Bifurcation along curves for the p-Laplacian in the unit ball

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The radial Dirichlet problem

We look for radial solutions of the Dirichlet problem

$$\begin{cases} -\Delta_p(u) = \lambda f(|x|, u) & \text{for } |x| \leqslant 1 \\ u = 0 & \text{for } |x| = 1 \end{cases}$$

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where $p > 2$, $\lambda > 0$, and $f \in C^1([0,1] \times \mathbb{R})$.

This amounts to solving

$$\begin{cases} -(r^{N-1}\phi_p(u'))' = \lambda r^{N-1} f(r,u), & 0 < r < 1 \\ u'(0) = u(1) = 0 \end{cases} \tag{P}$$

where $\phi_p(\xi) := |\xi|^{p-2}\xi,\ \xi \in \mathbb{R}$.

Our main assumptions are the following:

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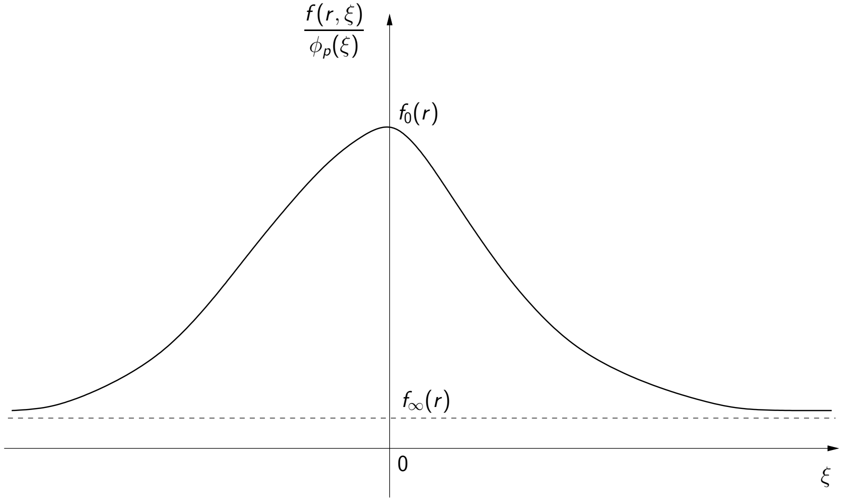
- there exist $f_0, f_\infty \in C^0[0,1]$ such that

$$\lim_{\xi \to 0} \frac{f(r,\xi)}{\phi_p(\xi)} = f_0(r) > 0 \quad \text{and} \quad \lim_{|\xi| \to \infty} \frac{f(r,\xi)}{\phi_p(\xi)} = f_\infty(r) > 0$$

uniformly for $r \in [0,1]$.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0
ξ

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Since

$$f_\infty(r) \leqslant \frac{f(r,\xi)}{\phi_p(\xi)} \leqslant f_0(r)$$

it follows by comparison principles in Walter (1998) that

$$\lambda_0 \leqslant \lambda \leqslant \lambda_\infty$$

for any positive/negative solution (λ, u) of (P).

Main result

With the above hypotheses and $p > 2$ we have

Theorem 1

There exist C^1 curves $u_\pm : (\lambda_0, \lambda_\infty) \to C^0[0,1]$ such that $u_\pm(\lambda)$ is the only positive/negative solution of (P)*, for all $\lambda \in (\lambda_0, \lambda_\infty)$.*

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Furthermore,

$$\lim_{\lambda \to \lambda_0} |u_\pm(\lambda)|_0 = 0 \quad \textit{and} \quad \lim_{\lambda \to \lambda_\infty} |u_\pm(\lambda)|_0 = \infty$$

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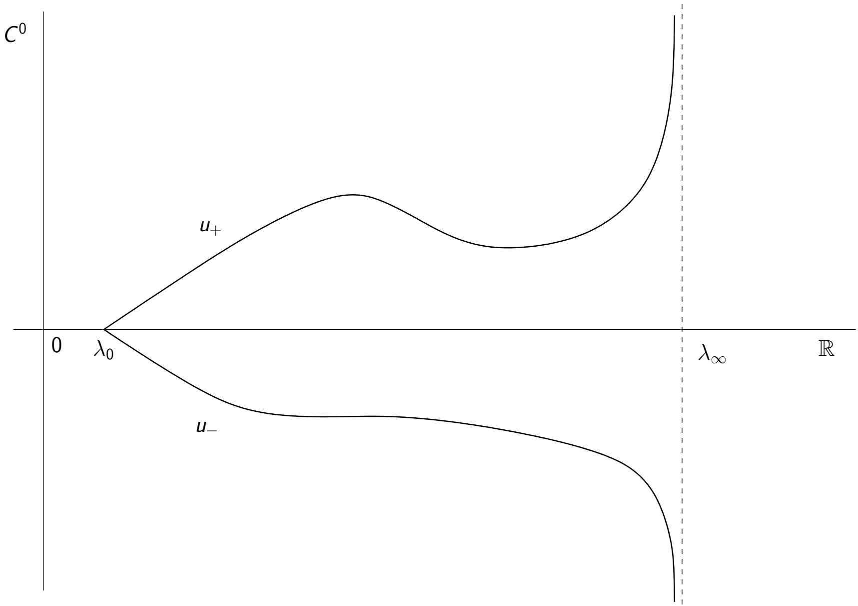
Of course, $u_\pm(\lambda) \in C^1[0,1]$ and $\phi_p(u_\pm(\lambda)') \in C^1[0,1]$.


C^0
u_+
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0
λ_0
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Proof

$$\begin{cases} -(r^{N-1}\phi_p(u'))' = r^{N-1}\,\lambda f(r,u), & 0 < r < 1 \\ \quad u'(0) = u(1) = 0 \end{cases} \tag{P}$$

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Letting $p' = \frac{p}{p-1}$ and defining $S_p : C^0[0,1] \to C^1[0,1]$ by

$$S_p(h)(r) := \int_r^1 \phi_{p'}\left(\int_0^s \left(\frac{t}{s}\right)^{N-1} h(t)\,\mathrm{d}t\right)\mathrm{d}s, \quad r \in [0,1]$$

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problem (P) is equivalent to

$$u = S_p(\lambda\, f(u)), \quad u \in C^0[0,1]$$

where $f(u)(r) \equiv f(r, u(r))$ (Nemitskii mapping).

Local bifurcation

Local bifurcation from $(\lambda_0, 0)$ for

$$F(\lambda, u) := u - S_p(\lambda f(u)) = 0$$

follows by a Crandall-Rabinowitz type of result stated in García-Melián & Sabina de Lis (2002).

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Then $v_0 > 0$ in $[0, 1)$ $\rightsquigarrow$ positive solutions for $s \in (0, \epsilon)$

negative solutions for $s \in (-\epsilon, 0)$

How does it work?

As in the $p = 2$ case solved by Crandall & Rabinowitz (1971), one applies the implicit function theorem to a function

$$\widetilde{F}(s, \lambda, z) := F(\lambda, s(v_0 + z))/s \quad \text{(and ext. by continuity at } s = 0\text{)}$$

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Here, $s \in \mathbb{R}$ and $(\lambda, z) \in \mathbb{R} \times Z$, with

$$Z := \{z \in C^0[0,1] : \int_0^1 r^{N-1} f_0(r) |v_0|^{p-2} v_0 \, z \, \mathrm{d}r = 0\}$$

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Note that

$$\mathrm{span}\{v_0\} \oplus Z = C^0[0,1]$$

The non-degeneracy condition in the implicit function theorem boils down to checking that the mapping

$$z \mapsto DS_p\big(\lambda_0\, f_0\, \phi_p(v_0)\big)\, f_0\, |v_0|^{p-2}\, z \qquad (\star)$$

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And this follows from the explicit form of $(\star)$ by an integration by parts argument, provided S_p is differentiable at $\lambda_0\, f_0\, \phi_p(v_0)$.

Differentiability

Recall: $p' = \frac{p}{p-1}$, $S_p : C^0[0,1] \to C^1[0,1]$,

$$S_p(h)(r) := \int_r^1 \phi_{p'} \left(\int_0^s \left(\frac{t}{s} \right)^{N-1} h(t)\, \mathrm{d}t \right) \mathrm{d}s, \quad r \in [0,1]$$

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Adapting Binding & Rynne (2007) yields the following result.

Theorem 2

Consider $u(h) := S_p(h)$ for $h \in C^0[0,1]$ and suppose that

$$u(h)'(x_0) = 0 \implies h(x_0) \neq 0$$

Then there exists a neighbourhood V of h in $C^0[0,1]$ such that $S_p : V \to W^{1,1}(0,1)$ is C^1.

Furthermore, $v = DS_p(h)w \iff v \in W^{1,1}(0,1)$ and

$$\begin{cases} -(p-1)\big(r^{N-1}|u(h)'(r)|^{p-2}v'(r)\big)' = r^{N-1}w(r), \quad 0 < r < 1 \\ v'(0) = v(1) = 0 \end{cases}$$

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Hence, Theorem 2 shows that S_p is differentiable at $\lambda_0\, f_0\, \phi_p(v_0)$.

Global continuation

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This follows by ODE arguments, using the monotonicity of f/ϕ_p.

Hence, through each solution $(\lambda, u) \in \mathcal{S}^{\pm}$ passes a unique local C^1 curve of solutions, that can be parametrized by λ.

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Local uniqueness at $(\lambda_0, 0)$ $\Longrightarrow$

there is two unique C^1 mappings $u_\pm : (\lambda_0, \lambda_\infty) \to C^0[0,1]$ s.t.

$$\mathcal{S}^{\pm} = \{(\lambda, u_\pm(\lambda)) : \lambda \in (\lambda_0, \lambda_\infty)\}$$

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• Can we obtain Crandall-Rabinowitz–like results in more general domains?

This seems difficult due to the issue of diffentiability/linearization of the inverse p-Laplacian, which would require good knowledge of the nodal sets of the eigenfunctions.