Fredholm Alternative for the p-Laplacian: Variational Approach (2nd plenary lecture)

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Recall that $\lambda_1 > 0$ is the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

$\varphi_1 > 0$ corresponding eigenfunction

We focus on the geometry of the energy

$$E_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega |u|^p - \int_\Omega f u, \quad u \in W_0^{1,p}$$

where $\lambda = \lambda_1$ or λ close to λ_1, $\int_\Omega f \varphi_1 = 0$ or "small"

$$\langle E_\lambda'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v - \lambda_1\int_\Omega |u|^{p-2}u\,v - \int_\Omega f v$$

critical point u_0 : $\langle E_\lambda'(u_0), v\rangle = 0 \quad \forall v$

u_0 is a weak sol'n of

$$-\Delta_p u = \lambda |u|^{p-2}u + f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0 .$$

We consider first $\lambda = \lambda_1$ & $\int_\Omega f\varphi_1 = 0$.

For $p = 2$ the geometry of

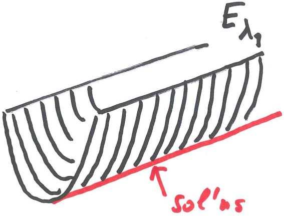
$$E_{\lambda_1}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_1}{2}\int_\Omega |u|^2 - \int_\Omega fu, \; u \in W_0^{1,2}$$

corresponds to "infinitely long" through with the bottom formed by one-dim. affine set of solutions to

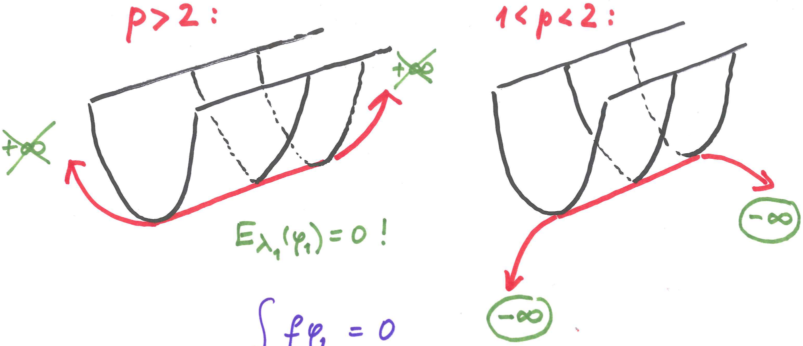
$$\begin{cases} -\Delta u = \lambda_1 u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

provided

$$\int_\Omega f\varphi_1 = 0.$$



For $p \neq 2$ the geometry of E_{λ_1} corresponds to a deformation of the trough which is different for $1 < p < 2$ and $p > 2$.



$E_{\lambda_1}(\varphi_1) = 0$!

$$\int_\Omega f\varphi_1 = 0$$

$p \neq 2$: critical points of E_{λ_1} are a priori bounded

$p > 2$: E_{λ_1} is bounded from below
it has a global minimizer
it is not coercive
$(E_{\lambda_1}(t\varphi_1) = 0 \quad \forall t \in \mathbb{R})$

$1 < p < 2$: E_{λ_1} is unbounded from below (and from above)
it has a saddle point
P.-S. condition not available

For $p > 2$ the following generalization of the Poincaré inequality was proved in

Fleckinger, Takáč (Adv. Diff. Eq. 2002):

$$u = \bar{u}\varphi_1 + \tilde{u}, \quad \bar{u} := \frac{\int_\Omega u\varphi_1}{\left(\int_\Omega \varphi_1^2\right)^{1/2}}, \quad \int_\Omega \tilde{u}\varphi_1 = 0$$

Poincaré $p = 2$: $$\int_\Omega |\nabla u|^2 - \lambda_1 \int_\Omega |u|^2 \geq (\lambda_2 - \lambda_1)\int_\Omega |\tilde{u}|^2$$

$p \neq 2$: $\exists c > 0$:

$$\int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p \geq c\left(\underbrace{|\bar{u}|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla\tilde{u}|^2}_{\text{extra term for } p\neq 2} + \int_\Omega |\nabla\tilde{u}|^p \right)$$

Consequently:

$$\int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p \geq \tilde{c}\left(|\bar{u}|^{p-2} \int_\Omega |\tilde{u}|^2 + \int_\Omega |\tilde{u}|^p \right).$$

The boundedness of E_{λ_1} from below follows:

$$E_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega f u$$

$$\geq \frac{\tilde{c}}{p}\left(|\bar{u}|^{p-2}\int_\Omega |\tilde{u}|^2 + \int_\Omega |\tilde{u}|^p \right) - \int_\Omega f\tilde{u}$$

$$\geq \left[\frac{\tilde{c}}{p}\left(\int |\tilde{u}|^p\right)^{\frac{1}{p'}} - \left(\int_\Omega |f|^{p'}\right)^{\frac{1}{p'}} \right] \left(\int_\Omega |\tilde{u}|^p\right)^{\frac{1}{p}} + \frac{\tilde{c}}{p}|\bar{u}|^{p-2}\int_\Omega |\tilde{u}|^2.$$

In Dr. + Takáč (Calc. Variations 2007):

$p > 2$: E_{λ_1} satisfies P.-S. condition on level $c < 0$ (it is violated on level $c = 0$)

$$\Downarrow$$

minimization argument and boundedness from below yields the existence of global minimizer

For $1<p<2$ the geometry of a saddle point was proved in Drábek, Holubová (J. Math. Anal. Appl. 2001):

$$\widetilde{W}_0^{1,p}(\Omega) := \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega u\varphi_1 = 0 \right\}$$

$$\exists\, c>0 : \quad \frac{1}{p}\int_\Omega |\nabla \tilde{u}|^p - \frac{\lambda_1}{p}\int_\Omega |\tilde{u}|^p \geq c \int_\Omega |\nabla \tilde{u}|^p$$

(proof by contradiction)

$$E_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla \tilde{u}|^p - \frac{\lambda_1}{p}\int_\Omega |\tilde{u}|^p - \int_\Omega \tilde{f}\tilde{u} \geq$$

$$\geq c\int_\Omega |\nabla \tilde{u}|^p - \left(\int_\Omega |\tilde{f}|^{p'}\right)^{\frac{1}{p'}} \left(\int_\Omega |\tilde{u}|^p\right)^{\frac{1}{p}}$$

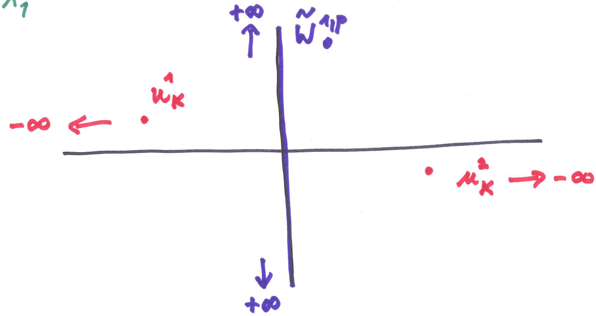
bdd below coercive on $\widetilde{W}_0^{1,p}$

On the other hand, $\forall K > 0 \ \exists \ u_K^1, u_K^2$

u_K^1 and u_K^2 are separated by $\tilde{W}_0^{1,p}$:

$$E_{\lambda_1}(u_K^i) < -K \quad , \quad i = 1,2$$

(E_{λ_1} is unbounded from below on $W_0^{1,p}(\Omega)$)



For $v_t = \varphi_1 + t^{-1} w$ we expand:

$$J(v_t) = \frac{1}{p}\int_\Omega |\nabla\varphi_1 + \frac{1}{t}\nabla w|^p - \frac{\lambda_1}{p}\int_\Omega |\varphi_1 + \frac{1}{t} w|^p$$

$$= \underbrace{\frac{1}{p}\int_\Omega |\nabla\varphi_1|^p - \frac{\lambda_1}{p}\int_\Omega |\varphi_1|^p}_{=0} + \underbrace{\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1\cdot\nabla\left(\frac{w}{t}\right) - \lambda_1\int_\Omega |\varphi_1|^{p-2}\varphi_1\frac{w}{t}}_{=0}$$

$$+ \underbrace{\frac{1}{2}\left(\int_\Omega (p-2)|\nabla\varphi_1|^{p-2}\left(\frac{\nabla\varphi_1}{|\nabla\varphi_1|}\cdot\nabla\left(\frac{w}{t}\right)\right)^2 + \int_\Omega |\nabla\varphi_1|^{p-2}\left|\nabla\frac{w}{t}\right|^2 - (p-1)\lambda_1\int_\Omega |\varphi_1|^{p-2}\left(\frac{w}{t}\right)^2\right)}_{=\frac{1}{t^2}(\text{bdd})} + o\left(\frac{1}{t^2}\right),\ t\to\infty$$

Choose w such that $\int_\Omega f w > 0$ and

set

$$u_t^1 = t^{\frac{1}{p}+\frac{1}{2}} v_t = t^{\frac{1}{p}+\frac{1}{2}}\left(\varphi_1 + \frac{w}{t}\right).$$

Then

$$E_{\lambda_1}(u_t^1) = J\left(t^{\frac{1}{p}+\frac{1}{2}} v_t\right) - \int_\Omega f u_t = t^{\frac{p}{2}+1} \underbrace{J(v_t)}_{\frac{1}{t^2}(\dots)+o\left(\frac{1}{t^2}\right)} - t^{\frac{1}{p}+\frac{1}{2}} \int_\Omega f \frac{w}{t}$$

$$= \underbrace{\frac{1}{2} t^{\frac{p}{2}-1} (\dots)}_{\to 0} + \underbrace{o\left(\frac{1}{t^2}\right)}_{\to 0} - \underbrace{t^{\frac{1}{p}-\frac{1}{2}}}_{\to\infty} \underbrace{\int_\Omega f w}_{>0}$$

$$\to -\infty \quad !$$

Similarly, if we choose $w : \int_\Omega f w < 0$

and consider

$$u_t^2 = - t^{\frac{1}{p}+\frac{1}{2}} v_t$$

we get $E_{\lambda_1}(u_t^2) \to -\infty$ as $t \to \infty$.

Moreover, u_t^1 and u_t^2 are separated by $\tilde{W}_0^{1,p}$ and for $t \gg 1$:

$$E_{\lambda_1}(u_t^{1,2}) < \inf_{\tilde{u} \in \tilde{W}_0^{1,p}} E(\tilde{u})$$

Observation. The saddle geometry of E_{λ_1} is preserved under small perturbations of f:

$$E_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega \tilde{f} u \quad , \int_\Omega \tilde{f}\varphi_1 = 0$$

has a saddle geometry

$$\Rightarrow \forall c : |c| \ll 1, \ c \neq 0, \ f = c\varphi_1 + \tilde{f}$$

$$E^c_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega f u$$

has still saddle geometry

Even if P.-S. condition is not available for E_{λ_1} $\left(\int_\Omega \tilde{f}\varphi_1 = 0 !\right)$, it holds true

for $E^c_{\lambda_1}$ with $c \neq 0$. (The proof relies on standard contradiction type argument.)

Combined with the geometry $(|c| \ll 1 !)$, we prove the existence of a saddle point of $E^c_{\lambda_1}$ $(c \neq 0)$ using the "classical arguments".

The proof of the existence of a critical point of E_{λ_1} $(c = 0)$ relies on the following argument:

We choose $c_1 < 0 < c_2$, $|c_i| \ll 1$, and consider

$$f_i = c_i \varphi_1 + \tilde{f}\,, \quad \int_\Omega \tilde{f} \varphi_1 = 0\,, \quad i = 1,2$$

$$E_{\lambda_1}^{c_i}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega f_i u$$

- satisfies P.-S. condition
- has geometry of saddle type

$\Longrightarrow \exists\ u_1, u_2 \in W_0^{1,p}$ critical points of $E_{\lambda_1}^{c_i}$, $i = 1,2$.

u_i are sol'ns of:

$$\begin{cases} -\Delta u = \lambda_1 |u|^{p-2} u + c_i \varphi_1 + \tilde{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

u_1 subsolution $\qquad$ u_2 supersolution

for

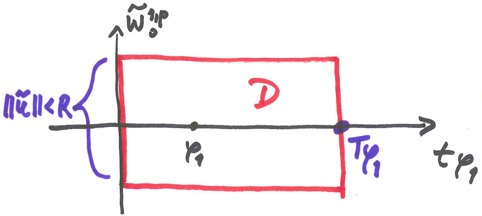
$$(*) \quad \begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u + \tilde{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Both well-ordered and non well-ordered case work in this case $\Rightarrow$ $\exists$ solution of $(*)$.

If $f = c\varphi_1 + \tilde{f}$, $c \neq 0$, $|c| \ll 1$,
more careful analysis (P. D., EJDE 2002)
of

$$E^c_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega f u$$

leads to : $\exists R>0 \ \exists T>0$ s.t.



$$\inf_D E^c_{\lambda_1}(u) < \inf_{\partial D} E^c_{\lambda_1}(u)$$

Since E_{λ_1} is bounded below on D,
E_{λ_1} is w. l. s. c. $\Rightarrow \exists u_D \in D$:

$$E^c_{\lambda_1}(u_D) = \min_D E^c_{\lambda_1}(u).$$

$E^c_{\lambda_1}$ is unbounded from below on $W^{1,p}_0$

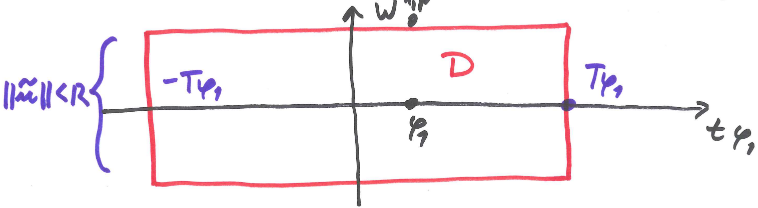
$\Rightarrow$ $E^c_{\lambda_1}$ has a mountain-pass geometry $(c \neq 0)$ + P.-S. cond.

$$\Rightarrow \exists u_0 \neq u_D,\ (E^c_{\lambda_1})'(u_0) = 0.$$

For $p > 2$, $\int_\Omega \tilde{f}\varphi_1 = 0 \qquad (c = 0)$

$$E_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega \tilde{f} u$$

- is bounded from below (as above)



$$\inf_D E_{\lambda_1}(u) < \inf_{\partial D} E_{\lambda_1}(u)$$

$$c \neq 0, \ |c| \ll 1, \ f = c\varphi_1 + \tilde{f}$$

$$E^c_{\lambda_1}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega (c\varphi_1 + \tilde{f})u$$

$$u = t\varphi_1 + \tilde{u}$$

$$= E_{\lambda_1}(u) \ - \underbrace{ct\int_\Omega \varphi_1^2}_{\substack{\text{small perturbation on } D \\ \text{but unbounded on } W^{1,p}_0}}$$

- $E^c_{\lambda_1}$ has a minimum over D
- $E^c_{\lambda_1}$ in unbounded from below on $W^{1,p}_0$
- $E^c_{\lambda_1}$ has a mountain-pass cr. point $\Rightarrow$ 2^{nd} sol'n

Fredholm alternative for the higher eigenvalues of the Dirichlet p-Laplacian:

1. Manásevich, Takáč (Proc. Lond. Math. Soc. 2002)
in 1 dimension, Ω = interval

2. Benedikt, Girg, Takáč (Nonlinear Anal. 2010)
in higher dimension $\Omega \subseteq \mathbb{R}^N$

Asymptotic analysis is more difficult when the eigenfunction changes sign in Ω (even if Ω = interval)!

OPEN PROBLEM # 1

$\sigma(\Delta_p)$ ... the set of all eigenvalues of
$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \; u|_{\partial\Omega} = 0 .$$

$\sigma(\Delta_p)$ contains a sequence

$$0 < \lambda_1 \quad \lambda_2 \leq \lambda_3 \; \cdots \; \lambda_n \to +\infty$$

$$\lambda_n = \inf \sup \ldots \quad (= \sup \inf \ldots)$$

For Ω bad domain, $\partial\Omega$ smooth

??? Find complete description of entire $\sigma(\Delta_p)$!!!

OPEN PROBLEM #2

Let u_λ be an eigenfunction of

$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

$u_\lambda \neq \varphi_1$! Ω bdd domain, $\partial\Omega$ smooth

??? Does u_λ satisfy the Unique continuation property in the sense that the zero set of u_λ has empty interior ???

Thank you for your attention!

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