The second eigenfunction of the p–Laplacian on the disc is not radial (40–minutes talk)

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Abstract

In this talk we report on the joint result of the speaker, Jiří Benedikt and Petr Girg where we prove that the second eigenfunction of the p-Laplacian, $p > 1$, on the disc is not radial. Our proof is a combination of asymptotic analysis for $p \to +\infty$ and the application of interval arithmetic.

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In

Benedikt, J., Drábek, P. and Girg, P., *The second eigenfunction of the p–Laplacian on the disc is not radial*, Nonlinear Anal. 75 (2012), 4422–4435

we combine analytic and computer aided rigorous mathematical proofs.

Eigenvalue problem and the first eigenvalue

Let $D \subset \mathbb{R}^2$ be the open unit disc centered at the origin. We consider the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is the p-Laplacian, $p > 1$, and λ is the spectral parameter.

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The existence of sign changing radial eigenfunctions associated with higher eigenvalues was shown in many papers.

Radial solutions and the second eigenvalue

Note that the radial eigenfunctions of (1) are determined by nonzero solutions $u = u(r)$ of the ordinary differential equation

$$-\left(r|u'|^{p-2}u'\right)' = \mu r|u|^{p-2}u \quad \text{in } (0,1) \tag{2}$$

subject to the boundary conditions

$$u'(0) = 0, \quad u(1) = 0. \tag{3}$$

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It is also well-known that there is the second eigenvalue of (1), $\lambda_2 > \lambda_1$. There are no eigenvalues of (1) in (λ_1, λ_2), and an eigenfunction associated with λ_2 has exactly two nodal domains in D.

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Note that the structure of the set of all eigenvalues of the p-Laplacian ($p \neq 2$) beyond λ_2 seems to be an interesting open problem.

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For the case $p = 2$, this fact follows from Payne, L. E., *On two conjectures in the fixed membrane eigenvalue problem*, J. Appl. Math. Physics (ZAMP) 24 (1973) and/or the Fourier method for the Laplacian on a disc. In this paper we present a different argument to prove this fact and generalize it for arbitrary $p > 1$.
It is important to note that the result for p sufficiently close to 1 follows from Parini, E., *The second eigenvalue of the p-Laplacian as p goes to 1*, Int. J. Differential Equations 2010 (2010). The proof of Parini's Theorem 6.1 is based on Cheeger's inequality and implies that a second eigenfunction of (1) is not radial provided $1 < p < p_0$, where p_0 is sufficiently close to 1.

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However, the value of p_0 is not quantified in Parini.

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(Huang, Y. X., On the eigenvalues of the p-Laplacian with varying p, Proc. Amer. Math. Soc. 125, no. 11 (1997).)

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The set of the scalars μ such that (2), (3) admits a nontrivial solution, consists of an unbounded increasing sequence

$$0 < \mu_1(p) < \mu_2(p) < \cdots .$$

Moreover, for any $k \in \mathbb{N}$, the set of solutions of (2), (3) for $\mu = \mu_k(p)$ is a one-dimensional space spanned by a solution Φ_k of (2), (3) with exactly $k-1$ zeros in $(0, 1)$, all of them simple. Furthermore, $\mu_k = \mu_k(p)$ as a function of $p \in (1, +\infty)$ is continuous for each $k \in \mathbb{N}$.

(Del Pino, M., Manásevich, R., *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations 92, no. 2 (1991).)

Preliminaries

In particular, we have $\lambda_1 = \mu_1$. The corresponding positive eigenfunction of (1) is obtained by rotation of Φ_1 around the origin. Similarly, by rotation of Φ_2, we obtain the second radial eigenfunction of the p-Laplacian which changes the sign exactly once in D and which corresponds to the eigenvalue μ_2. Our Main Result can be thus restated as

$$\lambda_2(p) < \mu_2(p) \tag{4}$$

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As mentioned above the inequality holds for $p = 2$. Hence by the continuous dependence of $\lambda_2(p)$ and $\mu_2(p)$ on p it holds also for p close to 2.

Preliminaries

Consider now the eigenvalue problem on a general domain

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We set $\lambda_1 = \lambda_1(\Omega)$ to emphasize the dependence of the principal eigenvalue λ_1 on the domain Ω.

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Let Ω_i, $i = 1, 2$, be bounded domains, $\Omega_1 \subsetneq \Omega_2, |\Omega_1| < |\Omega_2|$. Then $\lambda_1(\Omega_2) < \lambda_1(\Omega_1)$.

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The proof follows directly from the variational characterization of the principal eigenvalue.

Preliminaries

Let us consider the initial value problem

$$\begin{cases} -\left(r|u'|^{p-2}u'\right)' = r|u|^{p-2}u \quad \text{in } (0,+\infty), \\ u(0)=1, \quad u'(0)=0. \end{cases} \tag{6}$$

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According to Lemma 5.2 in Del Pino, M., Manásevich, R., *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations 92, no. 2 (1991), (6) has a unique solution defined on $[0,+\infty)$ which we denote by $J_{0,p} = J_{0,p}(r)$. Moreover, Lemma 5.3 from above paper implies that $J_{0,p}$ is oscillatory with zeros $0 < \nu_1(p) < \nu_2(p) < \cdots \to +\infty$, and Lemma 5.1 from above paper claims that these zeros are simple. Clearly, $\mu_k(p) = (\nu_k(p))^p$, $k = 1, 2, \dots$, and $\Phi_k(r) = J_{0,p}(\nu_k(p)r)$, $r \in [0,1]$, is the corresponding solution of (2), (3) from previous slide which has $k-1$ zeros in $(0,1)$.

Bessel function and its first two zeros

For $p = 2$, the equation in (6) can be written as

$$u'' + \frac{1}{r}u' + u = 0, \quad r \neq 0,$$

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We have

$$\nu_1(2) = 2.4048\cdots, \quad \nu_2(2) = 5.5201\cdots, \quad \dots$$

For $p \neq 2$, $p \in (1, +\infty)$, the solution $J_{0,p} = J_{0,p}(r)$ of (6) can be thus regarded as a generalization of the Bessel function $J_0 \equiv J_{0,2}$.

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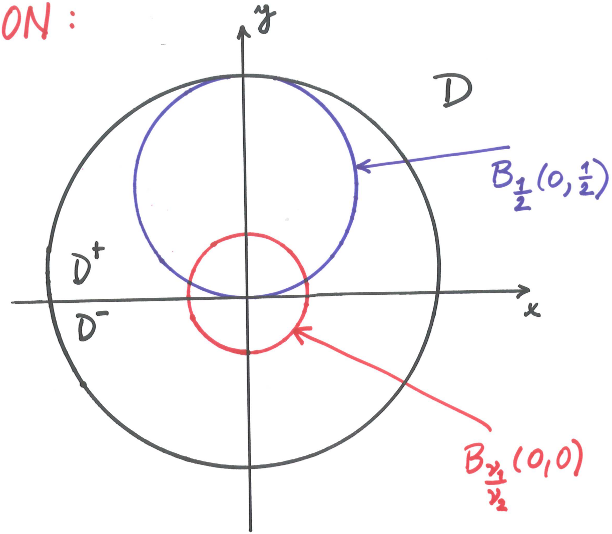
An observation that $2\nu_1(2) < \nu_2(2)$ appears to be important in the proof.

NOTATION :



$u^+ = u^+(x,y)$ positive eigenfunction associated with $\lambda_1(D^+)$

normalized by $\int_{D^+} |u^+|^p = 1$

$u^- = u^-(x,y) \stackrel{\text{def}}{=\!=} -u^+(x,-y)$

Extend u^+ (u^-) by zero to D^- (D^+) and set $u = u^+ + u^-$.

Fourier method $\Rightarrow$ u is the second eigenfunction on D for $p = 2$ $\Rightarrow$

$\lambda_1(D^+) = \lambda_1(D^-) = \lambda_2(D)$

? $p \neq 2$?

$$\mathcal{S} \stackrel{\text{def}}{=} \left\{ u \in W_0^{1,p}(D) : \int_D |u|^p = 1 \right\}$$

$$\mathcal{S}_1 \stackrel{\text{def}}{=} \left\{ u \in \mathcal{S} : u = \alpha u^+ + \beta u^-, \ |\alpha|^p + |\beta|^p = 1 \right\}$$

- $\mathcal{S}_1$ is homeomorphic to unit circle in R^2
- $\int_{D^\pm} |\nabla u^\pm|^p = \lambda_1(D^+) \implies$

$$\forall u \in \mathcal{S}_1 : \int_D |\nabla u|^p = |\alpha|^p \int_{D^+} |\nabla u^+|^p + |\beta|^p \int_{D^-} |\nabla u^-|^p$$

$$= \lambda_1(D^+)$$

$$\lambda_2 = \inf_{\mathcal{A} \in \mathcal{F}_2} \sup_{u \in \mathcal{A}} \int_D |\nabla u|^p$$

$\mathcal{F}_2$ is family of all sets in $\mathcal{S}$ which are homeomorphic to the unit circle in $\mathbb{R}^2$

$$\Rightarrow \quad \lambda_2 \leqq \int_D |\nabla u|^p = \lambda_1(D^+)$$

$$\forall u \in \mathcal{S}_1$$

Monotone dependence of $\lambda_1(\Omega)$ on the domain yields:

$$\lambda_1(D^+) < \lambda_1\left(B_{\frac{1}{2}}\left(0, \tfrac{1}{2}\right)\right)$$

Hence

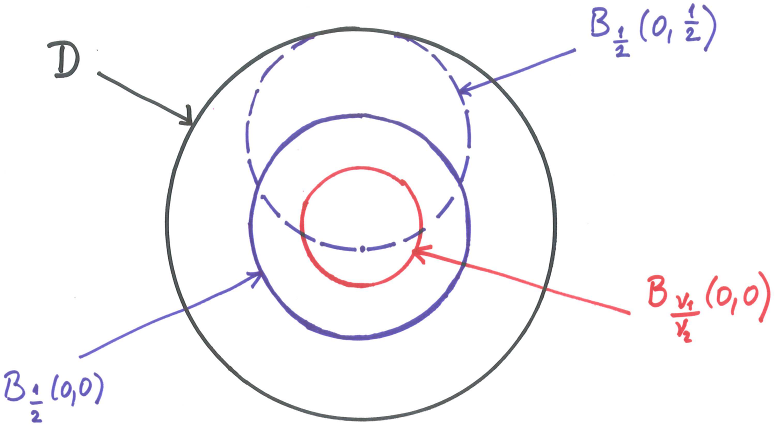
$$\lambda_2(D) \leqq \lambda_1(D^+) < \lambda_1\left(B_{\frac{1}{2}}\left(0, \tfrac{1}{2}\right)\right)$$

The second radial eigenfunction restricted to the ball $B_{\frac{\nu_1}{\nu_2}}(0,0)$ is the first eigenfunction for this ball:

$$\mu_2 = \lambda_1\left(B_{\frac{\nu_1}{\nu_2}}(0,0)\right)$$

If $\boxed{2\nu_1 < \nu_2}$ then monotone dependence of λ_1 on the domain and translation invariance of Δ_p implies

$$\lambda_1\left(B_{\frac{1}{2}}(0,\tfrac{1}{2})\right) = \lambda_1\left(B_{\frac{1}{2}}(0,0)\right) < \lambda_1\left(B_{\frac{\nu_1}{\nu_2}}(0,0)\right)$$



$$\lambda_2(D) \leq \lambda_1(D^+) < \lambda_1\left(B_{\frac{1}{2}}\left(0,\tfrac{1}{2}\right)\right) = \lambda_1\left(B_{\frac{1}{2}}(0,0)\right) < \lambda_1\left(B_{\frac{\nu_1}{\nu_2}}(0,0)\right) = \mu_2(D)$$

Summarizing all inequalities:

$$\lambda_2 < \mu_2 .$$

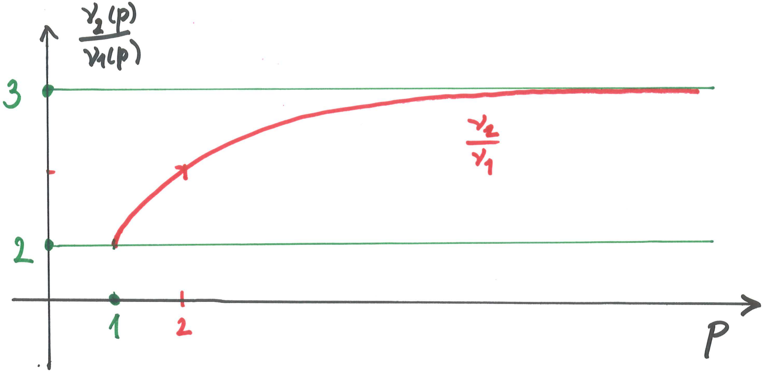
The key assumption was that

$$2\,\nu_1(p) < \nu_2(p).$$

This is well-known for $p = 2$.

Numerical simulations (not the proof!)

indicate :



Our proof consists of 3 steps:

1. We quantify the statement

   "p is close to 1"

   from Parini's paper and show that λ_2 is not radial provided

$$\boxed{1 < p \leqq 1.01}$$

2. We give an estimate for $\gamma_1(p)$ and $\gamma_2(p)$, the first two zeros of the solution

$$\begin{cases} -(r|u'|^{p-2}u')' = r|u|^{p-2}u & \text{in } (0,+\infty), \\ u(0) = 1, \ u'(0) = 0. \end{cases}$$

From an analytic estimate follows that $2\nu_1(p) < \nu_2(p)$ provided

$$\boxed{p > 226}$$

Actually, we show that $\frac{\nu_2}{\nu_1} \to 3_-$ as $p \to +\infty$.

3. We use an interval arithmetic to show that $2\nu_1(p) < \nu_2(p)$ provided

$$\boxed{p \in [1.01, 226].}$$

Ad. 1. To quantify the statement "p is close to 1"

reduces to finding suitable function $v \in W_0^{1,p}(D^+)$ such that

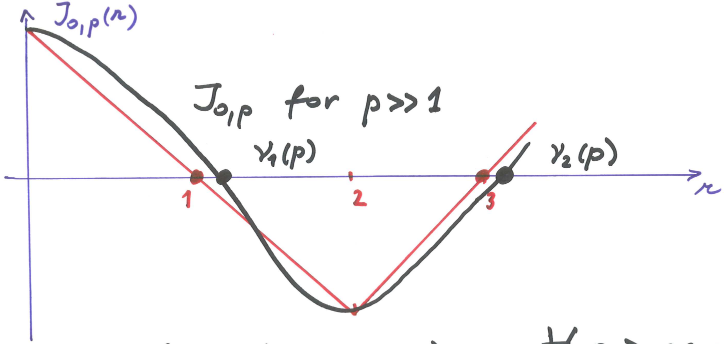
$$\frac{\int_{D^+} |\nabla v|^p}{\int_{D^+} |v|^p} \leq 3.5 .$$

We found a piecewise linear v which satisfies the inequality above.

Ad. 2.

$J_{0,p}$ solution of

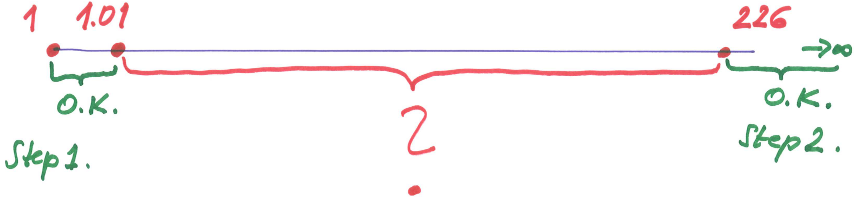
$$\begin{cases} -\left(r|u'|^{p-2}u'\right)' = r|u|^{p-2}u & \text{in } (0,+\infty), \\ u(0)=1,\ u'(0)=0 \end{cases}$$



$$2\nu_1(p) < \nu_2(p) \qquad \forall\, p \geqq 226$$

Ad. 3. We are left with compact interval

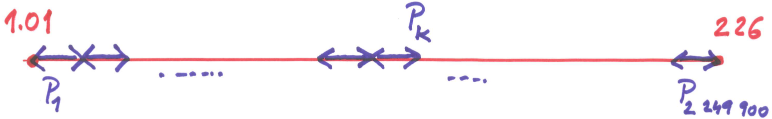
$$p \in [1.01, 226]$$



We cover $[1.01, 226]$ by finite number of subintervals

$$P_k \overset{\text{def}}{=} [1.0099 + 0.0001k,\ 1.01 + 0.0001k]$$

$$k \in \{1, 2, \dots, 2\,249\,900\}$$



The idea is to study discrete interval-valued dynamical system associated with

$$\begin{cases} -(r|u'|^{p-2}u')' = r|u|^{p-2}u & \text{in } (0,\infty), \\ u(0) = 1,\ u'(0) = 0. \end{cases}$$

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