MAT 238 Fall 2003 Vector calculus

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Scalar field: $f : \mathbf{R}^3 \to \mathbf{R}$, Vector field: $\underline{F} : \mathbf{R}^3 \to \mathbf{R}^3$

Line integral of scalar fields
- Notation: $\int_C f ds$
- Evaluation: $\int_a^b f(\underline{r}(t))|\underline{r}'(t)|\, dt$
- Mass of wire: $m = \int_C \rho\, ds$
- Center of gravity of a wire: $\bar{x} = \frac{1}{m}\int_C x\rho\, ds$
- Line integral of constant a function: $\int_C k\, ds = k \cdot \text{Length}\,(C)$

(Circulation) line integral of vector fields:
- Notation: $\int_C \underline{F} \cdot d\underline{r} = \int_C \underline{F} \cdot \underline{T}\, ds$
- Different notation: $\int_C F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$
- Evaluation: $\int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$
- Reversed path: $\int_{C_{rev}} \underline{F} \cdot d\underline{r} = -\int_C \underline{F} \cdot d\underline{r}$
- Application: Work of force field on moving object

Flux line integral (only in 2D, $\underline{F} : \mathbf{R}^2 \to R^2$)
- Notation: $\int_C \underline{F} \cdot \underline{N}\, ds$
- Evaluation: $\int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)^{\perp}\, dt$
- Reversed orientation: $\int_C \underline{F} \cdot (-\underline{N})ds = -\int_C \underline{F} \cdot \underline{N}\, ds$
- Application: Amount of fluid going through the boundary

Conservative vector fields
- Characterizations:

(a) $\underline{F}$ is conservative ($\int_C \underline{F} \cdot d\underline{r}$ only depends on the endpoints of C)
(b) $\underline{F}$ has a potential f ($\underline{F} = \nabla f$)
(c) $\oint_C \underline{F} \cdot d\underline{r} = 0$ for all simple closed curve C
(d) $\nabla \times \underline{F} = 0$ (if $\underline{F}$ is nice)

- How to find the potential: $f(x,y,z) = \int_C \underline{F} \cdot d\underline{r}$ where C is any curve connecting $(0,0,0)$ to (x,y,z)

Rotation ≡ curl
- Notation: $\text{curl}\,\underline{F} = \text{rot}\,\underline{F} = \nabla \times \underline{F}$
- Definition in 2D: $\frac{\partial}{\partial x}F_2(x,y) - \frac{\partial}{\partial y}F_1(x,y)$
- Interpretation in 2D: circulation density
- Definition in 3D: $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y,z) & F_2(x,y,z) & F_3(x,y,z) \end{vmatrix}$
- Interpretation in 3D: length is the maximal circulation density, direction is the direction of the maximum circulation density

Divergence
- Notation: $\operatorname{div} \underline{F} = \nabla \cdot \underline{F}$
- Definition: $\frac{\partial F_1(x,y,z)}{\partial x} + \frac{\partial F_2(x,y,z)}{\partial y} + \frac{\partial F_3(x,y,z)}{\partial z}$
- Interpretation: flux density, source or sink

Surface integral of scalar fields
- Notation: $\int \int_S f \, dS$
- Evaluation: $\int \int_R f(\underline{r}(u,v)) |\underline{r}_u(u,v) \times \underline{r}_v(u,v)| \, dvdu$
- Mass of aluminum foil: $m = \int \int_S \rho \, dS$
- Center of gravity of aluminum foil: $\bar{y} = \frac{1}{m} \int \int_S y\rho \, dS$
- Surface integral of a constant function: $\int \int_S k \, dS = k \cdot \text{Area}\,(S)$

Surface (flux) integral of vector fields
- Notation: $\int \int_S \underline{F} \cdot d\underline{S}$
- Alternative notation: $\int \int_S \underline{F} \cdot \underline{N} \, dS$
- Evaluation: $\int \int_R \underline{F}(\underline{r}(u,v)) \cdot (\underline{r}_u(u,v) \times \underline{r}_v(u,v)) \, dvdu$ or $r_v \times r_u$ depending on the orientation
- Interpretation: stuff going through the surface

Curl theorem
- Interpretation: the integral of circulation density is the circulation on the boundary

Stokes' theorem
- If $\underline{F} : \mathbf{R}^3 \to \mathbf{R}^3$ and C is the boundary of S (S is on the left of C) then $\int_C \underline{F} \cdot d\underline{r} = \int \int_S \nabla \times \underline{F} \cdot d\underline{S}$

Green's theorem (2D version of Stokes' theorem)
- If $\underline{F} : \mathbf{R}^2 \to \mathbf{R}^2$ and C is the boundary of R (R is on the left of C) then $\int_C \underline{F} \cdot d\underline{r} = \int \int_R \frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} dydx$

Divergence theorem
- Interpretation: the integral of flux density is the flux on the boundary
- Interpretation: amount coming in and out through the boundary (flux) is equal to the amount created and swallowed (divergence)

Gauss' theorem
- If $\underline{F} : \mathbf{R}^3 \to \mathbf{R}^3$ and S is the boundary (outward orientation) of the solid E then $\int \int_S \underline{F} \cdot d\underline{S} = \int \int \int_E \nabla \cdot \underline{F} \, dE$

2D version of Gauss' theorem
- If $\underline{F} : \mathbf{R}^2 \to \mathbf{R}^2$ and C is the boundary of R then $\int_C \underline{F} \cdot \underline{N} \, ds = \int \int_R \nabla \cdot \underline{F} \, dA$