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1. One

Problem 1.1. What is the smallest perimeter of a triangle that has area 12 and a side with length 6?

Problem 1.2. We pick a point on each side of a rectangle. What is the minimum perimeter of the resulting quadrilateral?

Problem 1.3. In the mid-1990s, Cuban government officials toyed with the idea of using only 11-centavos, 13-centavos, and 17-centavos stamps. What is the largest postage amount that cannot be formed using stamps of this value?

Problem 1.4. Construct a 6-by-6 matrix whose entries are only 1 or -1 so that the dot product of every pair of columns (if the columns are thought of as vectors), and every pair of rows, is 0, or explain why this cannot be done.

Problem 1.5. How many internal, disjoint triangles can be formed in the plane using 9 lines?

Problem 1.6. The last digit of the product of four consecutive odd numbers is 9. What digit do we have in the product before the last digit?

Problem 1.7. Find the smallest 2-digit number such that if you reverse its digits and add the two resulting numbers (and repeat if necessary) you do NOT form a palindromic number in less than 20 steps.

Problem 1.8. There are n red and n blue points in the plane. No three points are on the same line. Show that we can connect every red point to a different blue point with a straight line segment such that the line segments do not cross each other.

Problem 1.9. We paint the points of the plane with two colors so that every point is either red or green. Show that we can find a regular triangle with red corners or a regular triangle with green corners.

Problem 1.10. Consider a convex quadrilateral. Show that the sum of the diagonals is larger than half the perimeter.

2. Two

Problem 2.1. Prove the following statements.

(1) If x and y are positive numbers such that $xy = 1$ then $x + y \geq 2$.
(2) $|a + \frac{1}{a}| \geq 2$ for all nonzero numbers a.
(3) $\dfrac{x^2+2}{\sqrt{x^2+1}} \geq 2$.
(4) $\ln(a) + \log_a(e) \geq 2$ for all $a > 1$.
(5) $\dfrac{x^2}{1+x^4} \leq \dfrac{1}{2}$ for all real numbers x.

Problem 2.2. Let $a_1 \geq a_2 \geq a_3 > 0$ and $b_1 \geq b_2 \geq b_3 > 0$. Show that $a_1b_1+a_2b_2+a_3b_3$ is the maximum and $a_1b_3+a_2b_2+a_3b_1$ is the minimum of the set of all similar expressions. Generalize the result for n pairs of numbers.

Problem 2.3. Show that if a, b and c are positive real numbers then

(1) $a^2 + b^2 + c^2 \geq ab + bc + ca$
(2) $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$

Problem 2.4. Show that if a, b and c are real numbers then $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \leq \dfrac{a^8 + b^8 + c^8}{a^3b^3c^3}$.

Problem 2.5.

(1) Show that $\sqrt{xy} \leq \frac{x+y}{2}$ for all positive x and y.
(2) Let $x_1, \cdots, x_n$ be positive real numbers and $c = \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$. Apply Problem 2.2 to the numbers $a_1 = \frac{x_1}{c}$, $a_2 = \frac{x_1x_2}{c^2}$, ..., $a_n = \frac{x_1x_2\cdots x_n}{c^n}$ and $b_1 = \frac{1}{a_1}$, ..., $b_n = \frac{1}{a_n}$.

Problem 2.6. Show that $n! \leq \left(\frac{n+1}{2}\right)^n$ for all $n \geq 2$.

Problem 2.7. Apply Problem 2.5 to show that $\left(1 + \dfrac{1}{n}\right)^n \leq \left(1 + \dfrac{1}{n+1}\right)^{n+1}$ $\left(1 - \dfrac{1}{n}\right)^n \leq \left(1 - \dfrac{1}{n+1}\right)^{n+1}$ for all natural numbers n.

Problem 2.8.

(1) Apply the second part of Problem 2.7 to show that $\left(1 + \dfrac{1}{n}\right)^{n+1} \geq \left(1 + \dfrac{1}{n+1}\right)^{n+2}$ for all natural numbers n.
(2) Show that $x_n = \left(1 + \frac{1}{n}\right)^n$ is an increasing and bounded sequence. This implies that it is a convergent sequence. The limit is called e.

Problem 2.9. Show that $n! > \left(\frac{n}{e}\right)^n$.

Problem 2.10. Show the following.

(1) $\sqrt{\dfrac{x^2 + y^2}{2}} \geq \dfrac{x + y}{2}$ for all positive x and y.
(2) If a, b are positive numbers such that $a + b = 1$ then $(a + \frac{1}{a})^2 + (a + \frac{1}{b})^2 \geq \frac{25}{2}$.