Lecture I.:
(Simple)
Basic Variational and Topological Methods
(an introductory lecture for
graduate students and postdocs)

Lecture II.:
Regular and Singular Systems with the p- and q-Laplacians
(an advanced lecture, suitable also for
graduate students and postdocs)

Lecture III.:
Special Solution Methods for Singular Systems
(an advanced lecture)

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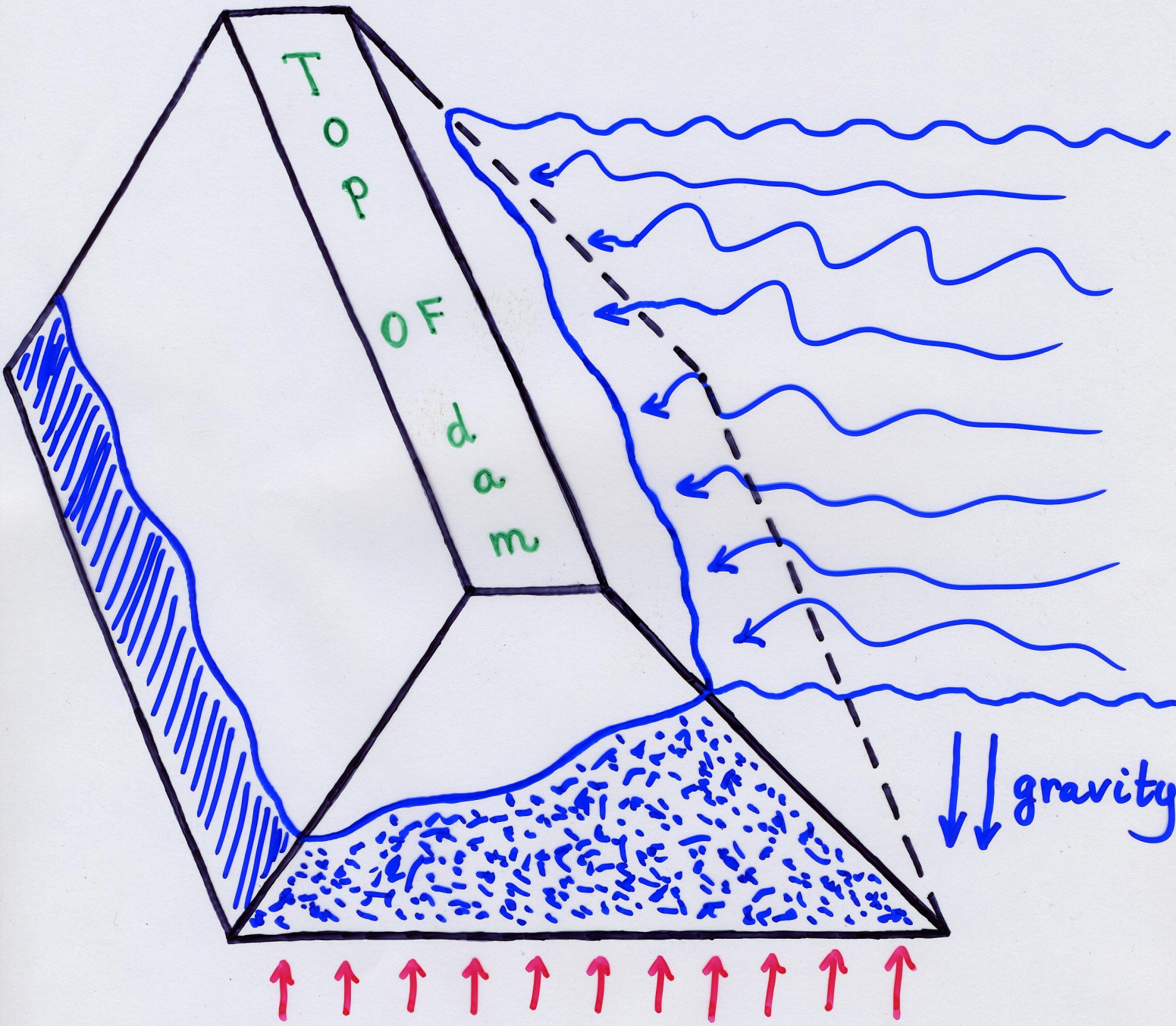
$u(x,t)$ – moisture $= \dfrac{\text{water}}{\text{volume}}$



$$\vec{\nabla} \cdot \left[\left(a \cdot |\vec{\nabla} u| + b \right) \vec{\nabla} u \right]$$

diffusion operator, $a, b \geqq 0$

Nonlinear Darcy's law:

(i) Forchheimer (1901)

$$(a + b\,|\vec{v}|)\,\vec{v} = -K(\Theta)\,\mathrm{grad}\,\Phi(\Theta)$$

$a, b \geqq 0$ – constants

(ii) Missbach (1937)

$$c\,|\vec{v}|^{q-2}\,\vec{v} = -K(\Theta)\,\mathrm{grad}\,\Phi(\Theta)$$

$c \geqq 0$ – a constant, $2 \leqq q \leqq 3$

$q = 2 \Rightarrow$ laminar flow

$q = 3 \Rightarrow$ turbulent flow

Lecture I.:
(Simple)
Basic Variational and Topological Methods
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Peter Takáč

Motivation from Mathematical Biology:

Gierer-Meinhardt reaction-diffusion system

The $(p-1)$-homogeneous quasilinear elliptic operator
$u \mapsto \Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$,
called the p-Laplacian, is defined for
$u \in W_0^{1,p}(\Omega)$ with values $\Delta_p u \in W^{-1,p'}(\Omega)$,
the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.
$-\Delta_p$ is called the *monotone* p-Laplacian.

System of two quasilinear elliptic equations:

$$(1) \qquad \begin{cases} -\Delta_p u = \lambda\, a(x)\, |u|^{\alpha_1} |v|^{\beta_1 - 1} v + f(x) & \text{in } \Omega\,; \\ -\Delta_q v = \mu\, b(x)\, |v|^{\alpha_2} |u|^{\beta_2 - 1} u + g(x) & \text{in } \Omega\,; \\ \qquad u = v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a **connected** C^2-manifold,
($\partial\Omega$ not necessarily connected ???),
$x = (x_1, \ldots, x_N)$ is a generic point in Ω,
$p, q \in (1, \infty)$ are given numbers,
$0 \le f, g \in L^\infty(\Omega)$ are given functions,
$a, b \in L^\infty(\Omega)$ are given functions satisfying

$$a_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \Omega} a(x) > 0\,, \qquad b_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \Omega} b(x) > 0,$$

and α_i, β_i are constants with $\alpha_i \ge 0$ and $\beta_i > 0$ for $i = 1, 2$.

If $f \equiv 0$ and $g \equiv 0$ on Ω, we may compare system (1) with a *homogeneous nonlinear eigenvalue problem* for the unknown pair of parameters $(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* = (0, \infty)^2$ associated with the unknown pair of nonnegative eigenfunctions $u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$.

Computing the partial derivatives

$$\begin{pmatrix} \frac{\partial f}{\partial u}, & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u}, & \frac{\partial g}{\partial v} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda\alpha_1\, a\, |u|^{\alpha_1-2}u|v|^{\beta_1-1}v\,, & \lambda\beta_1\, a\, |u|^{\alpha_1}|v|^{\beta_1-1} \\ \mu\beta_2\, b\, |v|^{\alpha_2}|u|^{\beta_2-1}\,, & \mu\alpha_2\, b\, |v|^{\alpha_1-2}v|u|^{\beta_1-1}u \end{pmatrix}$$

we observe that the system is **cooperative** (regular) if, e.g. $\alpha_1 < p-1$, $\alpha_2 < q-1$, $\beta_1 > 0$, $\beta_2 > 0$, and **competitive** (singular) if, e.g. $\alpha_1 < p-1$, $\alpha_2 < q-1$, $\beta_1 < 0$, $\beta_2 < 0$.

Given any $v \in C_0^0(\overline{\Omega})$, i.e. $v : \overline{\Omega} \to \mathbb{R}_+$ is a continuous function with $v = 0$ on $\partial\Omega$, let us consider the first equation in system (1),

$$\begin{cases} -\Delta_p u = \lambda\, a(x)\, |u|^{\alpha_1} |v|^{\beta_1 - 1} v + f(x) & \text{in } \Omega\,; \\ \qquad u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

i.e., for $u > 0$ and $v > 0$ in Ω,

$$(2) \quad -\Delta_p u = m(x)\,(u^+)^a + f(x) \text{ in } \Omega\,; \qquad u = 0 \text{ on } \partial\Omega\,,$$

where $a \in (-\infty, p-1)$ is a constant and $0 \le m \in L^\infty_{\text{loc}}(\Omega) \cap L^1(\Omega)$ is a weight function. Owing to $a < p-1$, Problem (2) is **subhomogeneous**.

Assume $0 \le m \in L^\infty(\Omega)$. A nonnegative solution $u \in W_0^{1,p}(\Omega)$ is obtained as a nonnegative critical point (the *global minimizer*) of the energy functional

$$(3) \qquad \mathcal{E}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x - \int_\Omega f(x)\, u \, \mathrm{d}x - \frac{1}{a+1} \int_\Omega (u^+)^{a+1}\, m(x) \, \mathrm{d}x$$

for $u \in W_0^{1,p}(\Omega)$. It is easy to see that this functional is coercive on $W_0^{1,p}(\Omega)$; by the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{a+1}(\Omega)$ provided $-1 < a < p-1$.

An alternative way for obtaining a nonnegative solution $u \in W_0^{1,p}(\Omega)$ is the following iteration procedure:

Let $u_0 \in W_0^{1,p}(\Omega)$ be the unique solution to

$$(4) \qquad -\Delta_p u_0 = f(x) \text{ in } \Omega\,; \quad u = 0 \text{ on } \partial\Omega\,.$$

We construct recursively a monotone increasing sequence of nonnegative functions $u_0 \le u_1 \le u_2 \le \dots$ in $W_0^{1,p}(\Omega)$, such that

$$(5) \qquad \begin{cases} -\Delta_p u_n = m(x)\, u_{n-1}^a + f(x) \text{ in } \Omega\,; \\ \qquad u_n = 0 \text{ on } \partial\Omega\,, \end{cases}$$

for $n = 1, 2, 3, \dots$. The inequality $u_{n-1} \le u_n$ a.e. in Ω implies $u_n \le u_{n+1}$ a.e. in Ω, by the *weak comparison principle* (see, e.g., Tolksdorf (1983)):

If $f, g \in L^\infty(\Omega)$ are given and $u, v \in W_0^{1,p}(\Omega)$ satisfy

$$(6) \qquad \begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \quad u|_{\partial\Omega} = 0\,; \\ -\Delta_p v = g(x) & \text{in } \Omega; \quad v|_{\partial\Omega} = 0\,, \end{cases}$$

then $f \le g$ a.e. in $\Omega \implies u \le v$ a.e. in Ω.

Moreover, the weak comparison principle implies:
$u_n \equiv u_n^{(m)} \in W_0^{1,p}(\Omega)$ is a monotone increasing function
of the weight $m \in L^\infty(\Omega)$, $m \ge 0$ a.e. in Ω, that is,
if $m, \tilde{m} \in L^\infty(\Omega)$, then we have

$$0 \le m \le \tilde{m} \text{ a.e. in } \Omega \implies 0 \le u_n^{(m)} \le u_n^{(\tilde{m})} \text{ a.e. in } \Omega.$$

The convergence of the monotone increasing sequence
$u_n \nearrow u$, say, in $L^p(\Omega)$ as $n \nearrow \infty$ is easily obtained from
the Lebesgue monotone convergence theorem.
The $W_0^{1,p}(\Omega)$-norms ($\implies$ the $L^p(\Omega)$-norms)
are bounded, by Hölder's inequality,

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla u_n|^p \,\mathrm{d}x = \int_\Omega m(x)\, u_{n-1}^a\, u_n \,\mathrm{d}x$$
$$\leq \|m\|_{L^\infty(\Omega)} \|u_{n-1}\|^a_{L^p(\Omega)} \|u_n\|_{L^{p/(p-a)}(\Omega)}$$
$$\leq C \|m\|_{L^\infty(\Omega)} \|u_{n-1}\|^a_{L^p(\Omega)} \left(\int_\Omega |\nabla u_n|^p \,\mathrm{d}x\right)^{1/p}$$

where $C \in \mathbb{R}_+$ is a Sobolev constant depending only on $0 < a < p-1$ and Ω. Hence, for $n = 1, 2, \ldots,$

$$\|u_n\|^{p-1}_{W_0^{1,p}(\Omega)} \leq C \|m\|_{L^\infty(\Omega)} \|u_{n-1}\|^a_{L^p(\Omega)}$$
$$\leq C \|m\|_{L^\infty(\Omega)} \|u_n\|^a_{L^p(\Omega)} .$$

The convergence of the sequence $u_n \to u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$ is then obtained from the convergence $\|u_n - u\|_{L^p(\Omega)} \to 0$, the weak convergence of a subsequence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, and the convergence $\|u_n\|_{W_0^{1,p}(\Omega)} \to \|u\|_{W_0^{1,p}(\Omega)}$ combined with the uniform convexity of $W_0^{1,p}(\Omega)$.

As a particular result, $u \equiv u^{(m)} \in W_0^{1,p}(\Omega)$ has the following properties:

Proposition. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary $\partial\Omega$ is a compact C^2-manifold, $1 < p < \infty$, and $0 < a < p-1$. Assume $0 \le m \in L^\infty(\Omega)$. Then the Dirichlet problem (2) has a unique nonnegative solution $u \in W_0^{1,p}(\Omega)$, $u = u^{(m)}$, where the function $u^{(m)} \in W_0^{1,p}(\Omega)$ has been constructed by the iteration procedure (4) and (5) described above. Furthermore, we have either $u^{(m)} = 0$ a.e. in Ω, or else

$$(7) \qquad u^{(m)} > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u^{(m)}}{\partial \nu} < 0 \quad \text{on } \partial\Omega\,.$$

Finally, given any two weights $m, \tilde{m} \in L^\infty(\Omega)$, we have

$$0 \le m \le \tilde{m} \text{ a.e. in } \Omega \implies 0 \le u^{(m)} \le u^{(\tilde{m})} \text{ a.e. in } \Omega.$$

((2))

$$-\Delta_p u = m(x)\,(u^+)^a + f(x) \text{ in } \Omega\,; \quad u = 0 \text{ on } \partial\Omega\,.$$

A closely related result has been obtained in Fleckinger, Hernández, Takáč, and de Thélin (1998).

The *strong comparison principle* (see, e.g., Tolksdorf (1983), Guedda and Véron (1989), and Cuesta and Takáč (1998)):

Theorem. If $f, g \in L^\infty(\Omega)$ and $u, v \in W_0^{1,p}(\Omega)$ satisfy

$$((6)) \qquad \begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \quad u|_{\partial\Omega} = 0\,; \\ -\Delta_p v = g(x) & \text{in } \Omega; \quad v|_{\partial\Omega} = 0\,, \end{cases}$$

then $f \leq g$ a.e. in Ω with $f \not\equiv g$ in Ω $\Longrightarrow$

$$(8) \qquad u < v \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega.$$

Of course, $\nu : \partial\Omega \to \mathbb{R}^N$ denotes the outer unit normal vector field on the boundary of Ω.

Moreover, the strong comparison principle implies: $u_n \equiv u_n^{(m)} \in W_0^{1,p}(\Omega)$ is a strictly monotone increasing function of the weight $m \in L^\infty(\Omega)$, $m \geq 0$ a.e. in Ω, that is, if $m, \tilde{m} \in L^\infty(\Omega)$, then we have

$$0 \leq m \leq \tilde{m} \text{ a.e. in } \Omega \text{ with } m \not\equiv \tilde{m} \text{ in } \Omega \implies$$

$$u_n^{(m)} < u_n^{(\tilde{m})} \text{ in } \Omega \quad \text{and} \quad \frac{\partial u_n^{(m)}}{\partial \nu} > \frac{\partial u_n^{(\tilde{m})}}{\partial \nu} \text{ on } \partial\Omega.$$

This is still an **OPEN PROBLEM**;
we will discuss it in the next lecture.

Lecture II.:
Regular and Singular Systems with the p- and q-Laplacians
(an advanced lecture, suitable also for graduate students and postdocs)

Peter Takáč

The **OPEN PROBLEM**:
Strong Comparison Principle.

Theorem. If $f, g \in L^\infty(\Omega)$ and $u, v \in W_0^{1,p}(\Omega)$ satisfy

$$((6)) \qquad \begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \quad u|_{\partial\Omega} = 0\,; \\ -\Delta_p v = g(x) & \text{in } \Omega; \quad v|_{\partial\Omega} = 0\,, \end{cases}$$

then $f \le g$ a.e. in Ω with $f \not\equiv g$ in Ω $\Longrightarrow$

$$((8)) \qquad u < v \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega.$$

Verified in some special cases:

1. General solutions in an interval $\Omega = (-R, R)$ or radially symmetric solutions in a ball $\Omega = B_R(0)$
(Cuesta and Takáč (1998));

2. Positive solutions in a domain $\Omega \subset \mathbb{R}^N$ whose boundary $\partial\Omega$ is a connected compact C^2-manifold
(Cuesta and Takáč (1998));

3. Positive solutions in a domain $\Omega \subset \mathbb{R}^N$ with a condition on the set $\{x \in \Omega : f(x) = g(x)\}$
(M. Lucia and S. Prashanth (2003));

4. General solutions in a domain $\Omega \subset \mathbb{R}^N$ with a condition on the set $\{x \in \Omega : f(x) = g(x)\}$
(M. Guedda and L. Véron (1989)).

5. General solutions in a domain $\Omega \subset \mathbb{R}^N$ with a condition on the set $\{x \in \Omega : \nabla u(x) = \nabla v(x) = \mathbf{0}\}$
(Damascelli, Pacella, and Ramaswamy (1999)).

COUNTEREXAMPLE to

Strong Comparison Principle:

Let $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, $p > 2$, and $\lambda < 0$.
Set $r = |x|$; hence, $0 \le r \le 1$. Recall $p' = \frac{p}{p-1} > 1$.
For every number θ satisfying $p' < \theta < \infty$, we define

$$u_\theta(x) \stackrel{\text{def}}{=} 1 - r^\theta \qquad \text{and} \tag{9}$$

$$\begin{aligned} f_\theta(x) \stackrel{\text{def}}{=} & \left[(p-1)(\theta-1) - 1 + N\right] \theta^{p-1} \\ & \times r^{(p-1)(\theta-1)-1} \\ & - \lambda(1 - r^\theta)^{p-1} \end{aligned} \tag{10}$$

for $x \in \overline{\Omega}$. Obviously, $u_\theta \in C^1(\overline{\Omega})$ and $f_\theta \in C^0(\overline{\Omega})$ satisfy $u_\theta > 0$ in Ω and $f_\theta > 0$ in $\overline{\Omega}$, together with $\psi_p(\nabla u_\theta) \equiv |\nabla u_\theta|^{p-2} \nabla u_\theta \in [C^1(\overline{\Omega})]^N$ and

$$\begin{cases} -\Delta_p u_\theta - \lambda u_\theta^{p-1} = f_\theta(x) & \text{in } \Omega\,; \\ u_\theta = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{11}$$

We observe that $u_\theta(0) = 1$, and $u_{\theta_1}(x) < u_{\theta_2}(x)$ whenever $0 < r < 1$ and $p' < \theta_1 < \theta_2 < \infty$.

We claim that, given any two numbers θ_1 and θ_2 satisfying $\frac{p}{p-2} \leq \theta_1 < \theta_2 < \infty$, there exists a constant $\lambda_p \equiv \lambda_p(\theta_1, \theta_2) > 0$ such that also $f_{\theta_1}(x) < f_{\theta_2}(x)$ for $0 < r \leq 1$ and every λ with $\lambda_p \leq -\lambda < \infty$.
Consequently, the SCP is *violated* for the boundary value problem (11).

To *prove* our claim, it suffices to verify that there exists a constant $\lambda_p \equiv \lambda_p(\theta_1, \theta_2) > 0$ with the following property: For all θ and λ satisfying $\theta_1 \leq \theta \leq \theta_2$ and $\lambda_p \leq -\lambda < \infty$, we have

$$\partial_\theta f_\theta(x) > 0 \quad \text{for } 0 < r \leq 1. \tag{12}$$

Notice that the partial derivative $\partial_\theta f_\theta \equiv \partial f_\theta / \partial \theta$ exists,

$$\begin{aligned}
(p-1)^{-1}\ \partial_\theta f_\theta &= [p(\theta-1)+N]\theta^{p-2}\ r^{(p-1)(\theta-1)-1} \\
&+ [p(\theta-1)+N-\theta]\theta^{p-1} \\
&\times\ r^{(p-1)(\theta-1)-1}\ \ln r \\
&+ \lambda(1-r^\theta)^{p-2}\ r^\theta\ \ln r \qquad \text{for } x \in \overline{\Omega},
\end{aligned}$$

and thus $\partial_\theta f_\theta \in C^0(\overline{\Omega})$. Inequality (12) is verified by a direct calculation ($\ln r < 0$).

System of two quasilinear elliptic equations:

$$(13) \qquad \begin{cases} -\Delta_p u = \lambda\, a(x)\, |u|^{\alpha_1} |v|^{\beta_1 - 1} v & \text{in } \Omega\,; \\ -\Delta_q v = \mu\, b(x)\, |v|^{\alpha_2} |u|^{\beta_2 - 1} u & \text{in } \Omega\,; \\ \qquad u = v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a **connected** C^2-manifold,
($\partial\Omega$ not necessarily connected ???),
$x = (x_1, \ldots, x_N)$ is a generic point in Ω,
$p, q \in (1, \infty)$ are given numbers,
$a, b \in L^\infty(\Omega)$ are given functions satisfying

$$a_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \Omega} a(x) > 0\,, \qquad b_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \Omega} b(x) > 0,$$

and α_i, β_i are constants with $\alpha_i \geq 0$ and $\beta_i > 0$ for $i = 1, 2$.

The $(p-1)$-homogeneous quasilinear elliptic operator
$u \mapsto \Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2} \nabla u)$,
called the p-Laplacian, is defined for
$u \in W_0^{1,p}(\Omega)$ with values $\Delta_p u \in W^{-1,p'}(\Omega)$,
the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

We view system (13) as
a *homogeneous nonlinear eigenvalue problem*
for the unknown pair of parameters
$(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* = (0, \infty)^2$ associated with
the unknown pair of nonnegative eigenfunctions
$u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$.
We will refer to such a couple (λ, μ)
as a "principal eigenvalue" of system (13).

Notice that system (13) is neither variational
nor of a Hamiltonian type, in general,
except for the cases when
either $\frac{\partial f}{\partial v} \equiv \frac{\partial g}{\partial u}$ or $\frac{\partial f}{\partial u} \equiv \frac{\partial g}{\partial v}$ where we have denoted by
$f(x, u, v)$ and $g(x, u, v)$, respectively,
the right-hand side of the first and second equations
in (13). Computing the partial derivatives

$$\begin{pmatrix} \frac{\partial f}{\partial v}, & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial u}, & \frac{\partial g}{\partial v} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda\beta_1\, a\, |u|^{\alpha_1}|v|^{\beta_1-1}\,, & \mu\beta_2\, b\, |v|^{\alpha_2}|u|^{\beta_2-1} \\ \lambda\alpha_1\, a\, |u|^{\alpha_1-2}u|v|^{\beta_1-1}v\,, & \mu\alpha_2\, b\, |v|^{\alpha_1-2}v|u|^{\beta_1-1}u \end{pmatrix}$$

we observe that the former case occurs
if and only if

$$\alpha_1 = \beta_2 - 1,\ \alpha_2 = \beta_1 - 1, \quad \text{and}$$
$$\lambda\beta_1 a(x) = \mu\beta_2 b(x) \ \text{ for a.e. } x \in \Omega\,,$$

whereas the latter case occurs if and only if either

$$\alpha_1 - 1 = \beta_2,\ \alpha_2 - 1 = \beta_1, \quad \text{and}$$
$$\lambda\alpha_1 a(x) = \mu\alpha_2 b(x) \ \text{ for a.e. } x \in \Omega\,,$$

or $\alpha_1 = \alpha_2 = 0$.
The special "superhomogeneous"
(often termed "superlinear") case
$\alpha_1 = \alpha_2 = 0$ and $\beta_1\beta_2 > (p-1)(q-1)$
was treated in
Clément, Manásevich, and Mitidieri
(Comm. P.D.E., 1993).
Systems with variational and Hamiltonian structures
are studied in de Thélin and Vélin (1991, 1993),
de Figueiredo (2008), for instance.
Our method does not require any variational or
Hamiltonian structure for system (13).

Variational structure:

For any $p, q \in (1, \infty)$ and $u \in W_0^{1,p}(\Omega)$, $v \in W_0^{1,q}(\Omega)$,

$$\mathcal{E}(u,v) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla v|^q \,\mathrm{d}x + \int_\Omega E(u,v)\,\mathrm{d}x\,,$$

where $\dfrac{\partial E}{\partial u} = f(u,v)$ and $\dfrac{\partial E}{\partial v} = g(u,v)\,,$

whence $\dfrac{\partial f}{\partial v} = \dfrac{\partial g}{\partial u}\,.$

Hamiltonian structure:

For $p = q = 2$ and $u, v \in W_0^{1,2}(\Omega)$,

$$\mathcal{H}(u,v) \stackrel{\text{def}}{=} \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega H(u,v)\,\mathrm{d}x\,,$$

where $\dfrac{\partial H}{\partial v} = f(u,v)$ and $\dfrac{\partial H}{\partial u} = g(u,v)\,,$

whence $\dfrac{\partial f}{\partial u} = \dfrac{\partial g}{\partial v}\,.$

We wish to apply a simplified version of a Kreĭn-Rutman theorem for homogeneous nonlinear mappings due to Takáč (Nonl. Anal., 1996).

Given any $f \in L^\infty(\Omega)$, we denote by $T_p(f) \equiv u \in W_0^{1,p}(\Omega)$ the unique weak solution of the boundary value problem

$$(14) \qquad -\Delta_p u = f(x) \ \text{ in } \Omega\,; \qquad u = 0 \ \text{ on } \partial\Omega\,.$$

It is well-known (DiBenedetto, Lieberman, Tolksdorf) that $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

We denote $X = [C_0^1(\overline{\Omega})]^2$, $X_+ = \{(f,g) \in X : f \geq 0 \text{ and } g \geq 0 \text{ in } \Omega\}$, and $\overset{\circ}{X}_+$ is the topological interior of X_+ in X. Finally, define the map $S : X \to X$ by $S(u,v) \overset{\text{def}}{=} (\tilde{u}, \tilde{v})$ with

$$\tilde{u} = T_p\left(a\,|u|^{\alpha_1}|v|^{\beta_1 - 1}v\right)\,,$$
$$\tilde{v} = T_q\left(b\,|v|^{\alpha_2}|u|^{\beta_2 - 1}u\right) \qquad \text{for } (u,v) \in X\,.$$

Homogeneity of S is obtained from that of $S^2 : u \longmapsto \tilde{v} \longmapsto v = \tilde{v} \longmapsto \tilde{u}$:
$[C_0^1(\overline{\Omega})]^\circ \longrightarrow [C_0^1(\overline{\Omega})]^\circ \longrightarrow [C_0^1(\overline{\Omega})]^\circ$.

A curve of principal eigenvalues

Lemma 1. **(i)** $(u,v) \in X_+ \setminus \{0\}$ is a weak solution of (13) for some $(\lambda,\mu) \in (\mathbb{R}_+^*)^2$
$\iff$ $(u,v) \in \overset{\circ}{X}_+$ and
$S(u,v) = (\lambda^{-1/(p-1)}u, \mu^{-1/(q-1)}v)$.
(ii) For all $\rho,\sigma \in \mathbb{R}_+$ and for all $(u,v) \in X_+$ we have

$$S(\rho u, \sigma v) = \Big((\rho^{\alpha_1}\sigma^{\beta_1})^{1/(p-1)} S_1(u,v)\,, \\ (\rho^{\beta_2}\sigma^{\alpha_2})^{1/(q-1)} S_2(u,v) \Big)\,.$$

(iii) If $(u,v) \in X_+$ solves (13) with (λ',μ') in place of (λ,μ), then for any $\rho,\sigma > 0$, the pair $(\rho u, \sigma v)$ solves (13) with (λ,μ) satisfying

$$\lambda = \rho^{p-1-\alpha_1}\sigma^{-\beta_1}\lambda'\,, \\ \mu = \sigma^{q-1-\alpha_2}\rho^{-\beta_2}\mu'\,. \tag{15}$$

1. The case when S is homogeneous

$$(\rho^{\alpha_1}\sigma^{\beta_1})^{1/(p-1)} = \rho\,, \qquad (\rho^{\beta_2}\sigma^{\alpha_2})^{1/(q-1)} = \sigma\,.$$

For $\rho = \sigma \in \mathbb{R}_+^* = (0,\infty)$ this means

- $\alpha_1 + \beta_1 = p - 1$ and $\alpha_2 + \beta_2 = q - 1$.

Look for principal eigenvalues of the map S via the Kreĭn-Rutman Theorem:

Theorem 1. Assume

$$\alpha_1 + \beta_1 = p - 1 \ \text{ and } \ \alpha_2 + \beta_2 = q - 1. \tag{16}$$

Then there exists $\Lambda > 0$ and a couple $(u_1, v_1) \in \overset{\circ}{X}_+$ such that system (13) possesses a positive weak solution $(u, v) \in X_+$ associated to some $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$ if and only if

$$\lambda^{\frac{1}{\beta_1}}\mu^{\frac{1}{\beta_2}} = \Lambda \tag{17}$$

Moreover, $(u, v) = c(u_1, v_1)$ for some constant $c > 0$.

2. The case when S^2 is homogeneous

We arrive at an alternative condition to Case 1:

- $\alpha_1 = \alpha_2 = 0$ and $\beta_1\beta_2 = (p-1)(q-1)$.

Another application – Kreĭn-Rutman Theorem

Let us assume $\alpha_1 < p-1$ and $\alpha_2 < q-1$. We introduce a new mapping $T : X_+ \to X_+$ defined by $T(u,v) \stackrel{\text{def}}{=} (J_1(v), J_2(u))$ where, for $(u,v) \in X$, $J_1(v) = \tilde{u}$ is the unique (weak) solution $\tilde{u}$ of

$$(18) \qquad \begin{cases} -\Delta_p \tilde{u} = a(x)|\tilde{u}|^{\alpha_1} v^{\beta_1} & \text{in } \Omega\,; \\ \tilde{u} = 0 & \text{on } \ \partial\Omega\,, \end{cases}$$

and $J_2(u) = \tilde{v}$ is the unique (weak) solution $\tilde{v}$ of

$$(19) \qquad \begin{cases} -\Delta_q \tilde{v} = b(x)|\tilde{v}|^{\alpha_2} u^{\beta_2} & \text{in } \Omega\,; \\ \tilde{v} = 0 & \text{on } \ \partial\Omega\,. \end{cases}$$

Existence is obtained by classical minimization while uniqueness follows from a convexity argument.

The regularity results combined with the strong maximum principle (Tolksdorf (1983), Vázquez (1984)) imply that the pair $(\tilde{u}, \tilde{v})$ belongs to $\overset{\circ}{X}_+$. Alternatively, the strict subhomogeneity may be used.

Let us consider the mapping T^2. Notice that

$$T^2(u,v) = (J_1 \circ J_2(u), J_2 \circ J_1(v)) \text{ for any } (u,v) \in X\,.$$

Hence, we can decouple T^2 and look for eigenvalues of each component. We see now that the condition (20) below for the homogeneity of $J_1 \circ J_2$ and $J_2 \circ J_1$ is less restrictive that the one found earlier for S^2. We will need the following result:

Lemma 2. Let us denote $V = C_0^1(\overline{\Omega})$. Then the mapping $J_i : V_+ \to V_+$ is nondecreasing for $i = 1, 2$.

Lemma 3. **(i)** $u, v \in V_+ \setminus \{0\}$ is a weak solution of (13) for some $(\lambda, \mu) \in (\mathbb{R}_+^*)^2 \iff$
$u, v \in \overset{\circ}{V}_+$ and $J_1(v) = \lambda^{\frac{-1}{p-1-\alpha_1}} u$, $J_2(u) = \mu^{\frac{-1}{q-1-\alpha_2}} v$.

(ii) For any $\rho, \sigma > 0$ we have

$$(J_1 \circ J_2)(\rho u) = \rho^{\frac{\beta_1\beta_2}{(p-1-\alpha_1)(q-1-\alpha_2)}} (J_1 \circ J_2)(u)\,,$$
$$(J_2 \circ J_1)(\sigma v) = \sigma^{\frac{\beta_1\beta_2}{(p-1-\alpha_1)(q-1-\alpha_2)}} (J_2 \circ J_1)(v)\,.$$

Sub- / Superhomogeneity Condition:

$$\frac{\beta_1\beta_2}{(p-1-\alpha_1)(q-1-\alpha_2)} \lesseqgtr 1\,.$$

Homogeneity Condition: $\cdots = 1$, i.e.,

$$\beta_1\beta_2 = (p-1-\alpha_1)(q-1-\alpha_2)\,. \tag{20}$$

The following main result holds:

Theorem 2. Assume $\alpha_1 < p-1$, $\alpha_2 < q-1$ and the homogeneity condition

$$((20)) \qquad \beta_1\beta_2 = (p-1-\alpha_1)(q-1-\alpha_2)\,.$$

Then there exist $\Lambda' > 0$ and $(u', v') \in \overset{\circ}{X}_+$ such that system (13) possesses a positive weak solution $(u, v) \in X_+$ associated to some $(\lambda, \mu) \in (\mathbb{R}^*_+)^2$ if and only if

$$(21) \qquad \lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}}\mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} = \Lambda'\,.$$

Moreover, $(u, v) = (\rho u', \rho\mu^{1/\beta_2} v')$ with some positive constant ρ.

Non-homogeneous systems

$$(22) \qquad \begin{cases} -\Delta_p u = \lambda a(x)|u|^{\alpha_1}|v|^{\beta_1-1}v + f(x) & \text{in} \quad \Omega; \\ -\Delta_q v = \mu b(x)|v|^{\alpha_2}|u|^{\beta_2-1}u + g(x) & \text{in} \quad \Omega; \\ u = v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

where $0 \leq f, g \in L^\infty(\Omega)$ are given functions.

Our aim is to study the solvability of system (22) in the following three cases:

(a) $\lambda, \mu > 0$ and (λ, μ) below or to the left of the eigenvalue curve $\mathcal{C}_1$,
(b) $(\lambda, \mu) \in \mathcal{C}_1$, and
(c) $\lambda, \mu > 0$ and (λ, μ) above or to the right of, but close to the eigenvalue curve $\mathcal{C}_1$.

We recall that the eigenvalue curve $\mathcal{C}_1$ has been defined in (21) with $\Lambda' \stackrel{\text{def}}{=} \Lambda_1^{-\sqrt{\beta_2/(q-1-\alpha_2)}}$.

When (λ, μ) lies below or to the left of $\mathcal{C}_1$

In order to assure the existence of $\mathcal{C}_1$ we assume the more general condition (20) jointly with $\beta_1, \beta_2 > 0$, $0 \leq \alpha_1 < p-1$, and $0 \leq \alpha_2 < q-1$.

Theorem 3. Let $f, g \in L^\infty(\Omega)$, $f \geq 0$, $g \geq 0$, and let $\lambda > 0, \mu > 0$ be such that

$$\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}} \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} < \Lambda'. \tag{23}$$

Then system (22) has a unique weak solution $(u, v) \in X_+$. If, moreover, $\alpha_1 = \alpha_2 = 0$ and $f + g \not\equiv 0$ then there is a unique solution in X.

Corollary 3. Consider $f, g \in L^\infty(\Omega)$ and let $\lambda > 0$, $\mu > 0$ be such that (23) holds. Then system (22) possesses at least one solution.

The case when $(\lambda, \mu) \in \mathcal{C}_1$

Theorem 4. Let $f, g \in L^\infty(\Omega)$, $f \geq 0, g \geq 0$, $f + g \not\equiv 0$, and let $\lambda > 0$, $\mu > 0$ be such that

$$\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}} \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} = \Lambda'.$$

Then system (22) has no solution in X_+. If, moreover, $\alpha_1 = \alpha_2 = 0$ then there is no solution in X.

When (λ, μ) lies above or to the right of $\mathcal{C}_1$ – An antimaximum principle for systems

Theorem 5. Let $\alpha_1 = \alpha_2 = 0$, $\beta_1\beta_2 = (p-1)(q-1)$, and $(\lambda_1, \mu_1) \in \mathcal{C}_1$. Consider two functions $0 \leq f, g \in L^\infty(\Omega)$ with $f + g \not\equiv 0$. Then there exists $\delta > 0$ such that, for all pairs $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda_1 < \lambda < \lambda_1 + \delta$ and $\mu_1 < \mu < \mu_1 + \delta$, every (weak) solution (u, v) of system (22) satisfies $u \ll 0$ and $v \ll 0$ in Ω, i.e., $-u, -v \in \overset{\circ}{V}_+$.

$$(22)\quad \begin{cases} -\Delta_p u = \lambda a(x)\, |u|^{\alpha_1=0} |v|^{\beta_1-1} v + f(x) & \text{in} \quad \Omega; \\ -\Delta_q v = \mu b(x)\, |v|^{\alpha_2=0} |u|^{\beta_2-1} u + g(x) & \text{in} \quad \Omega; \\ u = v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

where $0 \leq f, g \in L^\infty(\Omega)$ are given functions.

The homogeneity condition

$$(20)\quad \beta_1\beta_2 = (p-1-\alpha_1)(q-1-\alpha_2) = (p-1)(q-1)\,.$$

A singular quasilinear system

(**Gierer-Meinhardt** system in morphogenesis)

$$(24) \qquad \begin{cases} -\Delta_p u = \lambda\, a(x)\, u^{-a_1}\, v^{-b_1} + f(x) & \text{in } \Omega\,; \\ -\Delta_q v = \mu\, b(x)\, v^{-a_2}\, u^{-b_2} + g(x) & \text{in } \Omega\,; \\ \qquad u = v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold,

$\partial\Omega$ not necessarily connected,

$x = (x_1, \ldots, x_N)$ is a generic point in Ω,

$p, q \in (1, \infty)$ are given numbers,

$0 \le f, g \in L^\infty(\Omega)$ are given functions,

$a, b \in L^\infty(\Omega)$ are given functions satisfying

$$a_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} a(x) > 0\,, \qquad b_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} b(x) > 0,$$

and a_i, b_i are constants with $a_i \ge 0$ and $b_i > 0$ for $i = 1, 2$.

Lecture III.:
Special Solution Methods for Singular Systems
(an advanced lecture)

Peter Takáč

A singular quasilinear system

(**Gierer-Meinhardt** system in morphogenesis)

$$((24)) \qquad \begin{cases} -\Delta_p u = \lambda\, a(x)\, u^{-a_1}\, v^{-b_1} + f(x) & \text{in } \Omega\,; \\ -\Delta_q v = \mu\, b(x)\, v^{-a_2}\, u^{-b_2} + g(x) & \text{in } \Omega\,; \\ \qquad u = v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold,

$\partial\Omega$ not necessarily connected,

$x = (x_1, \ldots, x_N)$ is a generic point in Ω,

$p, q \in (1, \infty)$ are given numbers,

$0 \leq f, g \in L^\infty(\Omega)$ are given functions,

$a, b \in L^\infty(\Omega)$ are given functions satisfying

$$a_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} a(x) > 0\,, \qquad b_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} b(x) > 0,$$

and a_i, b_i are constants with $a_i \geq 0$ and $b_i > 0$ for $i = 1, 2$.

Application of Schauder's Fixed-point Theorem

Let us assume $a_1, a_2 > 0$. We introduce a mapping $T : \overset{\circ}{X}_+ \to \overset{\circ}{X}_+$ defined by $T(u,v) \overset{\text{def}}{=} (J_1(v), J_2(u))$ where, for $(u,v) \in \overset{\circ}{X}_+$,
$J_1(v) = \tilde{u}$ is the unique positive (weak) solution $\tilde{u}$ of

$$(25) \qquad \begin{cases} -\Delta_p \tilde{u} &= a(x)\, \tilde{u}^{-a_1} v^{-b_1} + f(x) \quad \text{in } \Omega\,; \\ \tilde{u} &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega\,, \end{cases}$$

and $J_2(u) = \tilde{v}$ is the unique positive (weak) solution $\tilde{v}$ of

$$(26) \qquad \begin{cases} -\Delta_q \tilde{v} &= b(x)\, \tilde{v}^{-a_2} u^{-b_2} + g(x) \quad \text{in } \Omega\,; \\ \tilde{v} &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega\,. \end{cases}$$

Clearly, both these problems are strictly subhomogeneous. Again, existence is obtained by classical minimization while uniqueness follows from a convexity argument for the following Dirichlet problem with a weight:

Given $1 < r < \infty$, $\delta > 0$, $\kappa \geq 0$, and a function $K \in L^1_{\text{loc}}(\Omega)$, we look for a *distributional* solution $u \in W^{1,r}_0(\Omega)$ to the equation in $W^{-1,r'}(\Omega)$, $r' = \frac{r}{r-1}$,

$$-\Delta_r u = K(x)\,(u+\kappa)^{-\delta} + f(x) \quad \text{in } \Omega. \tag{27}$$

First, we state a new regularity result for any positive weak solution to equation (27) (J. Giacomoni, I. Schindler, and P. Takáč (2012)):

Theorem 1. Let u be a positive weak solution to (27) with $\kappa = 0$ and $0 \leq K(x) \leq \text{const} \cdot d(x)^{-\omega}$ for $x \in \Omega$, where $\omega > 0$ is a constant.

(i) If $0 < \delta + \omega < 1$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. G+S+T (2007)

(ii) If $1 \leq \delta + \omega < 2 - \frac{1-\delta}{r}$ then $u \in C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. G+S+T (2012)

Proof. Hardy's inequality for $W_0^{1,r}(\Omega)$ and equivalent Campanato norms in Hölder spaces.

$$-\Delta_r u = K(x)\,(u+\kappa)^{-\delta} + f(x) \quad \text{in } W^{-1,\frac{r}{r-1}}(\Omega)\,. \tag{27}$$

Theorem 2. Let $1 < r < \infty$ and $0 \le \kappa < 1$, and let $K : \Omega \to \mathbb{R}_+$ be a function in $L^1_{\text{loc}}(\Omega)$ with the following additional integrability property:

(I) There is a function $\varphi_0 \in W_0^{1,r}(\Omega)$ such that $\varphi_0 > 0$ a.e. in Ω and

$$\int_\Omega K(x)\,(\varphi_0+\kappa)^{1-\delta}\,\mathrm{d}x < \infty \qquad \text{if } \delta \neq 1\,, \tag{28}$$

$$\int_\Omega K(x)\,[-\ln(\varphi_0+\kappa)]^+\,\mathrm{d}x < \infty \qquad \text{if } \delta = 1\,. \tag{29}$$

Then eq. (27) possesses a unique positive weak solution $u_\kappa \in W_0^{1,r}(\Omega)$. Moreover,
$0 < \kappa \le \kappa' < 1 \implies u_{\kappa'} \le u_\kappa$ a.e. in Ω.

The regularity results combined with
the strong maximum principle
(Tolksdorf (1983), Vázquez (1984))
imply that the pair $(\tilde{u}, \tilde{v})$ belongs to $\overset{\circ}{X}_+$.
Alternatively, the strict subhomogeneity may be used.

Let us to consider the mapping T^2. Notice that
for any $(u, v) \in X$,

$$T^2(u, v) = (J_1 \circ J_2(u), J_2 \circ J_1(v)) .$$

Theorem 3. Let $a_1, b_1, a_2, b_2 > 0$ satisfy

$$(p-1+a_1)(q-1+a_2) > b_1 b_2 \,. \qquad \text{(Subhomogeneity)}$$

Let us abbreviate

$$\gamma = \frac{p(q-1+a_2) - q b_1}{(p-1+a_1)(q-1+a_2) - b_1 b_2}\,, \tag{30}$$

$$\theta = \frac{q(p-1+a_1) - p b_2}{(p-1+a_1)(q-1+a_2) - b_1 b_2}\,, \tag{31}$$

and assume that

$$1 - \frac{1}{p} < \gamma < 1 \quad \text{and} \quad 1 - \frac{1}{q} < \theta < 1\,. \tag{32}$$

Then system (24) possesses a unique positive weak solution pair (u, v) in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ that satisfies the following inequalities with a constant $C > 0$: (Schauder's Fixed-point Theorem)

$$\begin{cases} C^{-1}\,\phi_{1,p}^{\gamma} \le u \le C\,\phi_{1,p}^{\gamma} & \text{and} \\ C^{-1}\,\phi_{1,q}^{\theta} \le v \le C\,\phi_{1,q}^{\theta} & \text{hold in } \Omega\,. \end{cases} \tag{33}$$

The normalized positive eigenfunction $\phi_{1,p}$ for the principal eigenvalue $\lambda_{1,p}$ of $-\Delta_p$:

$$(34) \qquad \begin{cases} -\Delta_p \phi_{1,p} = \lambda_{1,p} \, |\phi_{1,p}|^{p-2} \phi_{1,p} & \text{in } \Omega; \\ \phi_{1,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

$\phi_{1,p} \in W_0^{1,p}(\Omega)$ is normalized by $\phi_{1,p} > 0$ a.e. in Ω and $\int_\Omega (\phi_{1,p})^p \, \mathrm{d}x = 1$.

$\phi_{1,p}$ behaves like the distance function $d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$ near the boundary $\partial\Omega$, by the strong maximum principle (Vázquez (1984)).

For $\gamma > 0$ we have

$$(\phi_{1,p})^\gamma \in W_0^{1,p}(\Omega) \iff \gamma > \frac{1}{p'} = 1 - \frac{1}{p}.$$

Moreover, with $\mathfrak{c} = (1-\gamma)(p-1)$,

$$(35) \qquad \begin{aligned} & -\Delta_p(\phi_{1,p}^\gamma) \\ & = \gamma^{p-1} \left\{ \lambda_{1,p} \, \phi_{1,p}^{\gamma(p-1)} + \mathfrak{c} \, \phi_{1,p}^{-\mathfrak{c}-1} \, |\nabla \phi_{1,p}|^p \right\} \\ & = \gamma^{p-1} \, \phi_{1,p}^{-\mathfrak{c}-1} \left\{ \lambda_{1,p} \, \phi_{1,p}^p + \mathfrak{c} \, |\nabla \phi_{1,p}|^p \right\}, \end{aligned}$$

For $0 < \gamma < 1$,

$$-\Delta_p(\phi_{1,p}^{\gamma}) \approx \phi_{1,p}^{-(1-\gamma)(p-1)-1} \quad \text{in } \overline{\Omega}\,.$$

Definition. A pair of positive functions $(\underline{u}, \underline{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is **a pair of subsolutions** to system (24) (i.e., **a subsolution pair**) if and only if the following inequalities hold in the sense of distributions (i.e., Radon measures) in $W^{-1,\frac{p}{p-1}}(\Omega)$ and $W^{-1,\frac{q}{q-1}}(\Omega)$, respectively:

$(\underline{\mathrm{P}})$

$$\begin{cases} -\Delta_p \underline{u} \le \dfrac{1}{\underline{u}^{a_1}\underline{v}^{b_1}} & \text{in } \Omega; \quad u|_{\partial\Omega} = 0, \quad u > 0 \text{ in } \Omega, \\ -\Delta_q \underline{v} \ge \dfrac{1}{\underline{v}^{a_2}\underline{u}^{b_2}} & \text{in } \Omega; \quad v|_{\partial\Omega} = 0, \quad v > 0 \text{ in } \Omega. \end{cases}$$

Reversed inequalities: a pair of positive functions $(\overline{u}, \overline{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is **a pair of supersolutions** to system (24) (i.e., **a supersolution pair**) if and only ... $(\overline{\mathrm{P}})$.

Remark. It follows from the *weak comparison principle*
ineq-s ($\underline{\mathrm{P}}$) $\Longrightarrow$ $\underline{u} \leq J_1(\underline{v})$ and $\underline{v} \geq J_2(\underline{u})$ a.e. in Ω,
ineq-s ($\overline{\mathrm{P}}$) $\Longrightarrow$ $\underline{u} \geq J_1(\underline{v})$ and $\underline{v} \leq J_2(\underline{u})$ a.e. in Ω.

We have introduced the notions of
a pair of sub- and *supersolutions* to system (24) based on the inequalities $\underline{u} \leq J_1 \circ J_2(\underline{u})$ and $\overline{u} \geq J_1 \circ J_2(\overline{u})$, respectively, for the mapping $J_1 \circ J_2$,
while ignoring the reversed inequalities $\underline{v} \geq J_2 \circ J_1(\underline{v})$ and $\overline{v} \leq J_2 \circ J_1(\overline{v})$ for $J_2 \circ J_1$.

Existence of a pair of sub- and supersolutions, invariance of the conical shells

Four (4) alternatives are of interest.
We consider the following one:

Alternative 1. We search for positive solutions $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ to problem (24) by making the "Ansatz" that the function u (v, respectively)

should behave like a fixed (positive) power of the distance function $d(x)$ for $x \in \Omega$ near $\partial\Omega$, $u \sim \phi_{1,p}^{\gamma}$ and $v \sim \phi_{1,q}^{\theta}$ near $\partial\Omega$, for some constants $\gamma, \theta \in (0,1)$. Both $\phi_{1,p}$ and $\phi_{1,q}$ belong to $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ and, by the strong maximum principle (Vázquez (1984)), $\phi_{1,p} \sim d(x)$ and $\phi_{1,q} \sim d(x)$ near $\partial\Omega$. In our search for a pair of sub- and supersolutions to system (24) we choose the following special forms: $\left(m_1\phi_{1,p}^{\gamma},\, M_2\phi_{1,q}^{\theta}\right)$ for a pair of subsolutions and $\left(M_1\phi_{1,p}^{\gamma},\, m_2\phi_{1,q}^{\theta}\right)$ for a pair of supersolutions to system (24), where $0 < m_1 \le M_1$ and $0 < m_2 \le M_2$ are additional constants to be determined together with $\gamma, \theta \in (0,1)$. Of course, we need to choose these constants in such a way that the conical shell (36)

$$\begin{aligned}\mathcal{C} &\stackrel{\text{def}}{=} \Big\{(u,v) \in C_0(\overline{\Omega}) \times C_0(\overline{\Omega}) : \\ &\quad m_1\phi_{1,p}^{\gamma} \le u \le M_1\phi_{1,p}^{\gamma} \ \text{ and } \ m_2\phi_{1,q}^{\theta} \le v \le M_2\phi_{1,q}^{\theta}\Big\} \\ &= \left[m_1\phi_{1,p}^{\gamma},\, M_1\phi_{1,p}^{\gamma}\right] \times \left[m_2\phi_{1,q}^{\theta},\, M_2\phi_{1,q}^{\theta}\right]\end{aligned}$$

be invariant under the operator

$$T : (u,v) \mapsto T(u,v) \stackrel{\text{def}}{=} (J_1(v), J_2(u)) : \mathcal{C} \to W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \tag{37}$$

defined before Theorem 1 by eqs. (25) and (26), i.e., we need $T(\mathcal{C}) \subset \mathcal{C}$. A careful reader will find out that also the weaker condition $\mathcal{T}^2(\mathcal{C}) \subset \mathcal{C}$ would suffice; this, however, is more difficult to verify directly without knowing that $\mathcal{T}(\mathcal{C}) \subset \mathcal{C}$.

Hence, we wish to fulfill the following four inequalities:

$$T_1(M_2\phi_{1,q}^{\theta}) \geq m_1\phi_{1,p}^{\gamma} \quad \text{and} \quad T_2(m_1\phi_{1,p}^{\gamma}) \leq M_2\phi_{1,q}^{\theta}\,, \tag{38}$$

$$T_2(M_1\phi_{1,p}^{\gamma}) \geq m_2\phi_{1,q}^{\theta} \quad \text{and} \quad T_1(m_2\phi_{1,q}^{\theta}) \leq M_1\phi_{1,p}^{\gamma}\,. \tag{39}$$

In order to verify them, it suffices to show that $\left(m_1\phi_{1,p}^{\gamma},\, M_2\phi_{1,q}^{\theta}\right)$ is a pair of subsolutions and $\left(M_1\phi_{1,p}^{\gamma},\, m_2\phi_{1,q}^{\theta}\right)$ is a pair of supersolutions to system (24), provided all constants $\gamma, \theta \in (0,1)$, $0 < m_1 \leq M_1 < \infty$, and $0 < m_2 \leq M_2 < \infty$ are chosen

properly. Thus, all inequalities (38) and (39) will follow from the following four inequalities, respectively:

$$(40)\qquad -\Delta_p(m_1\phi_{1,p}^{\gamma}) \le \frac{1}{(m_1\phi_{1,p}^{\gamma})^{a_1}(M_2\phi_{1,q}^{\theta})^{b_1}} \qquad \text{in } \Omega\,,$$

$$(41)\qquad -\Delta_q(M_2\phi_{1,q}^{\theta}) \ge \frac{1}{(M_2\phi_{1,q}^{\theta})^{a_2}(m_1\phi_{1,p}^{\gamma})^{b_2}} \qquad \text{in } \Omega\,,$$

$$(42)\qquad -\Delta_q(m_2\phi_{1,q}^{\theta}) \le \frac{1}{(m_2\phi_{1,q}^{\theta})^{a_2}(M_1\phi_{1,p}^{\gamma})^{b_2}} \qquad \text{in } \Omega\,,$$

$$(43)\qquad -\Delta_p(M_1\phi_{1,p}^{\gamma}) \ge \frac{1}{(M_1\phi_{1,p}^{\gamma})^{a_1}(m_2\phi_{1,q}^{\theta})^{b_1}} \qquad \text{in } \Omega\,,$$

supplemented by the Dirichlet boundary conditions on $\partial\Omega$ and the positivity in Ω,

$$(44)\qquad \begin{aligned} &\phi_{1,p}|_{\partial\Omega} = \phi_{1,q}|_{\partial\Omega} = 0 \quad \text{and} \\ &\phi_{1,p} > 0,\ \phi_{1,q} > 0 \quad \text{in } \Omega\,. \end{aligned}$$

For $0 < \gamma < 1$,

$$-\Delta_p(\phi_{1,p}^{\gamma}) \approx \phi_{1,p}^{-(1-\gamma)(p-1)-1} \quad \text{in } \overline{\Omega}\,.$$