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Hours: MWF 11:20 – 12:20 MW 1:40 – 2:40

1. Let E, F and G be the points where the excircles of $\triangle ABC$ touch sides a, b and c respectively. Show that AE, BF and CG are concurrent. The point of concurrency is called the Nagel point. Hint: express the lengths in Ceva's theorem with s, a, b, and c.

Solution: Let E_b and E_c be the points where the excircle opposite to A touches the sides b and c respectively. Then we have $AE_c = AE_b$ since these are tangents to the excircle from A. We also have $BE_c = BE$ and $CE_b = CE$ for the same reason. Hence

$$\begin{aligned} a+b+c &= BE+CE+b+c \\ &= c+BE_c+b+CE_b \\ &= AE_c+AE_b \\ &= 2AE_c. \end{aligned}$$

Thus $BE = BE_c = AE_C - c = s - c$. Similar argument shows that

$$\frac{BE}{EC}\cdot\frac{CF}{FA}\cdot\frac{AG}{GB} = \frac{s-c}{s-b}\cdot\frac{s-a}{s-c}\cdot\frac{s-b}{s-a} = 1$$

and so the result follows from the converse of Ceva's theorem.

2. Show that the position vector of the incenter of $\triangle ABC$ is

$$\underline{j} = \frac{a\underline{a}+b\underline{b}+c\underline{c}}{a+b+c}.$$

Note that a, b, c are the lengths of the sides not the lengths of the position vectors.

Solution: Let A' be the intersection of the angle bisector of α and BC. We know that A' divides BC in the ratio $c:b$. So $\underline{a'} = \frac{b\underline{b}+c\underline{c}}{b+c}$. We have

$$\underline{j} = \frac{a\underline{a}+(b+c)\underline{a'}}{a+(b+c)}$$

and so $\underline{j}$ is the position vector of a point J on AA' which is the angle bisector of α. Similarly, J is a point on the other angle bisectors and so J must be the incenter.