

Limits of $f : \mathbf{R}^2 \rightarrow \mathbf{R}$

- Definition: $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ if $|f(x,y) - L| \rightarrow 0$ as $|(x,y) - (a,b)| = \sqrt{(x-a)^2 + (y-b)^2} \rightarrow 0$
- Continuous functions: if f is continuous (for example a rational function) then $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$
- Squeeze theorem: $f(x,y) \rightarrow L$ if $|f(x,y) - L| \leq g(x,y) \rightarrow 0$ for some $g(x,y)$
- $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ does not exist: if there are $q, r : \mathbf{R} \rightarrow \mathbf{R}^2$ such that $\lim_{t \rightarrow 0} q(t) = (a,b) = \lim_{t \rightarrow 0} r(t)$ and $\lim_{t \rightarrow 0} f(q(t)) \neq \lim_{t \rightarrow 0} f(r(t))$

Partial derivatives of $f : \mathbf{R}^n \rightarrow \mathbf{R}$

- Notation: $\frac{\partial}{\partial y} f(x,y) = f_y(x,y) = f_2(x,y) = D_y f(x,y) = D_2 f(x,y)$
- Definition: $f_y(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h}$

Derivative

- Notation: $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$, $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n) \in \mathbf{R}^n$
- Meaning: $f'(a) : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a linear functional that gives the change in the output of the local linearization of f at a
- Change in the output of a function: $f(x) - f(a) \approx f'(a)(x - a)$

Jacobian matrix

- Formula: $Jf(a) = \begin{pmatrix} D_1 f_1(a) & \cdots & D_n f_1(a) \\ \vdots & & \vdots \\ D_1 f_m(a) & \cdots & D_n f_m(a) \end{pmatrix}$
- Meaning: the matrix representation of the derivative
- Usage: $f'(a,b)(\Delta x, \Delta y) = Jf(a,b) \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

Chain rule $h(a) = f(g(a))$

- Function form: $h'(a) = f'(g(a)) \circ g'$
- Matrix form: $Jh(a) = Jf(g(a)) \cdot Jg(a)$
- Special case $\mathbf{R} \xrightarrow{g} \mathbf{R}^2 \xrightarrow{f} \mathbf{R}$: $h'(t) = f_x(g(t))g'_1(t) + f_y(g(t))g'_2(t)$

Directional derivative of $f : \mathbf{R}^2 \rightarrow \mathbf{R}$

- Definition: $D_{\underline{u}} f(x,y) = \frac{d}{dt} f(g(t))$ where $g(t) = \langle x,y \rangle + t\underline{u}$
- Formula: $D_{\underline{u}} f(x,y) = \nabla f(x,y) \cdot \underline{u}$
- Partial derivatives are special cases: $f_x(x,y) = D_{\underline{i}} f(x,y)$

Gradient vector

- Definition: $\nabla f(x,y) = \langle \frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y) \rangle$
- Direction of the gradient: gives the direction of the fastest increase
- Length of the gradient: gives the rate of the fastest increase
- Maximum directional derivative: $D_{\underline{u}_{max}} = |\nabla f(x,y)|$ when $\underline{u}_{max} = \frac{\nabla f(x,y)}{|\nabla f(x,y)|}$
- $\nabla f(x,y)$ is \perp to the level curve containing (x,y)
- $\nabla f(x,y,z)$ is \perp to the level surface containing (x,y,z)
- Normal vector of the tangent plane of $f : \mathbf{R}^2 \rightarrow \mathbf{R}$: $(f_x(x,y), f_y(x,y), -1)$

Extreme values of $f : \mathbf{R}^2 \rightarrow \mathbf{R}$

- Critical point (a, b) : if either $f_x(a, b) = 0 = f_y(a, b)$, or one of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ does not exist
- Local extrema: f has its local extrema at the critical points
- Absolute (or global) extrema: f has its absolute extrema at the critical points or at the boundary points of its domain
- Discriminant: $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2$
- Second derivative test: suppose $f_x(x, y) = 0 = f_y(x, y)$

$D(x, y)$	$f_{xx}(x, y)$	classification of (x, y)
+	+	local min
+	-	local max
-		saddle point
0		inconclusive

Lagrange multipliers

- One constraint: the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$ are taken at points satisfying $\nabla f(x, y) = \alpha \nabla g(x, y)$ and $g(x, y) = k$
- Two constraints: the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ and $h(x, y, z) = l$ are taken at points satisfying $\nabla f(x, y, z) = \alpha \nabla g(x, y, z) + \beta \nabla h(x, y, z)$, $g(x, y, z) = k$ and $h(x, y, z) = l$