  \end{cases}
  \]

  \[
  \lim_{|x| \to \infty} |D^\nu \omega(x)| |x|^{\frac{N-1}{2}} \exp|x| = b_\nu > 0 \quad \text{ for } \nu = 0, 1.
  \]

- The energy functional \( J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R} \) associated to problem \( (\rho_\infty) \) is given by
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The variational setting

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\[ \mathcal{N}^\phi := \left\{ u \in H_0^1(\Omega) : u \neq 0, \|u\|^2_V = |u|^p_p \right\} . \]

- Let \( \omega \) be the ground state of

\[(\phi_\infty) \quad \begin{cases} -\Delta u + u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N). \end{cases} \]

\[ \lim_{|x| \to \infty} |D^\nu \omega(x)| |x|^{\frac{N-1}{2}} \exp |x| = b_\nu > 0 \quad \text{for} \quad \nu = 0, 1. \]

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\[ J_\infty(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p} |u|^p_p \quad c_\infty := J_\infty(\omega) \]
Proposition

Let \((u_n)\) be a sequence in \(H^1_0(\Omega)\) such that \(u_n \rightharpoonup 0\) in \(H^1_0(\Omega)\), \(J_0(u_n) \to c > 0\) and \(J'_0(u_n) \to 0\) in \(H^{-1}(\Omega)\). Then there exist a sequence \((\zeta_n)\) in \(\Omega\), a closed subgroup \(K\) of finite index in \(\Gamma\), a nontrivial solution \(v\) to problem \((\varphi_\infty)\) and a sequence \((w_n)\) in \(H^1_0(\Omega)\) such that

\(\begin{align*}
\Gamma\zeta_n &= K \quad \text{for all } n \in \mathbb{N}, \\
|\zeta_n| &\to \infty \quad \text{and} \quad |\alpha \zeta_n - \hat{\alpha} \zeta_n| \to \infty \quad \text{if } \hat{\alpha} \alpha^{-1} \notin K, \quad \hat{\alpha}, \alpha \in \Gamma, \\
v(\alpha x) &= \phi(\alpha)v(x) \quad \text{for all } x \in \mathbb{R}^N, \alpha \in K, \\
\left\|u_n - w_n - \sum_{[\alpha] \in \Gamma/K} \phi(\alpha)v\alpha^{-1}(\cdot - \alpha \zeta_n)\right\| &\to 0, \\
w_n &\rightharpoonup 0 \quad \text{in } H^1_0(\Omega), \quad J_0(w_n) \to c - |\Gamma/K| J_\infty(v) \quad \text{and} \quad J'_0(w_n) \to 0 \quad \text{in } H^{-1}(\Omega).
\end{align*}\)
The $\phi$-equivariant Palais-Smale condition

**Definition**

$J_V$ satisfies the condition $(PS)^\phi_c$ on $N^\phi$ if every sequence $(u_n)$ such that

\[ u_n \in N^\phi, \quad J_V(u_n) \to c, \quad \nabla_{N^\phi} J_V(u_n) \to 0, \]

contains a convergent subsequence in $H^1_0(\Omega)$.

$\nabla_{N^\phi} J_V(u)$ is the orthogonal projection of $\nabla J_V(u)$ onto $T_u N^\phi$.

**Corollary**

$J_V$ satisfies the condition $(PS)^\phi_c$ on $N^\phi$ for all $c < |\Gamma/G| \ell c_\infty$.

\[ J^d_V := \{ u \in H^1_0(\Omega) : J_V(u) \leq d \} \]
Sketch of the proof of Theorem 2

- $J_V : \mathcal{N}^\phi \rightarrow \mathbb{R}$ is an even function, which is bounded from below and satisfies $(PS)_c^\phi$ on $\mathcal{N}^\phi$ for all $c < 2\ell c_\infty$.

- if $d < 2\ell c_\infty$, $J_V$ has at least

\[
\text{genus}(\mathcal{N}^\phi \cap J_V^d)
\]

pairs of critical points $\pm u$ with critical value $J_V(u) \leq d$.

\[\theta : \square \longrightarrow \mathcal{N}^\phi \cap J_V^d\]
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- $J_V : \mathcal{N}^\phi \to \mathbb{R}$ is an even function, which is bounded from below and satisfies $(PS)_c^\phi$ on $\mathcal{N}^\phi$ for all $c < 2\ell c_\infty$.

- If $d < 2\ell c_\infty$, $J_V$ has at least $\text{genus}(\mathcal{N}^\phi \cap J_V^d)$ pairs of critical points $\pm u$ with critical value $J_V(u) \leq d$.

$\theta : \Box \to \mathcal{N}^\phi \cap J_V^d$
Sketch of the proof of Theorem 2

- $J_V : \mathcal{N}^\phi \to \mathbb{R}$ is an even function, which is bounded from below and satisfies $(PS)_c^\phi$ on $\mathcal{N}^\phi$ for all $c < 2\ell c_\infty$.

- if $d < 2\ell c_\infty$, $J_V$ has at least $\text{genus}(\mathcal{N}^\phi \cap J_V^d)$ pairs of critical points $\pm u$ with critical value $J_V(u) \leq d$.

\[ \theta : \Box \longrightarrow \mathcal{N}^\phi \cap J_V^d \]
We define

\[ \theta : \mathbb{Z} \rightarrow \mathcal{N}^\phi \cap J^d_V \]

\[ \theta(y) := \pi \circ \chi \left( \sum_{[\alpha] \in \Gamma / \Gamma_y} \phi(\alpha) \omega_{R\alpha y} \right) \]

\[ d < 2\ell c_\infty \]

- \( \theta \) is continuous and satisfies \( \theta(gz) = \theta(z) \) for all \( g \in G \) and \( \theta(\gamma z) = -\theta(z) \) if \( \phi(\gamma) = -1 \).
\[ \hat{\theta} : Z/G \rightarrow \mathcal{N}^\phi \cap J^d_V \]

\[ \hat{\theta}(Gz) := \theta(z) \]

which satisfies \( \hat{\theta}((-1) \cdot Gz) = -\hat{\theta}(Gz) \) for all \( z \in Z \).

\[
\text{genus}(Z/G) \leq \text{genus}(\mathcal{N}^\phi \cap J^d_V)
\]
Sketch of the proof of Theorem 2

- \( \theta \) induces a continuous map:

\[
\hat{\theta} : Z/G \rightarrow \mathcal{N}^\phi \cap J^d_V
\]

\[
\hat{\theta}(Gz) := \theta(z)
\]

which satisfies \( \hat{\theta}((-1) \cdot Gz) = -\hat{\theta}(Gz) \) for all \( z \in Z \).

\[
\text{genus}(Z/G) \leq \text{genus}(\mathcal{N}^\phi \cap J^d_V)
\]


Thanks for your attention.