Fredholm Alternative for the p-Laplacian: Bifurcation Approach (1st plenary lecture)

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Conference "Variational and Topological Methods"
UNA Flagstaff, Arizona, June 6 – 9, 2012

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad 1<p<\infty$$

$$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial u}{\partial x_N}\right)^2}, \quad x \in \Omega \subseteq \mathbb{R}^N$$

$$u = u(x_1, \dots, x_N) : \Omega \to \mathbb{R}$$

$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,p} \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

principal eigenvalue of Δ_p

Ω bdd domain, $\partial\Omega$ smooth

φ_1 eigenfunction associated with λ_1

$\varphi_1 > 0$ in Ω, $\varphi \in C^{1,\alpha}(\overline{\Omega})$,

$\frac{\partial \varphi_1}{\partial \nu} < 0$ on $\partial\Omega$

$$-\Delta_p \varphi_1 = \lambda_1 |\varphi_1|^{p-2}\varphi_1 \text{ in } \Omega, \quad \varphi_1|_{\partial\Omega} = 0$$

We focus on existence and multiplicity

$$-\Delta_p u = \lambda |u|^{p-2}u + f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

for λ <u>close</u> to λ_1 !

Fredholm alternative $p = 2$:

$$-\Delta u = \lambda u + f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

1. $\lambda \neq \lambda_1$: $\forall f \; \exists!$ sol'n

2. $\lambda = \lambda_1$: 2a. $\int_\Omega f \varphi_1 = 0 \Rightarrow \exists \infty$ sol'ns

   2b. $\int_\Omega f \varphi_1 \neq 0 \Rightarrow \nexists$ sol'n

For $p \neq 2$ the situation is not that simple (it is more interesting!):

1. For $\lambda \neq \lambda_1$ (λ close λ_1), $\int_\Omega f\varphi_1 = 0$

$$\left.\begin{array}{ll} 1a. & 1 < p < 2,\ \lambda < \lambda_1 \\ 1b. & p > 2,\ \lambda > \lambda_1 \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists$ 3 sol'ns (at least one > 0, at least one < 0)

2. For $\lambda = \lambda_1$, $\int_\Omega f\varphi = 0 \Rightarrow \exists$ sol'n

All sol'ns are a priori bounded by a constant which depends on f !

3. For $\lambda = \lambda_1$, $\int_\Omega f\varphi_1 \neq 0$, $\left|\int_\Omega f\varphi_1\right| \ll 1$

$\Rightarrow \exists$ 2 sol'ns

For details see D., Girg, Takáč, Ulm (Ind. Univ. Math. J. 2004)

Bifurcation approach : $p = 2$

$$-\Delta u - \lambda u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

$$u = c\varphi_1 + u^T \qquad \int_\Omega u^T \varphi_1 = 0$$

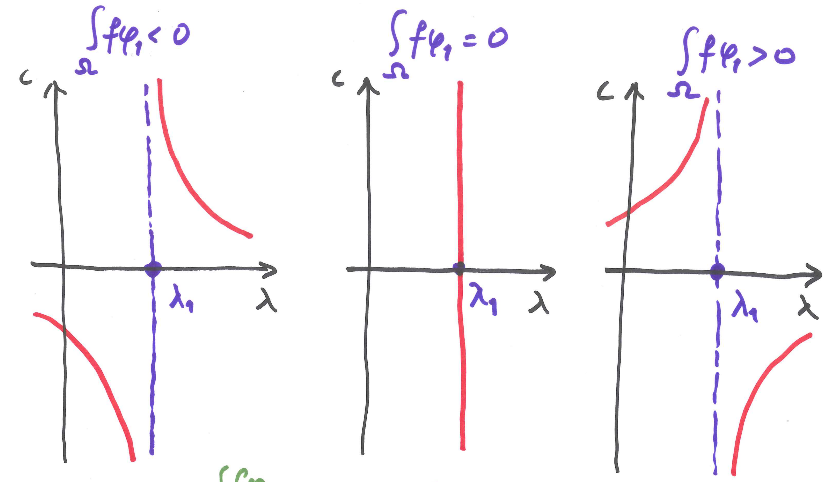
$$f = a\varphi_1 + f^T$$

$$\begin{cases} -\Delta u^T - \lambda u^T + (\lambda_1 - \lambda) c\varphi_1 = f^T + a\varphi_1 \quad \text{in } \Omega, \\ u^T|_{\partial\Omega} = 0 \end{cases}$$

$$(\lambda_1 - \lambda) c = a$$

$$c = \frac{a}{\lambda_1 - \lambda}, \quad \lambda \neq \lambda_1 \qquad \begin{cases} -\Delta u^T - \lambda u^T = f^T \text{ in } \Omega \\ u^T|_{\partial\Omega} = 0 \end{cases}$$

λ not an eigenvalue!

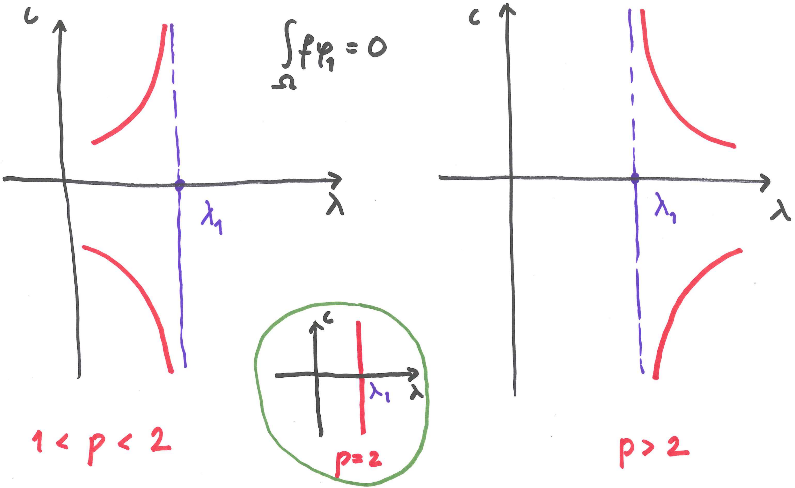


$\lambda \neq \lambda_1$: $c = \dfrac{\int_\Omega f\varphi_1}{\lambda_1 - \lambda}$

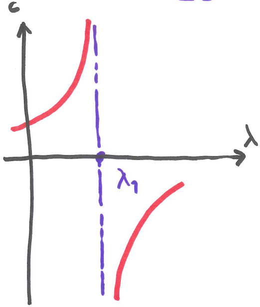
$\lambda = \lambda_1$: $\nexists\ c = \ldots$

$\lambda = \lambda_1$: $\exists\, \infty$ sol'ns

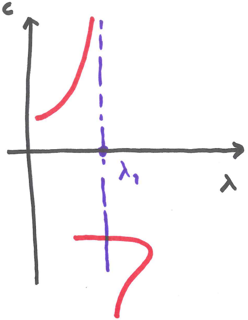
Motivated by the case $p = 2$ we studied the bifurcation from ∞ at $\lambda = \lambda_1$:



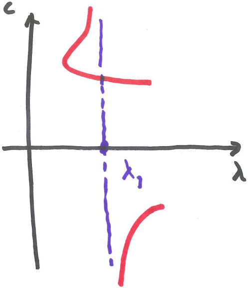
$$a = \int_\Omega f\varphi_1 > 0 :$$



$|a| \gg 1,\ p > 1$

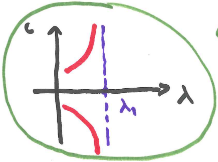


$|a| \ll 1,\ 1 < p < 2$

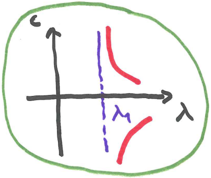


$|a| \ll 1,\ p > 2$

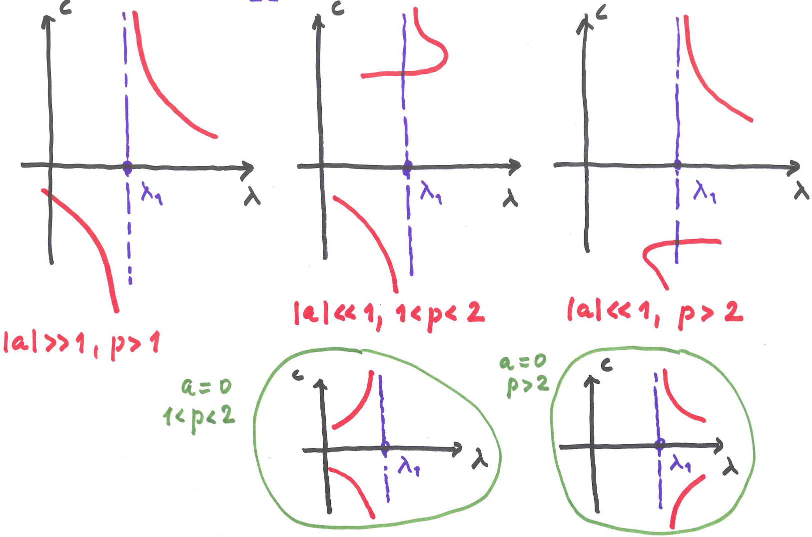
$a = 0$:
$1 < p < 2$:



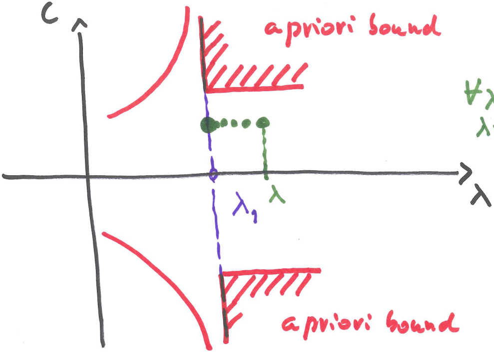
$a = 0$:
$p > 2$:



$$a = \int_\Omega f\varphi_1 < 0 \; :$$



$$a = \int_\Omega f \varphi_1 = 0$$: Fredholm alternative at λ_1 $(\lambda = \lambda_1)$

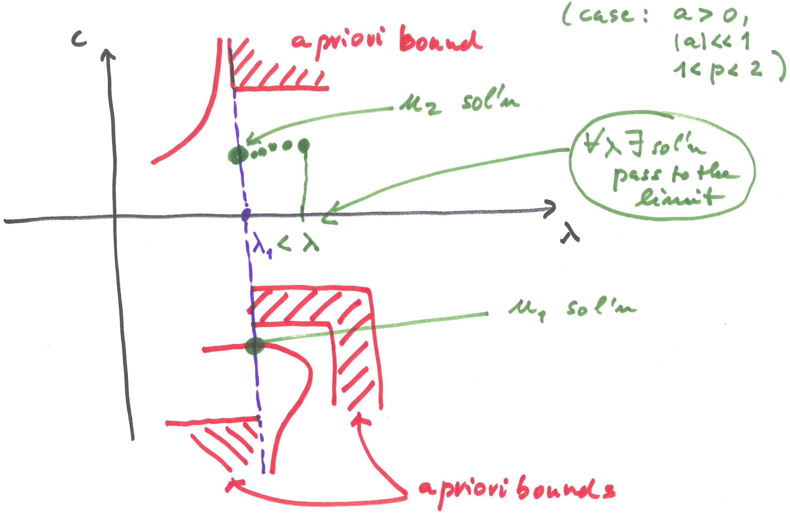


$\forall \lambda > \lambda_1$ $\exists$ sol'n $\oplus$ take the limit

$(1 < p < 2)$

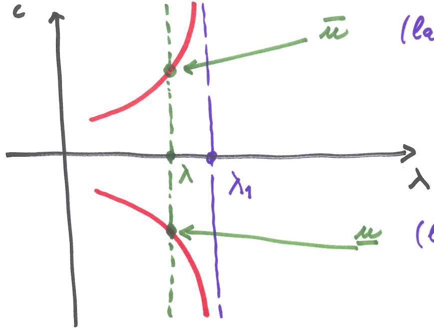
Similarly for $p > 2$!

$$a = \int_\Omega f\varphi_1 \neq 0\,, \quad \lambda = \lambda_1 \qquad (\exists\ 2\ \text{sol'ns})$$



$$a = \int_\Omega f\varphi_1 = 0, \quad \lambda < \lambda_1, \text{ close to } \lambda_1, \ 1<p<2:$$

$(\exists\ 3 \text{ sol'ns})$



(large positive sol'n)

↓

$\exists$ sol'n "between"
$\neq \bar{u}, \underline{u}$

↑

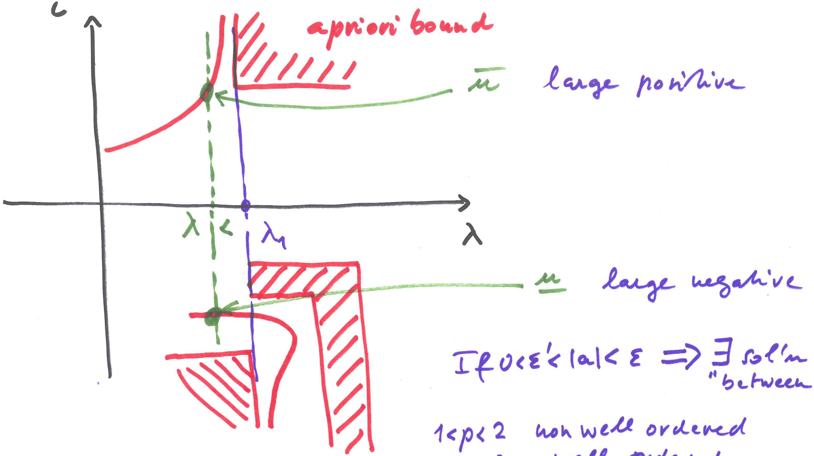
(large negative sol'n)

Remark: $\underline{u}, \bar{u}$ well-ordered for $p > 2$
non well-ordered for $1 < p < 2$

$\boxed{\text{Similarly for } p > 2!}$

$a = \int_\Omega f\varphi_1 \neq 0$, $\lambda < \lambda_1$ "close", $1<p<2$, $|a| \ll 1$

$(\exists\ 3$ sol'ns$)$



If $0<\varepsilon'<|a|<\varepsilon \Rightarrow \exists$ sol'n "between"

$1<p<2$ non well ordered
$p>2$ well ordered

Bifurcation from infinity at λ_1 for the p-Laplacian

$$\underbrace{\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v}_{(I(u),v)} - \lambda \underbrace{\int_\Omega |u|^{p-2} u v}_{(S(u),v)} = \underbrace{\int_\Omega f v}_{(F,v)}$$

$u \neq 0 \quad : \quad v = \dfrac{u}{\|u\|^2}$

$$I(v) - \lambda S(v) = \frac{F}{\|v\|^{2(p-1)}} = g(v)$$

$g(0) = 0$

$$I(v) - \lambda S(v) = \underbrace{g(v)}_{\text{higher order terms}}$$

$$\mathcal{I}(\varphi_1) - \lambda_1 \mathcal{S}(\varphi_1) = 0$$

bifurcation from $(\lambda_1, 0)$ for

$$\mathcal{I}(v) - \lambda \mathcal{S}(v) = g(v)$$

is transformed by $v = \dfrac{u}{\|u\|^2}$ to

bifurcation from (λ_1, ∞) for

$$\mathcal{I}(u) - \lambda \mathcal{S}(u) = F$$

Global bifurcation result from $(\lambda_1, 0)$ for the p-Laplacian: Drábek (Annali Mat. Pura Appl. 1991).

We get the existence of two global branches which bifurcate from

$(\lambda_1, +\infty)$ ... "asymptotically close" to $\tau\varphi_1$ with $\tau \gg 1$

and

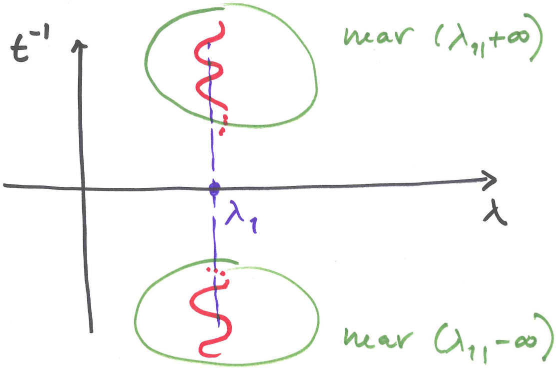
$(\lambda_1, -\infty)$ ... "asymptotically close" to $-\tau\varphi_1$ with $\tau \gg 1$

To describe these large sol'ns, we write

$$u = t^{-1}(\varphi_1 + v_t^T) = t^{-1}\varphi_1 + t^{-1}v_t^T$$

for $t \to 0$ dominates

We aslo write $\lambda = \lambda_1 + \mu$ and investigate sol'ns on bifurcation branches $(\lambda, u) = (\lambda + \mu, t^{-1}(\varphi_1 + v T))$ with μ, t small:



We investigate sol'ns $u_n = t_n^{-1}\varphi_1 + t_n^{-1} v_n^T$ of

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi = (\lambda_1 + \mu_n)\int_\Omega |u_n|^{p-2} u_n \phi + \int_\Omega f\phi \quad \forall\phi$$

with $t_n \to 0$ and $\mu_n \to 0$

and derive

$$\mu_n = -|t_n|^{p-2} t_n \int_\Omega f\varphi_1 + (p-2)|t_n|^{2(p-1)} \underbrace{Q_0}_{\geq 0}$$

$$+ (p-1)|t_n|^{2(p-1)} Q_1 \int_\Omega f\varphi_1 + o\left(|t_n|^{2(p-1)}\right)$$

For $\int_\Omega f\varphi_1 = 0$:

$$\mu_n = (p-2)|t_n|^{2(p-1)} \underbrace{Q_0}_{\geq 0} + o\left(|t_n|^{2(p-1)}\right)$$

Choose a test function

$$\phi = |t_n|^{p-2} t_n (\varphi_1 + v_n^T)$$

$$\int_\Omega |\nabla\varphi_1 + \nabla v_n^T|^p - \lambda_1 \int_\Omega |\varphi_1 + v_n^T|^p$$

$$= \mu_n \int_\Omega |\varphi_1 + v_n^T|^p + |t_n|^{p-2} t_n \int_\Omega f(\varphi_1 + v_n^T)$$

we have:

$$\int_\Omega |\nabla\varphi_1|^p - \lambda_1 \int_\Omega |\varphi_1|^p = 0$$

$$\int_\Omega |\nabla\varphi_1|^{p-2} \nabla\varphi_1 \cdot \nabla v_n^T - \lambda_1 \int_\Omega \varphi_1^{p-1} v_n^T = 0$$

$$p\int_\Omega \left(\int_0^1 A\left(\nabla(\varphi_1 + s v_n^T)\right)(1-s)\,ds\right)\nabla v_n^T \cdot \nabla v_n^T$$

$$- p(p-1)\lambda_1 \int_\Omega \left(\int_0^1 |\varphi_1 + s v_n^T|^{p-2}(1-s)\,ds\right)(v_n^T)^2$$

$$= \mu_n \left[\int_\Omega \varphi_1^p + p\int_\Omega \left(\int_0^1 |\varphi_1 + s v_n^T|^{p-2}(\varphi_1 + s v_n^T)\,ds\right) v_n^T\right]$$

$$+ |t_n|^{p-2} t_n \int_\Omega f\varphi_1 + |t_n|^{2(p-1)} \int_\Omega f V_n^T$$

$$V_n^T = \frac{v_n^T}{|t_n|^{p-2} t_n} \longrightarrow V^T \neq 0.$$

$$A(\vec{a}) = |\vec{a}|^{p-2}\left(\mathbb{I} + (p-2)\frac{\vec{a}\otimes\vec{a}}{|\vec{a}|^2}\right)$$

leads to the weighted norm:

$$\|v\| = \left(\int_\Omega \underbrace{|\nabla\varphi_1|^{p-2}}_{\text{weight}} \cdot |\nabla v|^2\right)^{\frac{1}{2}}$$

$$\int_\Omega \left(\int_0^1 |\varphi_1 + s v_n^T|^{p-2}(1-s)\,ds\right)(v_n^T)^2$$

leads to the weighted norm:

$$\|v\| = \left(\int_\Omega \underbrace{|\varphi_1|^{p-2}}_{\text{weight}} v^2\right)^{\frac{1}{2}}$$

Pass to the limit:

$$p\left(\int_\Omega A(\nabla\varphi_1)(\nabla V^T)^2 - (p-1)\lambda_1\int_\Omega \varphi_1^{p-1}|V^T|^2\right)$$

$$-\int_\Omega f V^T = \lim_{n\to\infty}\frac{1}{|t_n|^{p-2}t_n}\left(\frac{\mu_n}{|t_n|^{p-2}t_n}\int_\Omega \varphi_1^p + \int_\Omega f\varphi_1\right)$$

$$+\ p\left(\int_\Omega \varphi_1^{p-1}V^T\right)\lim_{n\to\infty}\frac{\mu_n}{|t_n|^{p-2}t_n}$$

$$\Downarrow$$

$$(p-2)\underbrace{\left[\int_\Omega A(\nabla\varphi_1)(\nabla v^T)^2-(p-1)\lambda_1\int_\Omega \varphi_1^{\,p-1}(v^T)^2\right]}_{Q_0=Q(v^T,v^T)>0 \text{ for } v^T\perp\varphi_1}$$

$$=-\underbrace{(p-1)\left(\int_\Omega f\varphi_1\right)\int_\Omega \varphi_1^{\,p-1}v^T}_{Q_1}$$

$$+\lim_{n\to\infty}\frac{1}{|t_n|^{p-2}t_n}\left(\frac{\mu_n}{|t_n|^{p-2}t_n}+\int_\Omega f\varphi_1\right)$$

and the formula follows:

$$\mu_n = -|t_n|^{p-2} t_n \int_\Omega f\varphi_1 + (p-2)|t_n|^{2(p-1)} Q_0$$

$$+ (p-1)|t_n|^{2(p-1)} Q_1 \int_\Omega f\varphi_1 + o\left(|t_n|^{2(p-1)}\right)$$

in particular: $\int_\Omega f\varphi_1 = 0$

$$\mu_n = (p-2)|t_n|^{2(p-1)} Q_0 + o\left(|t_n|^{2(p-1)}\right)$$

Thank you for your attention!

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