

1. Let A be an $n \times 1$ and B be a $1 \times n$ nonzero matrix. What can we say about the rank of AB ?

2. Let $A = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$. Show that $\det(A) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$.

Hint: Use induction on n . Apply row operations to simplify the cofactor expansion on the first column. Use algebraic identities and factor out as much as you can. Apply column operations to get a form where the inductive hypothesis can be applied.

3. Let A be an invertible matrix of size $n \times n$ containing only 0's and 1's. What is the maximum number of 1's A can have.

Hint: Consider the cases $n = 1, 2, 3, 4$ first to get a conjecture. To show invertibility use the cofactor expansion of the determinant and induction. Use properties of the determinant to show that more 1's is not possible.