Global bifurcation and stability for the asymptotically linear NLS

François Genoud
Heriot-Watt University
Edinburgh

Northern Arizona University
Flagstaff
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The asympotically linear NLS

We consider the nonlinear Schrödinger equation

$$i\partial_t\psi + \Delta\psi + f(x, |\psi|^2)\psi = 0 \tag{NLS}$$

for $\psi = \psi(t, x) : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}$, $N \geqslant 1$.

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For simplicity, we suppose that f has the form

$$f(x, s^2) = V(x)\frac{s^{p-1}}{1 + s^{p-1}}, \quad x \in \mathbb{R}^N,\ s \geqslant 0$$

for some $V \in C(\mathbb{R}^N, \mathbb{R}_+) \cap L^\infty(\mathbb{R}^N, \mathbb{R}_+)$ and $p > 1$.

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Then (NLS) is asympotically linear:

$$f(x, s^2) \to V(x) \quad \text{as} \quad s \to \infty$$

Standing waves

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A standing wave is a solution of the form

$$\psi(t,x) = \mathrm{e}^{i\lambda t}u(x) \quad \text{with} \quad \lambda \in \mathbb{R} \quad \text{and} \quad u \in H^1(\mathbb{R}^N)$$

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1. global bifurcation of solutions (λ, u_λ) of (SNLS);
2. orbital stability of the standing waves $\mathrm{e}^{i\lambda t}u_\lambda(x)$ of (NLS).

Asymptotic bifurcation

Consider the asymptotic linearization of (SNLS):

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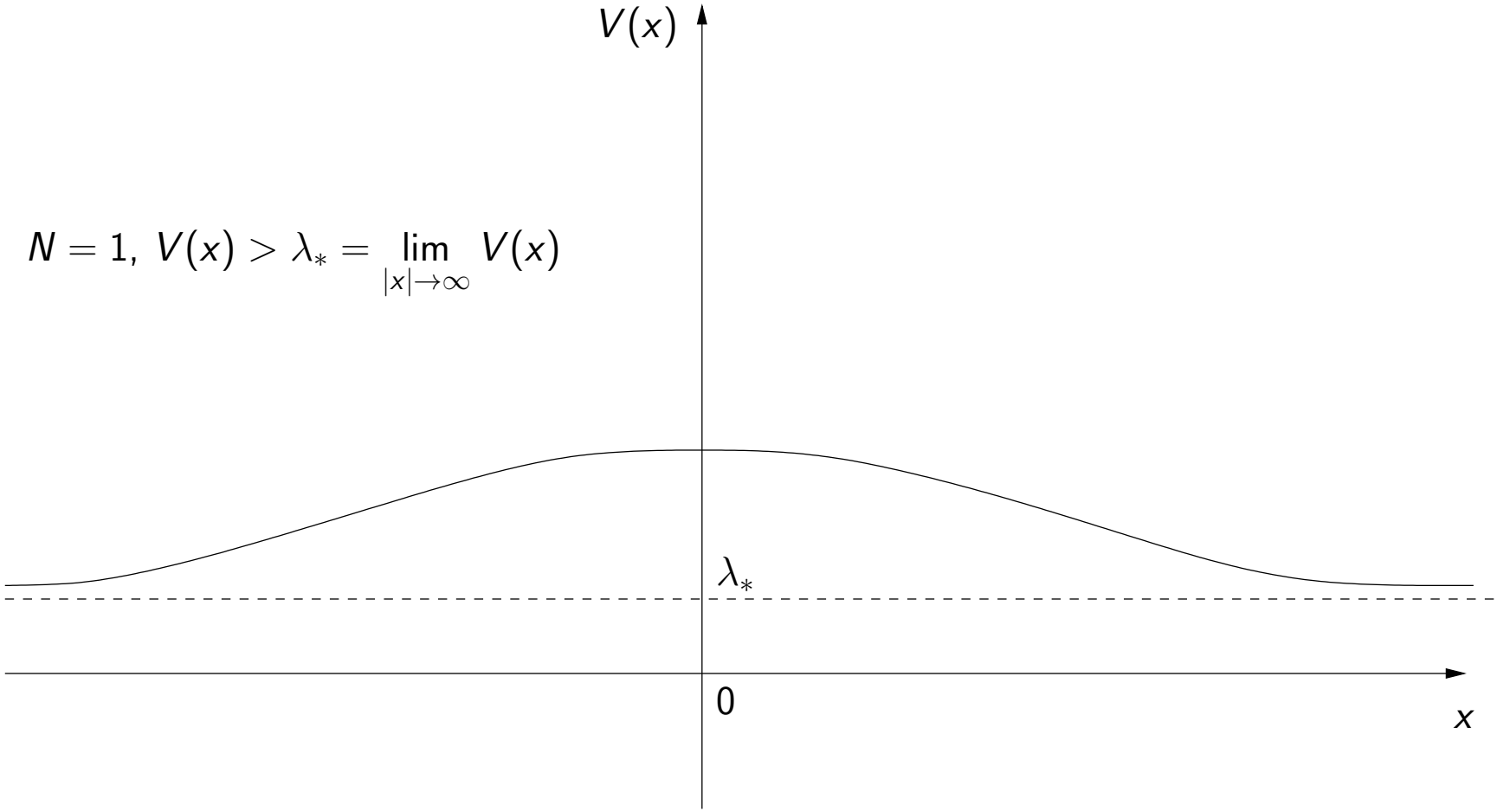
and λ_∞ is the principal eigenvalue of (AL).

$N = 1,\ V(x) > \lambda_* = \lim_{|x|\to\infty} V(x)$



Let $X := W^{2,q}(\mathbb{R}^N)$ and $Y := L^q(\mathbb{R}^N)$, $q \in [2, \infty) \cap (\frac{N}{2}, \infty)$.

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Defining it to be zero for $v = 0$, the map $v \mapsto g(\cdot, v/\|v\|^2)v$ is continuous from X to Y.

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However, the truncation

$$(\Delta + V - \lambda)v + \chi_{\{|x| \leqslant n\}} g(x, v/\|v\|^2)v = 0 \qquad (\star_n)$$

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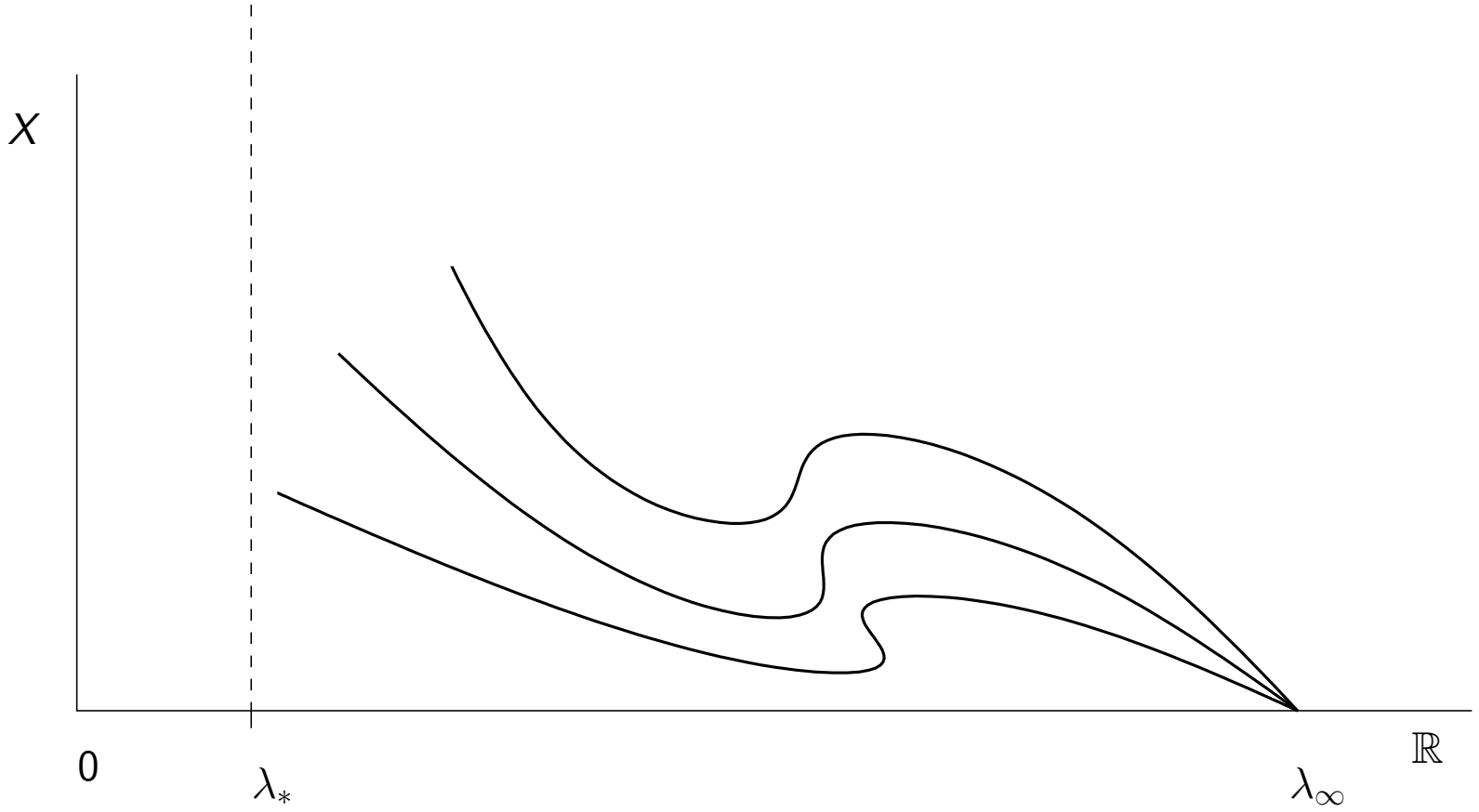
This global result is based on a recent degree theory of Rabier & Salter (2005), dealing with compact perturbations of Fredholm maps of index 0.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Then going back to the original variables via the inversion yields

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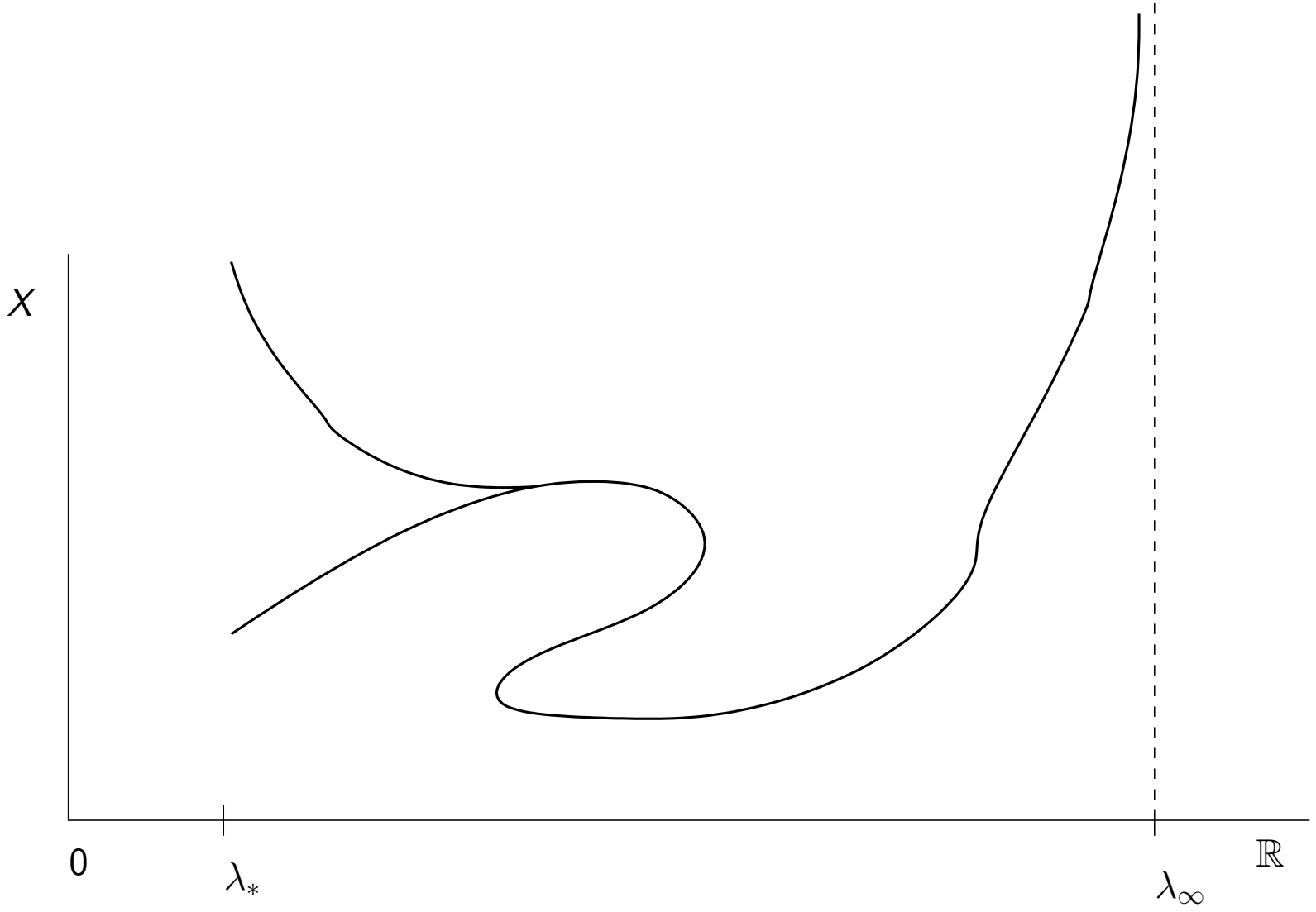
This method goes back to Rabinowitz (1973) and Toland (1973). Drabek, Takac, Girg etc. used it for quasilinear problems.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The one-dimensional case

$$u'' + f(x, u^2)u = \lambda u, \quad u \in H^1(\mathbb{R}) \tag{SNLS}$$

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Assuming V even and decreasing on $[0, \infty)$, the positive solution $(\lambda, u_\lambda) \in \mathcal{S}$ is unique, for all $\lambda \in (\lambda_*, \lambda_\infty)$, with

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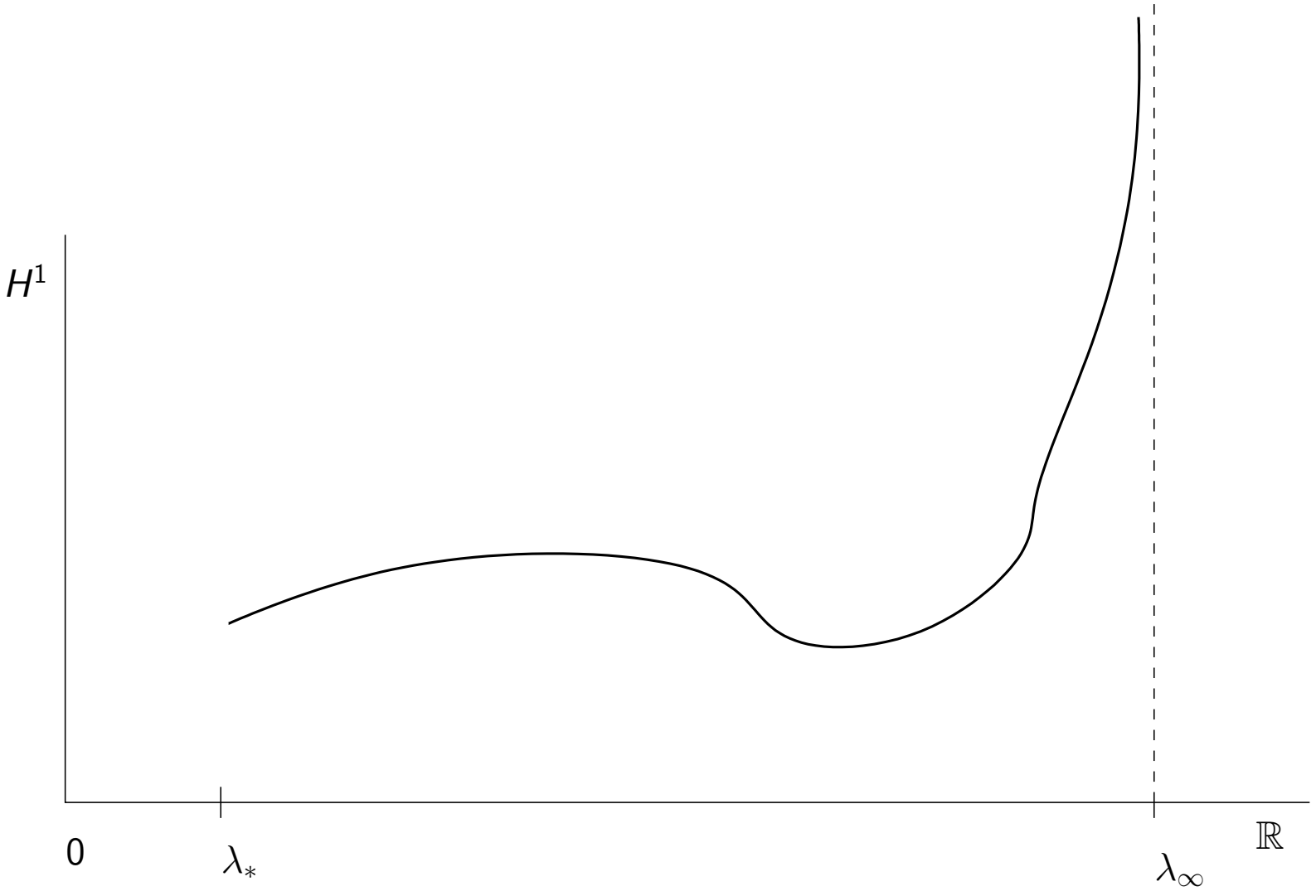
A compactness argument then shows that

$$\mathcal{S} = \{(\lambda, u_\lambda) : \lambda \in (\lambda_*, \lambda_\infty)\}$$

is a continuous curve.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Continuation down to $u = 0$

In addition we assume that

$$V(x) \sim |x|^{-b} \quad \text{as} \quad |x| \to \infty \quad \text{for some } b \in (0,1)$$

and $1 < p < 5 - 2b$.

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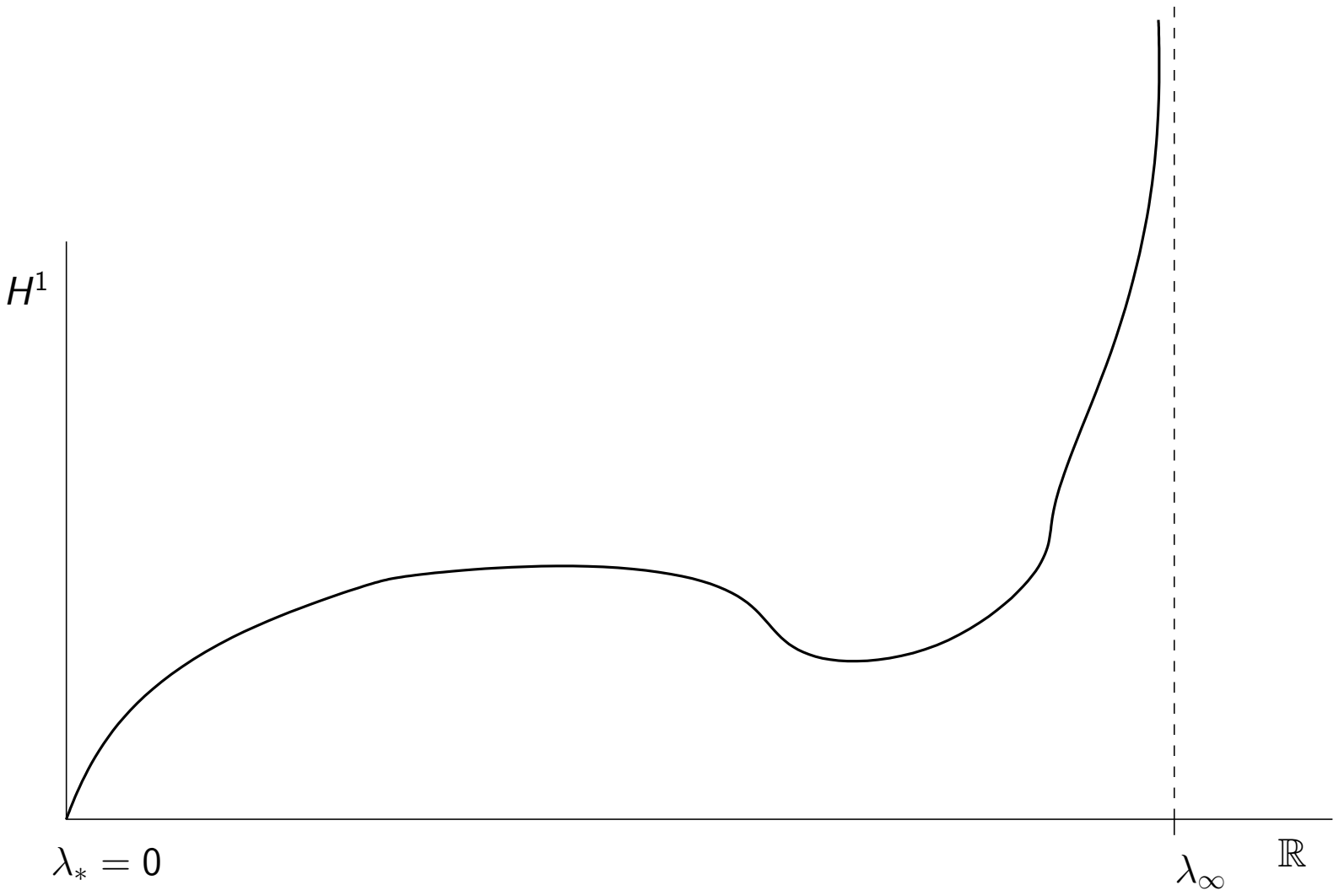
Then a perturbative argument, based on a rescaling of (SNLS) as $\lambda \to 0$, shows that we have bifurcation from $(0,0)$ in $\mathbb{R} \times H^1(\mathbb{R})$.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A smooth curve of solutions

If we further assume that V is C^1 and $V' < 0$ on $(0, \infty)$, then the linearized operator $T_\lambda : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$,

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Hence, applying the implicit function theorem at each point (λ, u_λ) shows that, in fact, we have a smooth curve of solutions, i.e.

$$(0, \lambda_\infty) \ni \lambda \mapsto u_\lambda \in H^1(\mathbb{R}) \quad \text{is } C^1$$

Theorem 2

Suppose $V \in C^1(\mathbb{R})$ *is even,* $V' < 0$ *on* $(0, \infty)$*, and*

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Then there exists $u \in C^1((0, \lambda_\infty), H^1(\mathbb{R}))$ *such that* (λ, u_λ) *is the unique positive even solution of* (SNLS) *for all* $\lambda \in (0, \lambda_\infty)$*, and*

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In fact, $u_\lambda \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ *with* $u_\lambda' < 0$ *on* $(0, \infty)$*, and* $u_\lambda, u_\lambda' \to 0$ *exponentially as* $|x| \to \infty$*.*

Orbital stability

We now go back to

$$i\partial_t \psi + \partial_{xx}^2 \psi + f(x, |\psi|^2)\psi = 0 \qquad \text{(NLS)}$$

for which we have solutions $\psi_\lambda(t,x) := \mathrm{e}^{i\lambda t} u_\lambda(x)$, $\lambda \in (0, \lambda_\infty)$.

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A standing wave $\psi(t,x) = \mathrm{e}^{i\lambda t}u(x)$ is orbitally stable iff
$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. for any solution φ of (NLS) we have

$$\|\varphi(0,\cdot) - u\|_{H^1} \leqslant \delta \implies \inf_{\theta\in\mathbb{R}} \|\varphi(t,\cdot) - \mathrm{e}^{i\theta}u\|_{H^1} \leqslant \varepsilon \quad \forall\, t \geqslant 0$$

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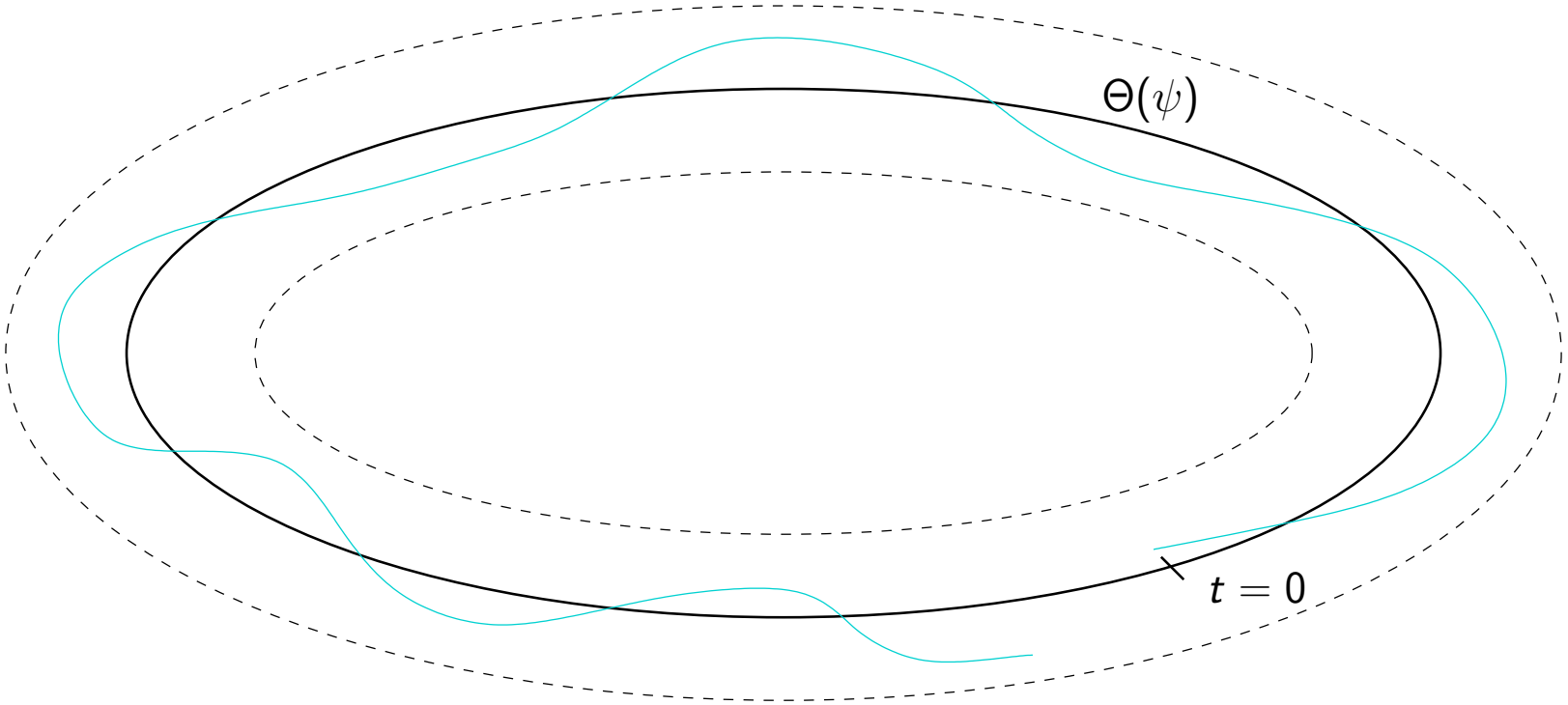
Roughly speaking: $\varphi(0,\cdot)$ close to $u \implies$

$\varphi(t,\cdot)$ close to the orbit $\Theta(\psi) := \{\mathrm{e}^{i\theta}u : \theta \in \mathbb{R}\}\ \forall\, t \geqslant 0$.


$\Theta(\psi)$
$t = 0$

The slope condition

In our setting, under appropriate conditions on the spectrum of the linearization of (NLS) at the standing wave ψ_λ, it follows from Grillakis-Shatah-Strauss (1987) that

$$\psi_\lambda \text{ orbitally stable} \iff \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda^2 \,\mathrm{d}x > 0$$

The slope condition

In our setting, under appropriate conditions on the spectrum of the linearization of (NLS) at the standing wave ψ_λ, it follows from Grillakis-Shatah-Strauss (1987) that

$$\psi_\lambda \text{ orbitally stable} \iff \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda^2 \,\mathrm{d}x > 0$$

Since $|u_\lambda|_{L^2} \to 0$ as $\lambda \to 0$ by Theorem 2, there is a small $\lambda > 0$ where the condition is verified. Hence we need only check that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda^2 \,\mathrm{d}x \neq 0 \quad \forall\, \lambda \in (0, \lambda_\infty)$$

Remarking that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda^2\,\mathrm{d}x = 2\int_{\mathbb{R}} u_\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} u_\lambda\,\mathrm{d}x$$

this follows by some integral identities derived from (SNLS) and the equation satisfied by $\xi := \frac{\mathrm{d}}{\mathrm{d}\lambda} u_\lambda$:

$$\xi'' + [f(x, u_\lambda^2) + 2\partial_2 f(x, u_\lambda^2) u_\lambda^2]\,\xi = \lambda\xi + u_\lambda$$

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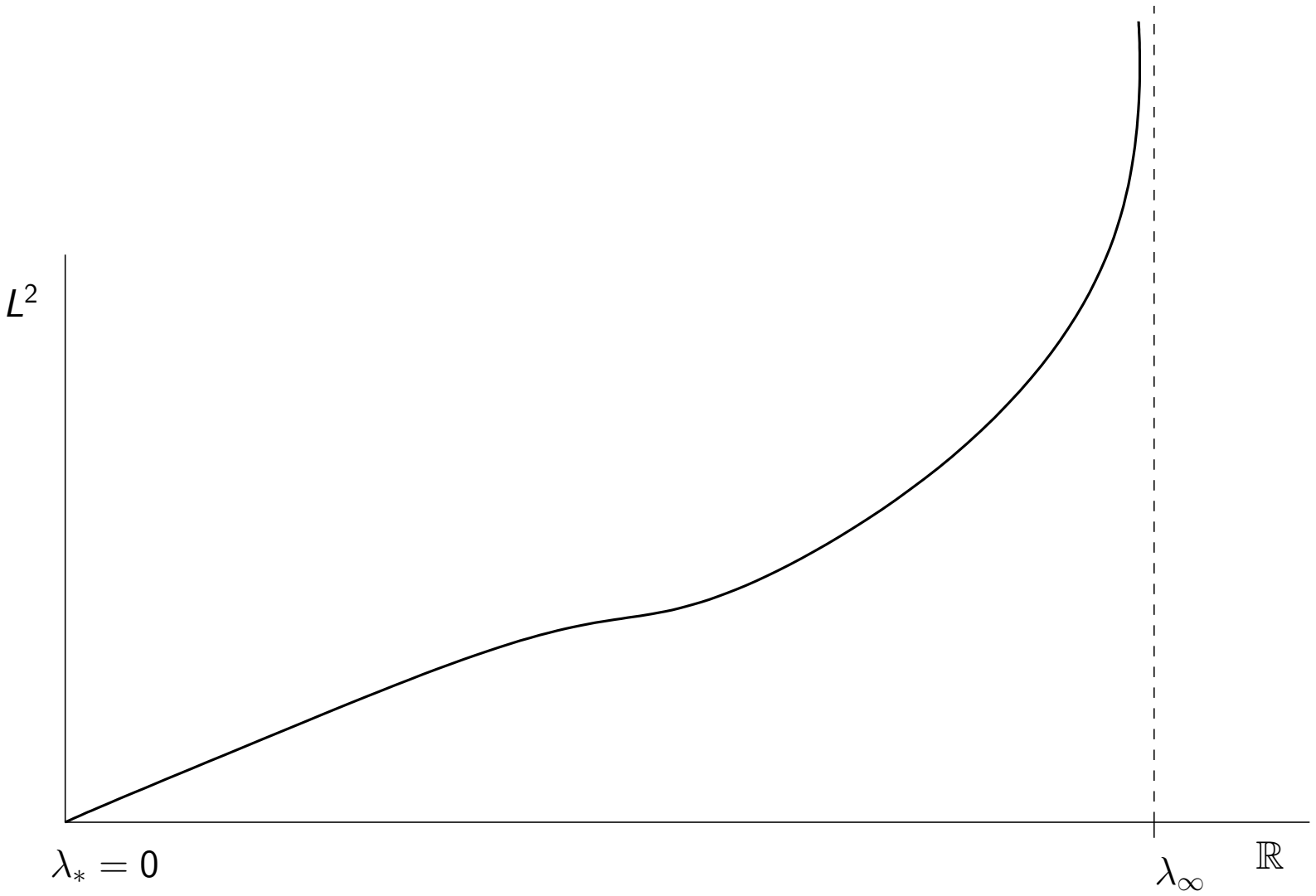
Theorem 3

Under the hypotheses of Theorem 2,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda^2\,\mathrm{d}x > 0 \quad \forall\,\lambda \in (\lambda_*, \lambda_\infty)$$

In particular, the standing wave ψ_λ is stable for all $\lambda \in (\lambda_*, \lambda_\infty)$.


L^2
$\lambda_* = 0$
λ_∞
$\mathbb{R}$

Previous results

A global branch of positive solutions was obtained by H. Jeanjean & Stuart (1999) for

$$u'' + q(|x|)u + f(|x|, u^2)u = \lambda u, \quad x \in \mathbb{R}$$

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The linear potential q allows one to get bifurcation from the principal eigenvalue of the linearization $u'' + q(|x|)u = \lambda u$ by the Crandall-Rabinowitz theorem.

Orbital stability along the branch was discussed via the slope condition in McLeod, Stuart & Troy (2003) but they were not able to deal with the asymptotically linear case.

Open problems

- Stability/instability in the presence of a linear potential q.

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Local bifurcation/stability can be proved at $\lambda = 0$ but global stability seems very challenging. Even the radial case is hard due to the term $\frac{N-1}{r} u'$ in the equation.