Global bifurcation and stability for the asymptotically linear NLS

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The asymptotically linear NLS

We consider the nonlinear Schrödinger equation

\[ i\partial_t \psi + \Delta \psi + f(x, |\psi|^2)\psi = 0 \tag{NLS} \]

for \( \psi = \psi(t, x) : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}, \ N \geq 1. \)
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For simplicity, we suppose that \( f \) has the form

\[ f(x, s^2) = V(x) \frac{s^{p-1}}{1 + s^{p-1}}, \quad x \in \mathbb{R}^N, \ s \geq 0 \]

for some \( V \in C(\mathbb{R}^N, \mathbb{R}_+) \cap L^\infty(\mathbb{R}^N, \mathbb{R}_+) \) and \( p > 1. \)
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Then (NLS) is asymptotically linear:

\[ f(x, s^2) \to V(x) \quad \text{as} \quad s \to \infty \]
Standing waves

\[ i\partial_t \psi + \Delta \psi + f(x, |\psi|^2)\psi = 0 \quad \text{(NLS)} \]

A standing wave is a solution of the form

\[ \psi(t, x) = e^{i\lambda t} u(x) \quad \text{with} \quad \lambda \in \mathbb{R} \quad \text{and} \quad u \in H^1(\mathbb{R}^N) \]
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1. global bifurcation of solutions \((\lambda, u_\lambda)\) of (SNLS);
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Main objectives:

1. global bifurcation of solutions \((\lambda, u_\lambda)\) of \((\text{SNLS})\);
2. orbital stability of the standing waves \(e^{i \lambda t} u_\lambda(x)\) of \((\text{NLS})\).
Asymptotic bifurcation

Consider the asymptotic linearization of (SNLS):

\[ \Delta u + V(x)u = \lambda u, \quad u \in H^1(\mathbb{R}^N) \quad \text{(AL)} \]

Let \( \lambda^* := \limsup_{|x| \to \infty} V(x) \in [0, \infty) \)

and suppose that \( \lambda_{\infty} := -\inf_{u \in H^1(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla u|^2 - V(x)u^2 \, dx / \int_{\mathbb{R}^N} u^2 \, dx > \lambda^* \)

Then \( \sigma_{\text{ess}}(\text{AL}) \subset (-\infty, \lambda^*] \) and \( \lambda_{\infty} \) is the principal eigenvalue of (AL).
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Then

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and \( \lambda_\infty \) is the principal eigenvalue of (AL).
\[ N = 1, \ V(x) > \lambda_* = \lim_{|x| \to \infty} V(x) \]
Let $X := W^{2,q}(\mathbb{R}^N)$ and $Y := L^q(\mathbb{R}^N)$, $q \in [2, \infty) \cap (\frac{N}{2}, \infty)$. Denote by $\| \cdot \|$ the usual norm of $X$. 
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Rewriting (SNLS) as

$$\Delta u + V(x)u + g(x, u)u = \lambda u \quad \text{with} \quad g(x, s) := f(x, s^2) - V(x)$$
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and performing the inversion $u \rightarrow v := u/\|u\|^2$ we get

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Bifurcation for (\( \star \)) from \( v = 0 \) at the eigenvalue \( \lambda_\infty \) of (AL) should yield asymptotic bifurcation for (SNLS).
Let $X := W^{2,q}(\mathbb{R}^N)$ and $Y := L^q(\mathbb{R}^N)$, $q \in [2, \infty) \cap (\frac{N}{2}, \infty)$.
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Bifurcation for $(\star)$ from $v = 0$ at the eigenvalue $\lambda_\infty$ of (AL) should yield asymptotic bifurcation for (SNLS).

Defining it to be zero for $v = 0$, the map $v \mapsto g(\cdot, v/\|v\|^2)v$ is continuous from $X$ to $Y$. 

The asymptotically linear NLS Flagstaff June 2012 5
Unfortunately, (AL) is not the linearization of (⋆) stricto sensu because $v \mapsto g(\cdot, v/\|v\|^2)v$ is not differentiable at $v = 0$. However, the truncation $(\Delta + V - \lambda) v + \chi\{|x| \leq n\} g(x, v/\|v\|^2)v = 0$ (⋆n) is linearizable at $v = 0$, with linearization (AL).

A global bifurcation theorem of Stuart & Zhou (2006) yields a connected set $C_n \subset \mathbb{R} \times X$ of positive solutions of (⋆n), bifurcating from $(\lambda_\infty, 0)$, for each $n \in \mathbb{N}$.

This global result is based on a recent degree theory of Rabier & Salter (2005), dealing with compact perturbations of Fredholm maps of index 0.
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Then going back to the original variables via the inversion yields

**Theorem 1**

There exists a connected set $S \subset \mathbb{R} \times X$ of positive solutions of (SNLS) with the following properties:

(i) $P S = (\lambda^*, \lambda^\infty)$, where $P (\lambda, u) := \lambda$.

(ii) $S$ is bounded away from $\mathbb{R} \times \{0\}$ in $\mathbb{R} \times X$.

(iii) If $\{ (\lambda_n, u_n) \} \subset S$ such that $\lambda_n \to \lambda$ as $n \to \infty$, then

$$\lim_{n \to \infty} |u_n|_{L^q(\mathbb{R}^N)} = \lim_{n \to \infty} |u_n|_{L^\infty(\mathbb{R}^N)} = \infty \iff \lambda = \lambda^\infty$$

This method goes back to Rabinowitz (1973) and Toland (1973).

Drabek, Takac, Girg etc. used it for quasilinear problems.
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(ii) \( S \) is bounded away from \( \mathbb{R} \times \{0\} \) in \( \mathbb{R} \times X \).
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This method goes back to Rabinowitz (1973) and Toland (1973). Drabek, Takac, Girg etc. used it for quasilinear problems.
The one-dimensional case

\[ u'' + f(x, u^2)u = \lambda u, \quad u \in H^1(\mathbb{R}) \]  

\[ f(x, s^2) = V(x) \frac{s^{p-1}}{1 + s^{p-1}} \]
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Assuming \( V \) even and decreasing on \([0, \infty)\), the positive solution \((\lambda, u_\lambda) \in S\) is unique, for all \( \lambda \in (\lambda_*, \lambda_\infty) \), with

\[ u_\lambda \text{ even} \quad \text{and} \quad u_\lambda' < 0 \text{ on } (0, \infty). \]
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\[ u_\lambda \text{ even and } u'_\lambda < 0 \text{ on } (0, \infty). \]

A compactness argument then shows that

\[ S = \{(\lambda, u_\lambda) : \lambda \in (\lambda_*, \lambda_\infty)\} \]

is a continuous curve.
The asymptotically linear NLS
Continuation down to $u = 0$

In addition we assume that

$$V(x) \sim |x|^{-b} \quad \text{as} \quad |x| \to \infty \quad \text{for some} \quad b \in (0, 1)$$

and $1 < p < 5 - 2b$. 
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In particular $\lambda_* = \lim_{|x| \to \infty} V(x) = 0$. 
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In particular $\lambda_* = \lim_{|x| \to \infty} V(x) = 0$.

Then a perturbative argument, based on a rescaling of (SNLS) as $\lambda \to 0$, shows that we have bifurcation from $(0, 0)$ in $\mathbb{R} \times H^1(\mathbb{R})$. 
The asymptotically linear NLS
A smooth curve of solutions

If we further assume that $V$ is $C^1$ and $V' < 0$ on $(0, \infty)$, then the linearized operator $T_{\lambda} : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$,

$$T_{\lambda} v := v'' + [f(x, u_{\lambda}^2) + 2 \partial_2 f(x, u_{\lambda}^2)u_{\lambda}^2]v - \lambda v$$

is an isomorphism, for any $\lambda \in (0, \lambda_{\infty})$. 
A smooth curve of solutions

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is an isomorphism, for any $\lambda \in (0, \lambda_\infty)$.

Hence, applying the implicit function theorem at each point $(\lambda, u_\lambda)$ shows that, in fact, we have a smooth curve of solutions, i.e.

$$(0, \lambda_\infty) \ni \lambda \mapsto u_\lambda \in H^1(\mathbb{R}) \quad \text{is} \quad C^1$$
Theorem 2
Suppose $V \in C^1(\mathbb{R})$ is even, $V' < 0$ on $(0, \infty)$, and

$$V(x) \sim |x|^{-b} \quad \text{as} \quad |x| \to \infty \quad \text{for some} \ b \in (0, 1)$$

with $1 < p < 5 - 2b$. 

In fact, $u_{\lambda} \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ with $u'_{\lambda} < 0$ on $(0, \infty)$, and $u_{\lambda}$ and $u'_{\lambda}$ approach 0 exponentially as $|x| \to \infty$. 

The asymptotically linear NLS Flagstaff June 2012 15
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with $1 < p < 5 - 2b$.

Then there exists $u \in C^1((0, \lambda_\infty), H^1(\mathbb{R}))$ such that $(\lambda, u_\lambda)$ is the unique positive even solution of (SNLS) for all $\lambda \in (0, \lambda_\infty)$, and

$$\lim_{\lambda \to 0} \| u_\lambda \|_{H^1} = 0, \quad \lim_{\lambda \to \lambda_\infty} \| u_\lambda \|_{H^1} = \infty$$
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$$\lim_{\lambda \to 0} \|u_\lambda\|_{H^1} = 0, \quad \lim_{\lambda \to \lambda_{\infty}} \|u_\lambda\|_{H^1} = \infty$$

In fact, $u_\lambda \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ with $u'_\lambda < 0$ on $(0, \infty)$, and $u_\lambda, u'_\lambda \to 0$ exponentially as $|x| \to \infty$. 
Orbital stability

We now go back to

\[ i \partial_t \psi + \partial_{xx}^2 \psi + f(x, |\psi|^2)\psi = 0 \]  

(NLS)

for which we have solutions \( \psi_\lambda(t, x) := e^{i\lambda t}u_\lambda(x), \lambda \in (0, \lambda_\infty). \)
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Definition 1

A standing wave \( \psi(t, x) = e^{i\lambda t} u(x) \) is orbitally stable iff

\[ \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. for any solution } \varphi \text{ of (NLS) we have} \]

\[ \| \varphi(0, \cdot) - u \|_{H^1} \leq \delta \implies \inf_{\theta \in \mathbb{R}} \| \varphi(t, \cdot) - e^{i\theta} u \|_{H^1} \leq \varepsilon \quad \forall \ t \geq 0 \]
Orbital stability

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**Definition 1**
A standing wave \( \psi(t, x) = e^{i\lambda t}u(x) \) is orbitally stable iff
\[ \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t. for any solution } \varphi \text{ of (NLS) we have} \]
\[ \| \varphi(0, \cdot) - u \|_{H^1} \leq \delta \implies \inf_{\theta \in \mathbb{R}} \| \varphi(t, \cdot) - e^{i\theta}u \|_{H^1} \leq \varepsilon \ \forall t \geq 0 \]

Roughly speaking: \( \varphi(0, \cdot) \) close to \( u \implies \varphi(t, \cdot) \) close to the orbit \( \Theta(\psi) := \{ e^{i\theta}u : \theta \in \mathbb{R} \} \ \forall t \geq 0 \).
In our setting, under appropriate conditions on the spectrum of the linearization of (NLS) at the standing wave $\psi_\lambda$, it follows from Grillakis-Shatah-Strauss (1987) that

$$\psi_\lambda \text{ orbitally stable } \iff \frac{d}{d\lambda} \int_{\mathbb{R}} u_\lambda^2 \, dx > 0$$
The slope condition

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\psi_\lambda \text{ orbitally stable } \iff \frac{d}{d\lambda} \int_{\mathbb{R}} u_{\lambda}^2 \, dx > 0
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Since \( |u_{\lambda}|_{L^2} \rightarrow 0 \) as \( \lambda \rightarrow 0 \) by Theorem 2, there is a small \( \lambda > 0 \) where the condition is verified. Hence we need only check that

\[
\frac{d}{d\lambda} \int_{\mathbb{R}} u_{\lambda}^2 \, dx \neq 0 \quad \forall \lambda \in (0, \lambda_\infty)
\]
Remarking that

\[ \frac{d}{d\lambda} \int_{\mathbb{R}} u_{\lambda}^2 \, dx = 2 \int_{\mathbb{R}} u_{\lambda} \frac{d}{d\lambda} u_{\lambda} \, dx \]

this follows by some integral identities derived from (SNLS) and the equation satisfied by \( \xi := \frac{d}{d\lambda} u_{\lambda} \):

\[ \xi'' + [f(x, u_{\lambda}^2) + 2 \partial_2 f(x, u_{\lambda}^2) u_{\lambda}^2] \xi = \lambda \xi + u_{\lambda} \]
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**Theorem 3**

*Under the hypotheses of Theorem 2,*

\[ \frac{d}{d\lambda} \int_{\mathbb{R}} u_\lambda^2 \, dx > 0 \quad \forall \lambda \in (\lambda_\ast, \lambda_\infty) \]

*In particular, the standing wave \( \psi_\lambda \) is stable for all \( \lambda \in (\lambda_\ast, \lambda_\infty) \).*
Previous results

A global branch of positive solutions was obtained by H. Jeanjean & Stuart (1999) for

$$u'' + q(|x|)u + f(|x|, u^2)u = \lambda u, \quad x \in \mathbb{R}$$

The linear potential $q$ allows one to get bifurcation from the principal eigenvalue of the linearization $u'' + q(|x|)u = \lambda u$ by the Crandall-Rabinowitz theorem.

Orbital stability along the branch was discussed via the slope condition in McLeod, Stuart & Troy (2003) but they were not able to deal with the asymptotically linear case.
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Open problems

- Stability/instability in the presence of a linear potential $q$.

- Continuation down to $(\lambda, u) = (0, 0)$ in case $\lambda^* = \lim_{|x| \to \infty} V(x) > 0$.

  The implicit function theorem allows one to extend the branch down to $\lambda = 0$ but the behaviour of $\|u_\lambda\|$ as $\lambda \to 0$ is not clear yet.

- Variational characterizations of positive solutions are available, see e.g. Costa & Tehrani (2001). This might give useful informations.

- Higher dimensions.

  Lack of uniqueness results for the continuation down to $u = 0$.

  Local bifurcation/stability can be proved at $\lambda = 0$ but global stability seems very challenging. Even the radial case is hard due to the term $N - 1 r u'$ in the equation.
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