Arrangements of hyperplanes: Resonance varieties over C and over finite fields

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July 27, 2001

Abstract

The objective of this paper is to discuss those arrangements of hyperplanes that support resonance varieties. The method for determining whether such resonance varieties exist will be explained, and some of the new examples found during the summer REU program will be presented and discussed. Interesting data arises when an arrangement of hyperplanes does not have resonance varieties over C, the complex field, yet does have resonance varieties over a finite field Z_p, for some prime p. Therefore, this paper will build up to examining these types of arrangements, and will discuss the implications in the broader area of all resonance varieties. To understand this paper, some knowledge of linear algebra, specifically matrices, will be needed.

Introduction - Coloring Arrangements and Matroids

Peter Orlik in *Arrangements of Hyperplanes* [6] (also see [5]), defines hyperplanes and arrangements very aptly:

<u>Def</u>: Let **K** be a field and let $V_{\mathbf{K}}$ be a vector space of dimension j. A **hyperplane** H in $V_{\mathbf{K}}$ is a vector subspace of dimension (j-1). An **arrangement** $A_{\mathbf{K}} = (A_{\mathbf{K}}, V_{\mathbf{K}})$ is a finite set of hyperplanes in $V_{\mathbf{K}}$.

For the purposes of this paper we will be looking mostly at the projective plane over $\mathbb{C}$, meaning the hyperplanes are planes in three-dimensional space, but will appear as lines in two-dimensional projective space, looking at the affine plane z=1 from above.

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* This paper was part of the 2001 summer REU program at Northern Arizona University, under the supervision of Mike Falk.

The places where hyperplanes intersect are the points of interest. Using the intersection points, a matroid dot picture can be drawn representing the arrangement. Matroids can be defined by many different sets of axioms; in this case, however, we just need to know that points lying on the same line of a matroid correspond to an intersection point of the corresponding hyperplanes in the arrangement. (For the basics defining a matroid and the properties of a matroid, see references [1] and [7].) Here is an easy example. First, the arrangement is presented, and then the corresponding matroid dot picture:

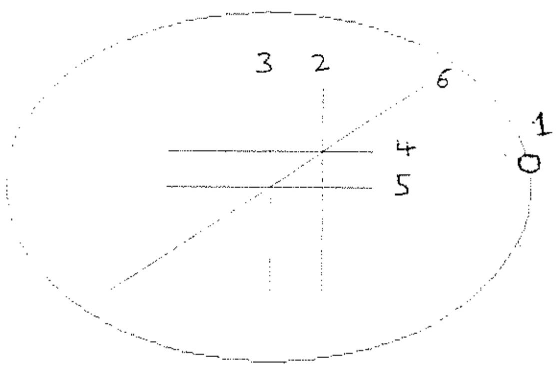

Figure 1a

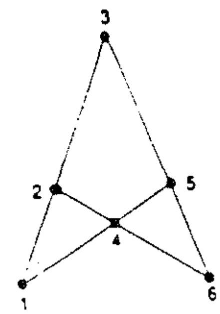

Figure 1b (from [2])

The fact that the points 1,4, and 5 lie on the same line signify that hyperplanes 1,4, and 5 intersect at a point, indicated by the small circle around the intersection. The point 3 denotes the hyperplane at infinity, at which 5 intersects it, 1 and 2 intersect it together, and 4 and 6 intersect it together. Some rules to remember are that parallel lines meet at infinity, and a circle signifies that the hyperplane at infinity is included in the arrangement.

Now that we have a general understanding of arrangements and matroids, the task is to find certain arrangements that follow certain rules. The rules enable us to find arrangements that might give us resonant varieties. For research purposes, we actually work backwards when finding arrangements with certain properties. Therefore, the reasons for the rules given make much more sense when working forwards, although understanding why the rules follow from working forward is very difficult. In this introduction, and in the next section, Resonant Varieties over C, I will state the rules needed to find resonant varieties. Then, in the section entitled Working Forward, I will begin to hint at why the rules work. However, I will not attempt to go into detail about why the rules came into play. Instead, the section will begin to give a broad overview of how the rules fit in.

Now, the rules these hyperplane arrangements must follow is this:

1. Each hyperplane is assigned a certain color.
2. If an intersection of n hyperplanes has n-1 of them the same color, then all n of them must be the same color.

From these rules, it is clear that for an intersection of only 2 hyperplanes, both must be the same color. Also, remember again that parallel lines meet at infinity.
These rules can also be expressed for the matroid dot pictures as follows:

1. Each dot is given a certain color.
2. If a line with n dots has n-1 of them the same color, then all n of them must be the same color.

So lines containing only two points must have both points the same color. One cannot choose where lines are and are not. If there are two dots on a page, there is a line between them. When following these rules, we hope to find a multi-colored arrangement. An arrangement with just one color will not give rise to any resonance. Figure 1a is an example, in fact one of the most famous examples, of an arrangement that follows these rules and gives something multi-colored, but the majority of arrangements do not. Here is one that must be mono-colored, due to too many intersections of just two hyperplanes.

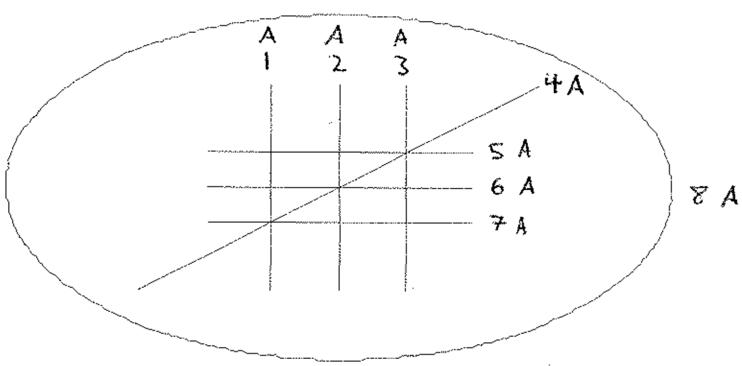

Figure 2

Without loss, hyperplane 1 is colored A, and then hyperplanes 5 and 6 must be colored A (each meet in a two point intersection with 1), so 3 and 7 must be colored A, so 2 must be colored A; then 4 (which meets in a three point intersection with 3 and 5, which are both A) is colored A, so finally 8 is colored A (because of the two-point intersection with 4, or because of the four-point intersection with 1,2,3, or with 5,6,7).

Now that we know how to color an arrangement (or matroid), we can go on to finding resonance varieties.

Resonance Varieties over C

Once we have an example of an arrangement that satisfies the conditions above, we can check it for resonance varieties. To do this, we set up a matrix. The columns are labeled 1 through n, representing each of the n hyperplanes. Then the rows are listed by the multi-colored intersections. These are called the multi-colored flats. This is because in the matroid, they all lie on a single line, showing that they are linearly dependent.
The actual definition of a flat, taken from Brylawski and Kelly's article ([1]), is this:

> For an arbitrary matroid *M* on a set *S*, we say that a $\in$ *S* *depends* on $A \subseteq S$ if either a $\in$ *A* or $B \cap$ a is dependent for some independent subset *B* of *A*. We define the *closure* $\bar{A}$ of a subset *A* of *S* as the set of all points that depend on *A*. The *closed sets*, or *flats*, of *M* are the sets *A* for which $\bar{A} = A$.

In our "dot pictures," dependence is defined as follows: a set of points is dependent if one of them lies in the line or plane determined by the rest.
Now that the rows and columns are labeled, we simply fill in the matrix with 1s and 0s as follows: For the k^{th} column (labeled k and representing the k^{th} hyperplane), if the multi-colored flat in row m includes k, write a 1 in that row and column. Otherwise, write a 0. For example, referring to Figure 1, the row labeled by the multi-colored flat 1,2,3 would have a 1 in the first, second, and third column, and a 0 in the fourth, fifth, and sixth column. The last row has all ones, corresponding to the multi-colored flat that is the entire arrangement.

This matrix is then row-reduced and the rank and a basis for the nullspace are found. We require that the nullspace is two-dimensional or greater. Given two non-parallel vectors in the nullspace, call them λ and μ, and taking two columns that correspond to two hyperplanes of the same color, we ask that the determinant of the 2x2 matrix formed should be zero, for any two columns for hyperplanes of the same color. Once all these conditions are satisfied, the arrangement supports resonance weights λ and μ, and the set of all such resonant weights forms the resonance variety.
I will now show an example to review these steps. As Figure 1 is a famous example, done many times before, I present to you now a figure found during research by Cahmlo Olive:

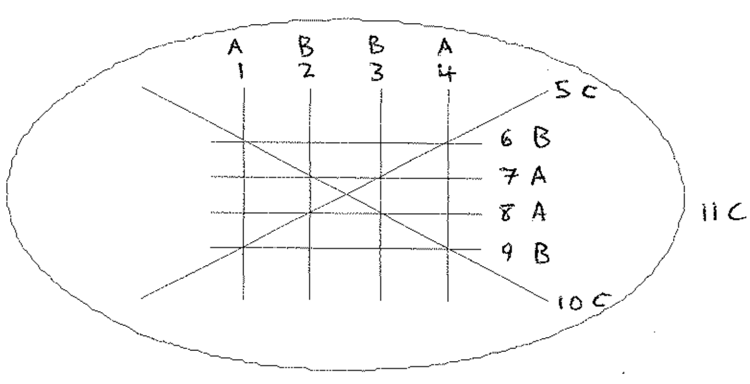

Figure 3

And here is its corresponding matrix:

	1	2	3	4	5	6	7	8	9	10	11
1 6 10	1	0	0	0	0	1	0	0	0	1	0
4 5 6	0	0	0	1	1	1	0	0	0	0	0
4 9 10	0	0	0	1	0	0	0	0	1	1	0
1 5 9	1	0	0	0	1	0	0	0	1	0	0
2 5 8	0	1	0	0	1	0	0	1	0	0	0
2 7 10	0	1	0	0	0	0	1	0	0	1	0
3 5 7	0	0	1	0	1	0	1	0	0	0	0
3 8 10	0	0	1	0	0	0	0	1	0	1	0
6 7 8 9 11	0	0	0	0	0	1	1	1	1	0	1
1 2 3 4 11	1	1	1	1	0	0	0	0	0	0	1
1 - 11	1	1	1	1	1	1	1	1	1	1	1

Matrix 3a

And here is the matrix after being row-reduced:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Matrix 3b

As you can see, the matrix has rank 9, so the nullspace has dimension 2. Here is a basis for the nullspace:

$$\begin{pmatrix}
-1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 2 \\
-1 & 1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & 0
\end{pmatrix}$$

Checking mono-colored 2x2 determinants, hyperplanes 1,4,7, and 8 are all colored A, and the 2x2s given by the two rows and columns 1,4 then 1,7 then 1,8 then 4,7 then

4,8 then 7,8 are all zero. Same for the Bs (hyperplanes 2,3,6, and 9) and the Cs (hyperplanes 5,10 and 11).

It is interesting to note that for λ, A is always represented by -1, B by 0, and C by 1, although the last C is represented by 2, signifying that the line at infinity should be doubled so that the flats 1,2,3,4,11 and 6,7,8,9,11 all have the same number of each color. For μ, A is represented by -1, B by 1, and C by 0. Also, if a column contains only zeros, then the hyperplane corresponding to that column is not needed, and a smaller hyperplane subarrangement that supports the same resonant weights can be found by taking out the hyperplane.

Because a certain number in a vector of the nullspace always represents a color, the 2x2 determinants will always be zero, because you will always get something of the form:

$$\begin{vmatrix} \alpha & \alpha \\ \beta & \beta \end{vmatrix} \quad \text{Or if a line is doubled, tripled, or more:} \quad \begin{vmatrix} \alpha & k\alpha \\ \beta & k\beta \end{vmatrix}$$

The determinants of these types of 2x2 matrices will always give zero.
If the nullspace is of dimension greater than 2, find at least two vectors (not parallel) where the 2x2s are all zero. Then try to add on more vectors where this works. As we will see, the nullspace of dimension greater than 2 shows up much more when working over Z_P than over C, although we will see one example over C.

Once all these conditions are met, the arrangement supports resonant weights.

One more interesting property to note. We can compute the defining polynomials of these hyperplane arrangements. Picking a center of the arrangement as the origin, for the purposes of the defining polynomial, the line x=0 is written as just x, x+y=0 as x+y, x+1=0 as x+z (since we are looking at the projective plane at z =1), and so on. For Figure 3 we see:

A: $(y+z)(y-z)(x+2z)(x-2z)$
B: $(x+z)(x-z)(y+2z)(y-2z)$
C: $(x+y)(x-y)z^2$ (since the line at infinity is doubled)

Here A-B=-3C. For arrangements that work over $\mathbb{C}$ according to the rules stated, you will always get this linearly dependent relationship, where the defining equation of one color can be expressed as a linear combination of the others.

Now we are ready to hint at why these rules work and some of the definitions of the terminology used, by working forward.

Working Forward

We start with a definition of resonant varieties:

Def: λ is resonant if and only if there exists μ not parallel with λ and $a_\lambda \wedge a_\mu = 0$ in the Orlik-Solomon algebra.

For more on the Orlik-Solomon algebra, denoted OS(A), and how it relates to resonant varieties, the reader is encouraged to read the first two sections of [3]. Therefore, I need not go through what is needed to understand this definition, including Orlik-Solomon algebras OS(A) and the like. One easy thing to say about OS(A) is that it is anti-commutative, that is, $a_1 \wedge a_2 = -a_2 \wedge a_1$.

The main theorem about resonant varieties, which gives the rules we use working backward, is this:

Thm: Say λ is resonant with μ, and let $A_{\lambda,\mu} = \text{supp } \lambda \cup \text{supp } \mu$,
Then define a partition π of $A_{\lambda,\mu}$ as follows:
i~j (that is, column i and column j are the same color) if and only if $\begin{vmatrix} \lambda_i & \lambda_j \\ \mu_i & \mu_j \end{vmatrix} = 0$

Then we say the partition π is a neighborly partition; that is, the partition satisfies rules 1 and 2, and that λ, μ is in L_π, and $\langle\lambda, \mu\rangle = 0$ over π. L_π is in the nullspace of the matrix, which is why we check the nullspace for at least two vectors where the necessary 2x2s are zero.

This last statement is what gives us that the determinants of the 2x2s for the same color have to be zero. The theorem gives us the rules described in the introduction for what constitutes a neighborly partition. It is too complicated to try to explain in this paper how this is accomplished, or how the other rules described are set up, but I invite the reader to consult [2] and [3] for more information on working forward.

Resonant Varieties over Z_p

Examples with the potential of giving us more insight into these resonant varieties are those that do not give resonant varieties over C but do over a particular finite field. The rest of this paper will focus on these examples, and what they could mean in terms of certain theorems or conjectures. First, a short introduction on what it means for a hyperplane arrangement to have resonant varieties over Z_p.

Introduction: Working over Z_p

As explained in the section entitled Resonant Varieties over C, the last few steps to check that a hyperplane arrangement has resonant varieties is to set up the matrix listing the multi-colored flats, then row-reduce. If the nullspace is of dimension 2 or higher, and if all the 2x2 determinants in L_π for mono-colored hyperplanes are zero, then the arrangement has resonant varieties.
In a few examples a hyperplane arrangement has a neighborly partition, but does not have two-dimensional nullspace when the matrix is row-reduced. However, it might still have two-dimensional nullspace or greater over a finite field.
For those who do not know how to row-reduce mod p, the rules are not much more difficult than row-reducing over C. You need to do what is usually called integer row reduction. How this differs from regular row-reduction is as follows:

* You may not multiply or divide a row or column by any number, after all, you may be multiplying or dividing by zero!
* You can, however, add or subtract a multiple of one row to another to get rid of leading numbers, since possibly adding zeros to a row will not change anything.

With these rules, you can see that not all leading numbers will be able to be changed into ones. That is expected. Once a matrix is sufficiently integer row-reduced, look for rows or columns where all of the numbers have a common prime denominator. Reducing over this prime, then, will turn that complete row or column into zeros, adding to the rank of the matrix. A small example should suffice:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

Example of integer-row-reducing

As you can see, over C this matrix has rank 2 (the second row will be divided by 3 to give the vector (0, 1,2)). However, over Z_3 the second row turns to all zeros, giving the matrix rank 1.

What does it mean if an arrangement does not have resonant varieties over C, but does over Z_p? We are still not completely sure, but the examples in this text will bring us closer to letting us make conjectures and prove theorems about resonant varieties over Z_p that we already know or suspect for resonant varieties over C.

Examples

The rest of this paper will be dedicated to the examples of arrangements that do not work over C but do over Z_p, as well as certain conjectures that we might deduce from these examples.

Example 1

The first example we will look at is the seven-point triangle, with five colors:

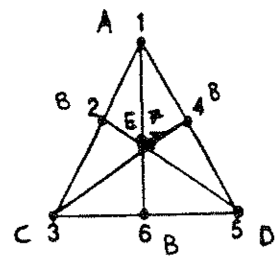

Figure 4

The matrix has nullity = 0 over C, but nullity = 3 over Z_2:

$$\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&0&0&1&0&0&1\\ 1&0&0&0&1&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&1&0&0\\ 0&0&1&0&0&1&1\\ 1&1&1&1&1&1&1 \end{pmatrix} \quad \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \quad \begin{pmatrix} 1&0&0&0&1&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&0&0&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

Matrix 4 | Matrix 4 Row-reduced over C | Matrix 4 Row-reduced over Z_2

A basis for the nullspace over Z_2 is:

$v_1 = (1,1,0,1,1,0,0)$
$v_2 = (1,0,1,1,0,1,0)$
$v_3 = (0,1,1,1,0,0,1)$

We see, looking at the 2x2s for the mono-colored dots 2,4, and 6, that v_1 and v_3 give us the results we want, the determinants equaling zero. Also, we see that $v_1 + v_3 = (1,0,1,0,1,0,1)$ gives a zero determinant for the 2x2 matrices for v_1, v_2, and v_3. However, we see that for v_1 and v_3, the 6th column contains all zeros, meaning the dot labeled 6 is not needed. Taking away that point, we get a copy of the famous 6-point matroid introduced at the beginning of this paper (Fig. 1). So we have found nothing new so far.

Similarly, comparing v_2 and $v_1 + v_3$, the second column is all zeros. Taking away that point gives another copy of Figure 1. Finally, comparing v_1 with $v_1 + v_3$ or v_3 with $v_1 + v_3$ tells us to take out that 6th dot again. So for all of the possibilities, the appropriate vectors in the nullspace tells us to take the subarrangement of the seven-point triangle that turns out to be Figure 1. While this triangle does have resonant varieties over Z_2, the nullspace gives us nothing new to work with in exploring resonant varieties over finite fields.

What is most interesting here is that $v_1 + v_3 = (1,0,1,0,1,0,1)$ forms a resonant pair with v_1 v_2 and v_3, hence with every vector in L_π. This does not occur over $\mathbb{C}$. More information on this example can be found in [4].

Example 2

This example is the "MacLane Matroid," which has eight points. Although it is a dot picture, some curved lines have to be drawn to show some of the lines with three points, such as the line containing the points 1, 2, and 6. This picture represents a set of points in $\mathbb{C}^3$ where the points on the curved lines are actually collinear. Here is a picture of the MacLane Matroid:

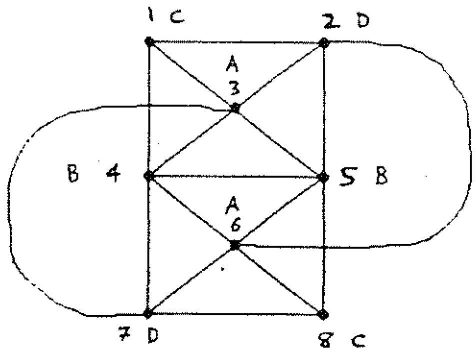

Figure 5

One can check that the nullity of the matrix over C is zero, but over Z_3 the nullspace has dimension 2. A basis for the nullspace is:

$v_1 = (0, -1, -1, -1, 1, 1, 1, 0)$
$v_2 = (-1, 0, -1, 1, -1, 1, 0, 1)$

Since the mono-colored columns are columns 1 and 8, 2 and 7, 3 and 6, and 4 and 5, the 2x2 matrices all have determinant zero. However, in this example, it is hard to say that the numbers in v_1 and v_2 correspond to the colors that the points should be, such as say, A should be represented by -1, B by 1 and so on.
For example, consider columns 4 and 5. These points lie on a 2-point line, so they must be colored the same. However, column 4 and 5 list the numbers as -1 and 1, respectively for v_1. Complicating matters further, the numbers show up as 1 and -1 respectively for v_2.
A way to account for this anomaly would be to say some of the lines need to be doubled. Since we are working mod 3, -1 is the same as 2. If we double column 1, column 3, column 4, and column 7, then A and B would both be represented by 1, D would be represented by -1, and C (the color C) by 0, for v_1. For v_2 then, the numbers are A=1, B=-1, C=1, D=0. However, there seems to be no clear logic in doubling these points, such as giving the same number of colors for each line. Therefore, work still has to be done in explaining this example, and deducing what we can say about resonant varieties mod p.

Example 3

This example is an exciting one. It was discovered in two ways. First, by Cahmlo Olive, who came up with this picture of the hyperplane arrangement:

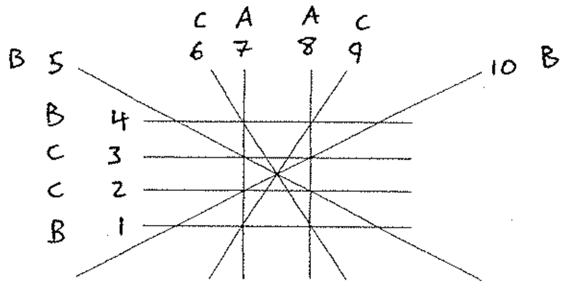

Figure 6

Also, this ten-line hyperplane arrangement is a sub-arrangement of the 15-line icosahedron arrangement pictured here:

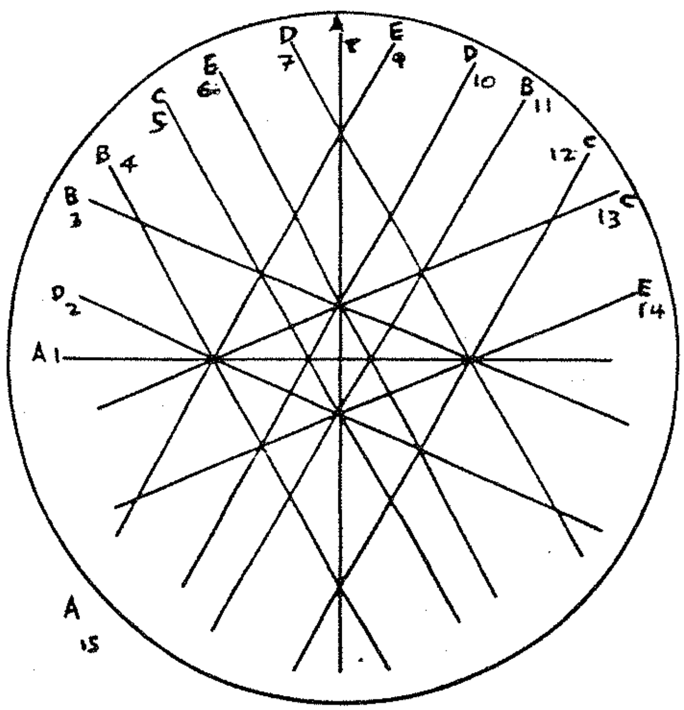

Figure 7

While the matrix for this arrangement has zero nullity over C, over Z_2, the nullspace has dimension 5. For convenience, I will move around the columns of the nullspace so that the

the first three numbers correspond to the As, the next three to the Bs, and so on. The nullspace then is:

$v_1 = (1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0)$
$v_2 = (0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0)$
$v_3 = (1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0)$
$v_4 = (1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 1)$
$v_5 = (1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0)$
[____] [____] [____] [____] [____]
A B C D E

From this, one can check that using v_1 and v_3, the 2x2 determinants turn out to be zero for all the pairs of mono-colored hyperplanes. Also, for v_1 we get that A, B, D and E are all represented by the number 1, while the number zero represents C. For v_2, A, C, and E are all represented by 1, while B and D are zero. But we also get some columns of all zeros, telling us to take away the third A numbered, the third B, the third C, the second D, and the third E.

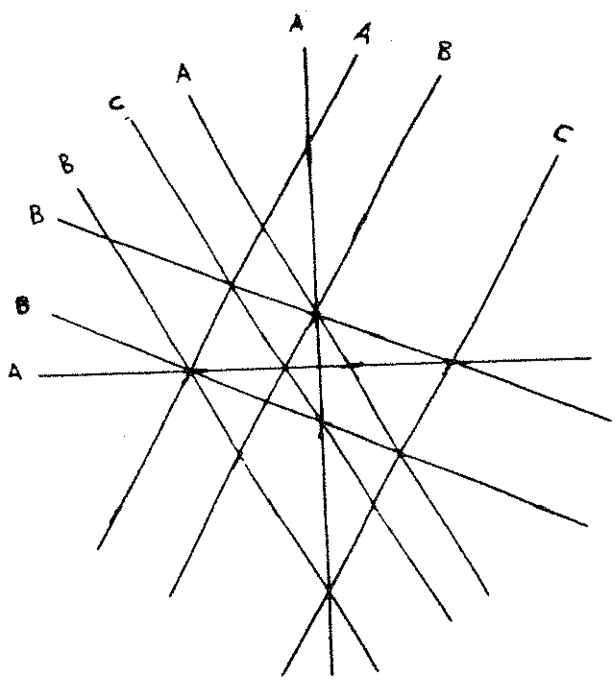

Figure 8

When we do this, some of the colors collapse, turning all the Es into As and all the Ds into Bs. Once we do this, with a little renumbering, we can get all of the same multi-colored flats as in Figure 6, meaning the matrix is the same, and the arrangements are the same, since they are recognized by their intersection points in the matrix. So Figure 6 and Figure 8 are the same, but Figure 8 was obtained by taking the subarrangement of the 15-line icosahedron example specified by the nullspace basis.

Here is Figure 8 with the appropriate lines taken away, alongside Figure 6. Although they do not look similar, since there intersections contain the same numbers, they have the same underlying matroid. As Figure 6 looks easier to work with, we will continue with the example using that figure:

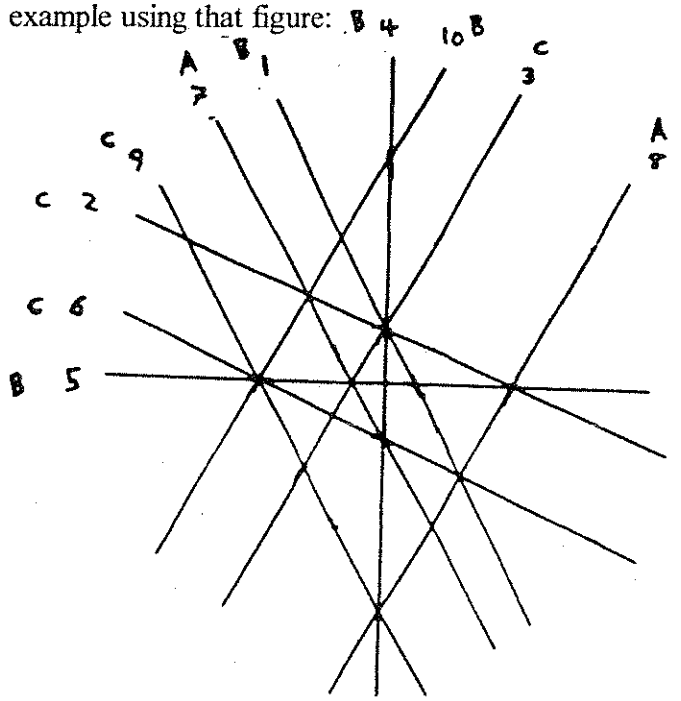

Figure 8b

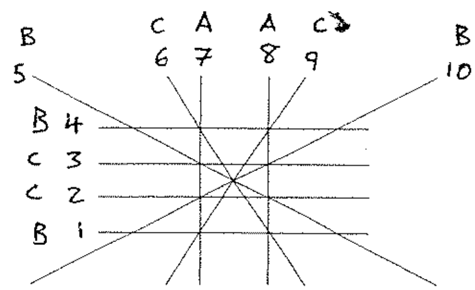

Figure 6 (again)

The nullspace over C for the matrix corresponding to this example is of dimension 1. Remember the nullspace had to have dimension 2 or higher, so this example has no resonant varieties over C. Over Z_2, however, the matrix has nullspace with dimension 3. The vectors for a basis for the nullspace are:

$v_1 = (1, 1, 0, 0, 1, 1, 0, 0, 1, 1)$
$v_2 = (1, 1, 0, 0, 0, 0, 1, 1, 0, 0)$
$v_3 = (0, 0, 1, 1, 1, 1, 0, 0, 0, 0)$

[B] [C] [A]

For these three vectors, no combination of two give the required zero determinants for mono-colored 2x2s. But a linear combination of the vectors might work. Since we are working mod 2, there are not that many linear combinations of these three vectors to look at. The linear combinations (besides the three listed above) are:

$v_1 + v_2 = (0, 0, 0, 0, 1, 1, 1, 1, 1, 1)$
$v_1 + v_3 = (1, 1, 1, 1, 0, 0, 0, 0, 1, 1)$
$v_2 + v_3 = (1, 1, 1, 1, 1, 1, 1, 1, 0, 0)$
$v_1 + v_2 + v_3 = (0, 0, 1, 1, 0, 0, 1, 1, 1, 1)$

Any of $v_1 + v_2$, $v_1 + v_3$, or $v_2 + v_3$ together with another of these three will work, but all three together will not. For example, we will use $v_1 + v_2$ and $v_1 + v_3$. For $v_1 + v_2$, B is represented by 0, and A and C are represented by 1. For $v_1 + v_3$, A and B are represented by 1, and C is represented by 0.

There are also other interesting points to note. Every multi-colored flat has either: 1. one of each color, or 2. two Bs and two Cs. So while not every multi-colored flat always has the same number of hyperplanes of each color, every multi-colored flat does have the same number of hyperplanes of each color **mod 2**. There is a possible conjecture here about when a hyperplane arrangement will not have resonant varieties over C but will over Z_p. For example, I would not be surprised if there were a 15-line arrangement with 3 As, 6 Bs and 6 Cs, which does not work over C but does over Z_3.

Other Examples

There are still a couple of other examples that did not make it deeply into my research, but that could be checked out further, possibly for the next draft. One is the 9-point "ceva," the dot picture of which looks like this:

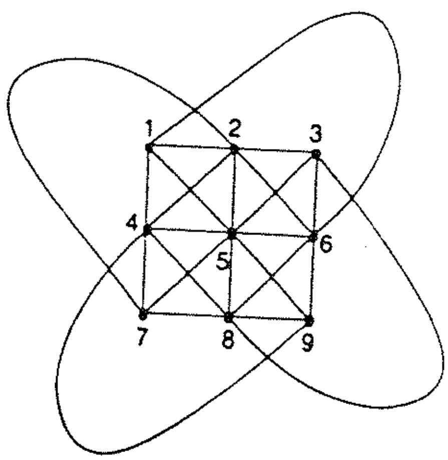

Figure 9 (from [2])

As you can check, there are no two point lines, and different ways of coloring it. The nullspace has dimension 2 over C, but dimension 3 over Z_3. Not much has yet been determined about the nullspace or the 2x2s.

Also, you can make this picture a line picture instead of a dot picture, colored like this:

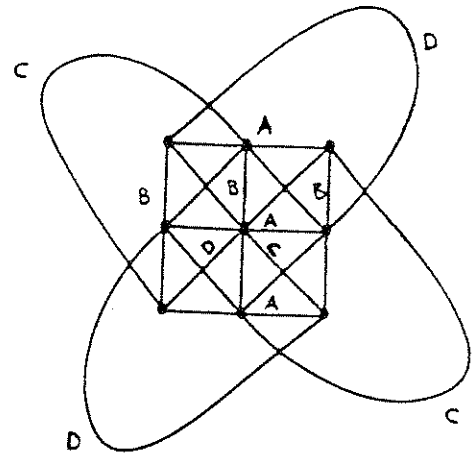

Figure 10 (from [2])

Here the nullspace has dimension 3 over C, which is unusual. All mono-colored 2x2s vanish, and specific colors correspond to specific numbers. In short, this picture has resonant varieties over C. Over Z_3, the nullspace has dimension 5. I have yet to find a pair of linear combinations of the basis vectors that will give zero determinants for the 2x2s. For more information on this example, see [2].

Finally, here are two more examples taken from [2]. Just like the seven-point triangle, these also have copies of Figure 1 in them. The first figure has 5 copies of Figure 1, while the second figure has 4 copies. There are some predictions about what might happen in different finite fields. Here are the figures:

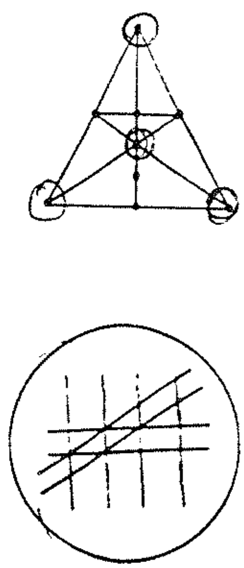

Figure 11

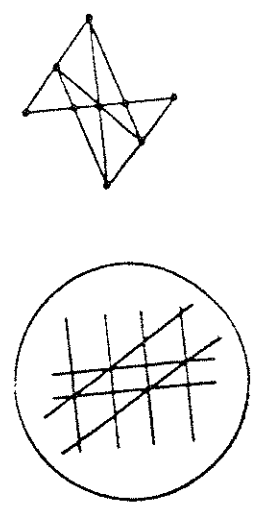

Figure 12

These examples may give some ideas about what more there could be to do in checking what theorems may be present about resonant varieties over Z_p.

Further Research with Resonant Varieties over Finite Fields

While there are theorems about resonant varieties over C (see [2] and [3]), some of these theorems do not seem to work over Z_p. So these theorems need to be examined more to see how it will fit for finite fields, or how a theorem could be generalized to fit all fields. It would also be useful to know when a hyperplane arrangement will not work over C but will over Z_p. After this, it would be useful to know what it means to have resonant varieties over a finite field, and how this will help us better understand the Orlik-Solomon algebras. Before this is accomplished, however, some interesting questions need to be answered, such as: Do specific numbers in the nullspace still correspond to specific colors? What does it mean if the nullspace does not produce two vectors where the mono-colored 2x2s are all zero? These are just a couple of the questions that need to be answered to have a better understanding of resonant varieties over finite fields.

Conclusion

We have looked at a certain type of hyperplane arrangement, one with a neighborly partition and a matrix with a certain nullspace. By finding these arrangements with resonant varieties, we can better understand the Orlik-Solomon algebra. Still to be done, however, is to find the connections and the differences between finding resonant varieties over C and those not over C but over Z_p. While theorems and conjectures are made for resonant varieties over C, some may have to be changed for working over finite fields, and explained as to what the changes signify. This paper is the first step in this process, presenting examples over Z_p that point out the similarities and the differences from the examples over C. With this knowledge, we can conjecture and try to prove what it is that these examples and the other unfound ones will show.

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