GRAPHIC HYPERPLANE ARRANGEMENTS

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1. Introduction and notation

Let $\mathcal{A}$ be an arrangement of hyperplanes, that is $\mathcal{A} = \{H_1, \cdots, H_n\}$, a finite set of hyperplanes in $\mathbb{C}^\ell$. Each hyperplane is the kernel of a linear transformation, $\alpha_i : \mathbb{C}^\ell \to \mathbb{C}$, or equivalently, the null space of a matrix, $a_i = [a_{i_1} \cdots a_{i_\ell}]$. We call a subset, $S \subseteq \mathcal{A}$ dependent if and only if the corresponding linear transformations are linearly dependent, i.e. $\{\alpha_i | H_i \in S\}$ is a linear dependent set.

We will focus on *graphic* arrangements, that is, arrangements which get their defining equations from some simple graph, Γ. Assume we are given a graph, Γ with edge set, E, then the defining equations of the arrangement are:

$$\{z_i - z_j = 0 \mid (i, j) \in E\}.$$

Given Γ, the number of edges in the graph corresponds to the number of hyperplanes as each edge is attributes one defining equation. In addition, the rank of the graph, the maximal spanning tree, represents the dimension that we are in.

To each arrangement, we have the main algebraic object, the Orlik-Solomon (O.S.) algebra, $A(\mathcal{A})$. For each hyperplane in our arrangement we assign a basis vector, e_i. Let $\mathcal{E}$ be the exterior algebra generated by these basis vectors, $\bigwedge(e_1, \cdots, e_n)$. We define a linear mapping,

$$\partial : \mathcal{E}^p \to \mathcal{E}^{p-1} \text{ by } \partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{n} (-1)^{k-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p},$$

where $\widehat{e_{i_k}}$ indicates we omit this element. For notational simplicity, for some element $s = (i_1, \cdots, i_n)$, we denote $e_{i_1} \wedge \cdots \wedge e_{i_p}$ by e_s. Let $\mathcal{I}$ be the ideal generated by ∂e_s where s is a dependent set, i.e.

$$\mathcal{I} = < \partial e_s | s \text{ is a dependent set} >.$$

We take the quotient of $\mathcal{E}/\mathcal{I}$ and we obtain the O.S. algebra, $A(\mathcal{A})$. It is important to note that this algebra is graded. We know that the exterior algebra, $\mathcal{E}$ has a natural grading. Additionally, $\mathcal{I}$ is generated by homogeneous elements and so the resulting quotient

$$\mathcal{E}/\mathcal{I} \cong A(\mathcal{A})$$

is a graded algebra. This allows us to decompose

$$A(\mathcal{A}) \cong A^0 \oplus A^1 \oplus \cdots \oplus A^n$$

where n is the dimension of our arrangement, i.e. the number of hyperplanes. One of the most crucial theorems in the theory of hyperplane arrangements follows.

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Theorem 1.1. *Let $\mathcal{A}$ be an arrangement of hyperplanes. Let $M = \mathbb{C}^\ell - \bigcup_{H\in\mathcal{A}} H$ be the complement of the union of hyperplanes in complex space. Then $A(\mathcal{A}) \cong \mathcal{H}^*(M)$, that is the O.S. algebra is isomorphic to the cohomology of the complement.*

This theorem tells us that the cohomology is calculated from combinatorial data, as the only information in the quotient of $\mathcal{E}/\mathcal{I}$ consists of the generators for the hyperplanes and the dependent sets. In addition, for graphic arrangements, this tell us that if we have two isomorphic graphs, Γ_1 and Γ_2, and we take the associated arrangements, $\mathcal{A}_{\Gamma_1}$ and $\mathcal{A}_{\Gamma_2}$ they will have (linearly) isomorphic O.S. algebras and thus isomorphic cohomology. A natural question to then ask is whether or not two (graphic) arrangements with isomorphic O.S. algebras (or cohomologies) are generated from two isomorphic graphs. The quick answer is no, an entire class of counterexamples to such a statement can be found in **Orlik-Solomon algebras and Tutte Polynomials**. However, we try to find sufficient conditions such that this statement can be true. This serves as the motivation for the work contained in this paper.

2. Resonance Varieties

Definition 2.1. For our purposes a graph has a non-trivial resonance variety if and only if every one of its edges is in a K3. For our purposes we will call these graphs (and subgraphs) non-trivial (note that this is not the conventional use of the term). Hence we use the following definitions considering only non-trivial graphs and subgraphs.

Definition 2.2. We say that two non-trivial subgraphs H and K are edge-disjoint if they do not share any edges.

Definition 2.3. A non-trivial graph G, is edge-joint if there does not exist two edge-disjoint non-trivial subgraphs, H and K, such that $H \cup K = G$.

Definition 2.4. A non-trivial subgraph H, of a non-trivial graph G, is an edge-joint component if it is a maximal edge-joint non-trivial subgraph. That is, for any non-trivial subgraph K such that $K \not\subset H$ we have that the union $K \cup H$ is edge-disjoint.

Theorem 2.5. *Given a graphic arrangement $\mathcal{A}$, let $\mathcal{B}$ be the arrangement formed by removing all hyperplanes from $\mathcal{A}$ which are not in a K3 in the associate graph. Let H be the graph associated with $\mathcal{B}$ and $e(H)$ and $c(H)$ denote the number of the edges in H and the number of edge-joint components of H respectively, then* $\dim\langle\mathcal{R}(\mathcal{A})\rangle = e(H) - c(H)$.

Proof. We know via earlier discussions that the only components in $\mathcal{A}$ which contribute to $\mathcal{R}(\mathcal{A})$ are those which form a K3 or a K4 in the graph associated with $\mathcal{A}$. Thus we need only consider the components in $\mathcal{A}$ which belong to at least a K3 to obtain $\mathcal{R}(\mathcal{A})$, in other words $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{B})$. Thus we need only concern ourselves with graphic arrangements corresponding to graphs which have every edge belonging to at least one K3. To prove our theorem, it suffices to show that if $\mathcal{B}$ is a graphic arrangement such that in the associated graph H, every edge belongs to a triangle and H is an edge-joint graph then $\dim\langle\mathcal{R}(\mathcal{B})\rangle = e(H) - 1$. Our original theorem follows from this as an immediate consequence of summing up the previous equality for every edge-joint component of H.

It is trivially known that $\dim \langle \mathcal{R}(\mathcal{B}) \rangle \leq e(H)-1$ due to the fact that $\langle \mathcal{R}(\mathcal{B}) \rangle \subseteq \{x \in \mathbb{C}^n \mid \sum_{i=1}^{i=n} x_i = 0\}$. It remains to be shown that $\dim \langle \mathcal{R}(\mathcal{B}) \rangle \geq e(H) - 1$. It can be shown by induction that $\dim \langle \mathcal{R}(\mathcal{B}) \rangle \geq e(H) - 1$. Let $\mathcal{T}$ be the set of all triangles $\{t_1, t_2, \cdots, t_m\}$, in H, and let H_i be the subgraph of H induced by $\{t_1, \cdots, t_i\}$. Let $\mathcal{K}$ be the set of all K3s and K4s in H. Then since $\mathcal{R}(\mathcal{B}) = \bigcup_{\forall x \in \mathcal{K}} L_x$ we have $\mathcal{R}(\mathcal{B}) \supseteq \bigcup_{\forall x \in \mathcal{T}} L_x$. It suffices to show that $\dim \left\langle \bigcup_{\forall x \in \mathcal{T}} L_x \right\rangle \geq e(H) - 1$.

The linear subspace corresponding to a single K3 in H is the space $L_t = \{\lambda \in \mathbb{C}^n | \lambda_i + \lambda_j + \lambda_k = 0\}$ where i, j, and k correspond to the edges of the $t_i \in H$. This linear subspace is spanned by the following basis vectors:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 \end{pmatrix}$$

Without loss of generality, the first three rows are i, j, and k. Thus our induction hypothesis:

$$\dim \langle \bigcup_{\forall x \in \mathcal{T}} L_x \rangle \geq e(H) - 1$$

holds trivially for $i = 1$, the case of the triangle, where $e(H) - 1 = 2$ and $\dim \langle L_t \rangle = 2$. Now we show that if it is true for H_i it must be true for H_{i+1}. It should first be noted that since H is edge-joint, it is always possible to establish a sequence $t_1, \cdots, t_m$ such that $\forall i \in \{1, \cdots, i-1\}$, H_i and t_{i+1} share at least one edge (else all remaining triangles would be edge-disjoint from H_i and hence H would be edge-disjoint). This means we need not consider the case where $e(H_{i+1}) = e(H_i) + 3$. Now consider the case when $e(H_{i+1}) = e(H_i) + 2$. In this case the space $\left\langle \bigcup_{\forall x \in H_{i+1}} L_x \right\rangle$ has a basis formed by the row span of the following matrix:

$$\begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,l} & 0 & 0 \\ h_{2,1} & h_{2,2} & \cdots & h_{2,l} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{2*i,1} & h_{2*i,2} & \cdots & h_{2*i,l} & 0 & 0 \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & -1 \end{pmatrix}$$

It should be noted that all columns corresponding to edges not in the subgraph H_{i+1} have been removed for convenience because they contain only zero entries. It is readily seen upon observation that the last two rows cannot be reduced via row operations and thus that $\dim \left\langle \bigcup_{\forall x \in H_{i+1}} L_x \right\rangle = \dim \left\langle \bigcup_{\forall x \in H_i} L_x \right\rangle + 2$. Thus if

$\dim\left\langle \bigcup_{\forall x\in H_i} L_x \right\rangle \geq e(H_i)$ then $\dim\left\langle \bigcup_{\forall x\in H_{i+1}} L_x \right\rangle \geq e(H_{i+1})$

Now consider the case where $e(H_{i+1}) = e(H_i)+1$. In this case the space $\left\langle \bigcup_{\forall x\in H_{i+1}} L_x \right\rangle$ has a basis formed by the row span of the following matrix:

$$\begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,l} & 0 \\ h_{2,1} & h_{2,2} & \cdots & h_{2,l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{2*i,1} & h_{2*i,2} & \cdots & h_{2*i,l} & 0 \\ 0 & \cdots & 1 & -1 & 0 \\ 0 & \cdots & 1 & 0 & -1 \end{pmatrix}$$

It is obvious upon examination that row reduction could reduce at most one of the last two rows. Thus $\dim\left\langle \bigcup_{\forall x\in H_{i+1}} L_x \right\rangle \geq \dim\left\langle \bigcup_{\forall x\in H_i} L_x \right\rangle + 1$. Since $e(H_{i+1}) = e(H_i) + 1$ we have: that $\dim\left\langle \bigcup_{\forall x\in H_{i+1}} L_x \right\rangle \geq e(H_{i+1})$.

The only remaining case our induction hypothesis must be proven for is when $e(H_{i+1}) = e(H_i)$. However this is trivial because $\dim\left\langle \bigcup_{\forall x\in H_{i+1}} L_x \right\rangle \geq \dim\left\langle \bigcup_{\forall x\in H_i} L_x \right\rangle$ which implies $\dim\left\langle \bigcup_{\forall x\in H_{i+1}} L_x \right\rangle \geq e(H_{i+1})$.

This concludes our proof that $\dim\langle\mathcal{R}(\mathcal{A})\rangle = e(H) - c(H)$.

□

The following is a useful corollary:

Corollary 2.6. *Given a* 2-connected*, parallel-indecomposable graph* Γ*, such that each edge is contained in a K3,* $\dim(\mathcal{R}^1(\mathcal{A})) = e(\Gamma) - 1$.

3. Polymatroids

$$\mathcal{R}^1 = Span\{L_1 \cup \cdots \cup L_k\} \bigcup M_1 \cup \cdots \cup M_n\}$$

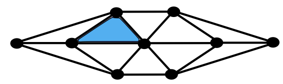
We know that each L_i corresponds to a K3 and each M_i corresponds to a K4.

The **polymatroid** is a function that assigns to each subspace of $\mathcal{R}^1$ the dimension of its span.

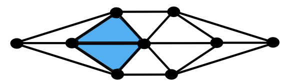
Key point: look at **Wheel Graphs**

Wheel graphs correspond to **degenerate sets** where we get a *less than expected* dimensional first resonance variety.

Examples

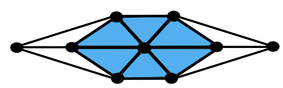


A K3 contributes a **2** dimensional component to $\mathcal{R}^1$.

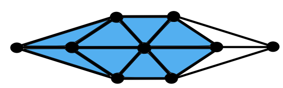


Here is the **parallel connection** of two K3s. We are able to prove the following:

Theorem 3.1. *Parallel connection of a K3 corresponds to adding an additional 2 dimensional component to* $\mathcal{R}^1$.

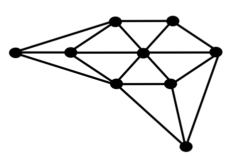


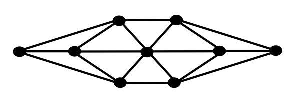
A **wheel graph** on 7 vertices corresponds to an **11** dimensional component of $\mathcal{R}^1$, even though we would expect a 12 dimensional component.



If we consider the left-most wheel graph we get a component of dimension 7. Adding the remaining K3s in succession, we add three K3s by parallel connection $(3 \cdot 2 = 6)$ and the final K3 forms a wheel graph, so we only gain a 1 dimensional component, resulting in a total omponent of dimension **14**.

4. Possible Counterexample





The graphic arrangements corresponding to both of these graphs have:

(a) Equivalent Chromatic Polynomials
(b) Same Polymatroid
(c) Isomorphic Quadratic O.S. Algebras

(a) indicates that the dimension of each grading of the O.S. algebra corresponding to both arrangements are the same.
In fact, many other computable invariants are the same for both arrangements (ϕ_3 etc ...).

5. Realization Spaces of Graphic Matroids and of Rank-3 Truncations of these Matroids

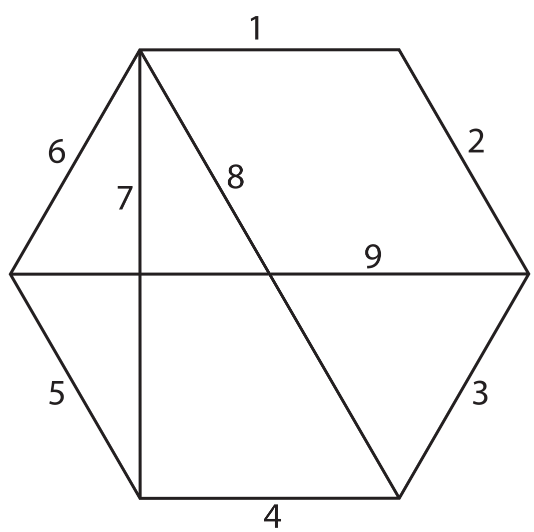
From a graph Γ, we can construct its corresponding graphic arrangement $\mathcal{A}_\Gamma$. This naturally gives us the complement of the arrangement: $\mathrm{M}(\mathcal{A}_\Gamma)$. From here we can construct the matroid and then use the matroid to construct the Orlik-Solomon Algebra, the Resonance Varieties (specifically R^1), and finally the polymatroid. It is our goal to find out how much information regarding the graph Γ can be learned directly from the polymatroid. In other words, how far can we backtrack in the process stated above.

Realization spaces deal with a small part of the puzzle. Specifically, they enable the construction of the complement of the graphic arrangement from the matroid, up to diffeomorphism. We are able to do this by applying Randell's Isotopy Theorem: if the realization space of a graphic matroid is connected, then applying Randell's Isotopy Theorem we get back the complement of the arrangement $\mathrm{M}(\mathcal{A}_\Gamma)$.

Let us begin with the definition of a realization space.

Definition 5.1 (Realization Space)**.** Let M be a rank l matroid on n points. Let $\tilde{R}(M) = \{B^{n\times l} \in \mathbf{C}^{n\times l} \mid \text{the row-dependence matroid of B is M}\}$. Construct the following equivalence relation $\sim$: $B \sim \tilde{B} \iff \tilde{B} = DBC$ where C is a nonsingular $l \times l$ matrix (which deals with change of variables), and D is a nonsingular diagonal $n \times n$ matrix (which deals with scaling each of the rows of B). The **realization space** of the matroid M is $R(M) = \tilde{R}(M)/\sim$.

Example 5.2. We examine the following graph and construct its realization space.



First we construct the set of circuits:

$$\{(567), (478), (1238), (4568), (1269), (3459), (12347), (123456)\}$$

Since this graph has rank 5, it is possible for us to choose five edges such that they are linearly independent in our representing matrix B. In the graph, this means that these five edges do not form a circuit.

Let us choose $\{1, 2, 3, 4, 5\}$ as our basis. By rescaling of rows, in other words using D and C from above, we may relabel our basis to be the standard basis for a rank-5 matrix with five columns. We let $i = e_i$ for $i \in \{1, 2, 3, 4, 5\}$. We may also choose a sixth edge to close off the circuit and make the set of six edges linearly dependent such that no five of them are dependent. We let 6 be this edge, which makes the largest circuit in our graph. So 6 is a linear combination of our basis row-vectors. Using D and C again we can relabel 6 to be the row vector $\begin{pmatrix}1 & 1 & 1 & 1 & 1\end{pmatrix}$.

Now let us construct 7. We see it is a linear combination of $\overline{56}$, as well as $\overline{1234}$. Because it is a linear combination of $\overline{1234}$ we see that the fifth entry in 7 must be zero, as 1, 2, 3, and 4 all contribute 0 in this entry. As 7 is a linear combination of $\overline{56}$ we see that the first four entries must all be the same. By rescaling, we get that $7 = \begin{pmatrix}1 & 1 & 1 & 1 & 0\end{pmatrix}$.

We now construct 8: a linear combination of $\overline{456}$, $\overline{47}$, and $\overline{123}$. Since 8 is a linear combination of $\overline{123}$, we see that 8 must have zeros as its fourth and fifth entries. Since 8 is a linear combination of $\overline{47}$ we see that 8's first three entries must be the same. With this requirement, 8 automatically becomes a linear combination of $\overline{456}$. By rescaling these first three entries, we get that $8 = \begin{pmatrix}1 & 1 & 1 & 0 & 0\end{pmatrix}$.

Finally, we construct 9, a linear combination of $\overline{126}$, and $\overline{345}$. Since 9 is a linear combination of $\overline{345}$, 9 must necessarily have zeros as its first two entries. Because 9 is a linear combination of $\overline{126}$ we see that its last three entries must be the same. By rescaling, we get that $9 = \begin{pmatrix}0 & 0 & 1 & 1 & 1\end{pmatrix}$.

So our B matrix looks like:

$$B = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 1\end{pmatrix}$$

.

Notice that there are no free variables in this matrix. So it defines one point in $\mathcal{C}^{9\times 5}$. Trivially, this point, and hence the realization space of our matroid, is connected. More so, the matroid of our graph is projectively unique, due to the realization space being a single point.

Definition 5.3 (Binary Matroid)**.** A **binary matroid** is a matroid that can be representable over the two-element field. For our purposes, the important part of this definition is that this is a generalization of graphic matroids, which are thus representable over a two-element field.

As it turns out, in *Uniquely Representable Combinatorial Geometries* Brylawski and Lucas proved that all binary matroids are projectively unique with respect to the equivalence relation $\sim$ defined earlier.

In order to further study realization spaces of graphic matroids, we altered our study of realization spaces. Due to our focus on only the first resonance variety, we are unable to detect dependencies larger than the 3-circuits. Brylawski and

Lucas's proof, combined with the lack of information given by the first resonance variety, altered our study to focus on the rank-3 truncations of matroids and their corresponding realization space.

In other words, any set of 4 or more row-vectors is automatically linearly dependent. Because of this, we can assume that every edge is in a K_3 in our graph. For if an edge e were not in a K_3, we could remove it from the graph when constructing our matrix B. Once all of the dependencies with the edited graph are accounted for, we may add a row vector to B to represent e such that this last row does not create a linear dependence relation with any two rows above it.

We now seek to prove that all rank-3 truncations of graphic matroids have connected realization spaces. If true, we could apply Randell's Isotopy Theorem and construct the complement of the corresponding arrangement $\mathrm{M}(\mathcal{A}_\Gamma)$.

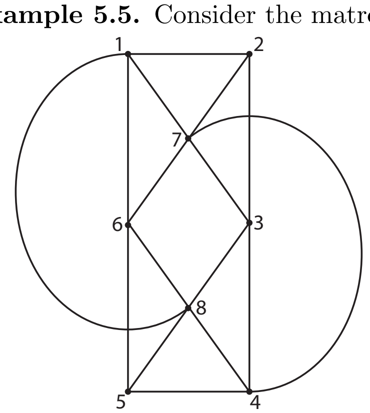
To illustrate our motivation for proving that these rank-3 truncations have connected realization spaces we follow with an example of a non-graphic matroid whose realization space is not connected.

But first let us state the Cayley Algebra Formula which is used to uniquely construct a row of the B matrix.

Definition 5.4 (Cayley Algebra Formula)**.** Given a row vector in 3-space x_m, such that x_m is a linear combination of $\overline{x_i x_j}$ and $\overline{x_k x_l}$, we can uniquely construct x_m using the following formula:

$$x_m = (x_i \vee x_j) \wedge (x_k \vee x_l) = -det(x_i, x_j, x_k) \cdot x_l + det(x_i, x_j, x_l) \cdot x_k$$

Example 5.5. Consider the matroid:



First, lets us construct the set of 3-circuits:

$$\{(128), (583), (234), (173), (457), (165), (276), (684)\}$$

As in the above examples, we may label three linearly independent points in the matroid as our basis, and a fourth one to make the set linearly dependent. Let $\{1, 2, 4\}$ be our standard 3-basis and $5 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$.

Noting that 8 is a linear combination of $\overline{12}$ we see that 8 must necessarily have a zero in its third entry. We may rescale its first entry to 1 and denote its second entry by a such that $a \neq 0$. Hence, $8 = \begin{pmatrix} 1 & a & 0 \end{pmatrix}$.

Using the Cayley Algebra Formula (CAF) we can uniquely construct 3 such that $3 = \begin{pmatrix} 0 & a-1 & 1 \end{pmatrix}$. Again, applying CAF we can uniquely construct 7 to get that $7 = \begin{pmatrix} 1-a & 1-a & 1 \end{pmatrix}$. Similarly, using CAF we get that $6 = \begin{pmatrix} 1-a & 1 & 1 \end{pmatrix}$.

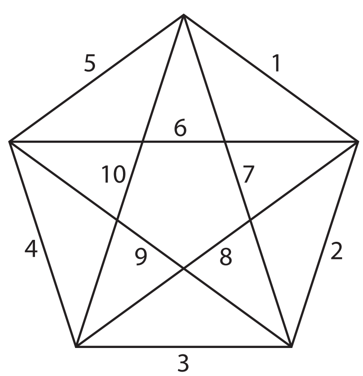
Now we need to check the last circuit (684). Computing the determinant of these three row vectors we arrive at the equation: $a^2 - a + 1 = 0$. This implies that $a = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Hence our B matrix looks like:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a-1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1-a & 1 & 1 \\ 1-a & 1-a & 1 \\ 1 & a & 0 \end{pmatrix} \text{ such that } a = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

These are two isolated points in the complex realization space. Hence, the realization space of this matroid is not connected, as there is no path inside the space to connect these two points.

Next we choose a graphic matroid, specifically $K-5$, and show that its realization space is connected.

Example 5.6. Consider the K_5 graph labelled as such:



We begin by constructing the set of 3-circuits:

$$\{(283), (165), (127), (579), (269), (349), (468), (1810), (3710), (4510)\}$$

Again, we can choose three independent row vectors to be our basis and a fourth vector to have ones as its three entries. We choose 1, 2, 6, 8 to be these vectors; the first three are our basis.

We first construct 3. Noting that it is a linear combination of $\overline{28}$ we see that the first and third entries must be the same so we may rescale them to 1 and relabel the second entry as a. Hence, we get $3 = \begin{pmatrix} 1 & a & 1 \end{pmatrix}$ such that $a \neq 0$.

We construct 5 in a similar manner, noting that it is a linear combination of $\overline{16}$ and get that $5 = \begin{pmatrix} 1 & 0 & b \end{pmatrix}$ such that $b \neq 0$. Likewise, noting that 7 is a linear combination of $\overline{12}$ we get that $7 = \begin{pmatrix} 1 & c & 0 \end{pmatrix}$ such that $c \neq 0$.

Using CAF and that 9 is a linear combination of $\overline{57}$ and $\overline{26}$ we get that $9 = \begin{pmatrix} 0 & -c & b \end{pmatrix}$. Also using CAF on the linear dependence relations (349) and (468) we get that $4 = \begin{pmatrix} c & a & ab - b + c \end{pmatrix}$ such that $ab - b + c \neq 0$. We construct the last row vector 10 by using CAF and the linear dependence relations (1810) and (3710) to get that $10 = \begin{pmatrix} a - 1 - c & -c & -c \end{pmatrix}$, such that $a - 1 - c \neq 0$.

To check that all the 3-circuits in above set are dependent we calculate the determinant of the last three circuit: (4510). It is automatically equal to zero as everything cancels to zero. Hence our B matrix is:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & a & 1 \\ c & c & ab-b+c \\ 1 & 0 & b \\ 0 & 0 & 1 \\ 1 & c & 0 \\ 1 & 1 & 1 \\ 0 & -c & b \\ a-1-c & -c & -c \end{pmatrix}$$

such that $a \neq 0$, $b \neq 0$, $c \neq 0$, $ab - b + c \neq 0$, $a - 1 - c \neq 0$.

As we've ensured that all the 3-circuits have determinant equal to zero and our only constraints on our variables are inequalities we're left with a space isomorphic to $(\mathbf{C}-0)^3$ with a little bit of adjustment to account for the constraints: $ab-b+c \neq 0$ and $a - 1 - c \neq$. This space is a Zariski-open set. Then it immediately follows that this space is connected. Hence, the realization space of K_5 is connected and we may apply Randell's Isotopy Theorem to reconstruct the complement of the corresponding arrangement.

6. Realization Spaces, continued

Now we will move towards offering a simple algorithm for verifying if the realization space of the rank-3 truncation of a matroid is path connected. We will also offer a heuristic argument to show that the realization spaces of rank-3 truncations of k-4 free graphs are path connected.

First let us propose the following definition for the rank-3 truncation of graphic matroids:

Definition 6.1. Given a graphic matroid, M, with graph G of order n the realization space of its rank-3 truncation is:

$$\tilde{R}(M') = \{x \in \mathbb{C}^{3n} | \det(x_i, x_j, x_k) = 0 \iff \text{the edges } i, j, \text{ and } k \text{ form a K3 in } G \}$$

As before we are interested in $\tilde{R}$ quotient out all relations $\sim$ i.e. $R(M') = R/\sim$

where x_i, x_j, x_k are the ith, jth and kth row of R. This follows simply because three rows are dependent iff the determinant of the rows is zero. Furthermore, trivially any four rows of x are dependent and any four elements in a rank-3 truncation are dependent. Since we are dealing with simple graphs: i, j, and k are contained in a K3 $\iff$ they are dependent of rank 2 in the associated matroid. Hence i, j, and k are in a K3 $\iff$ the determinant of the three rows is zero (note that since this is a simple graph all rank 1 subsets are independent and all subsets of order 2 are dependent i.e. no loops or multiple edges so the only non-rank 3 contribution to dependent subsets come from K3's).

With this reasoning, $R(M')$ is a set of equations and in-equations, or in the terms of algebraic geometry, an algebraic variety intersected with the complement of an algebraic variety, i.e.
$R(M') = \{x \in C^{3n} | det(x_i, x_j, x_k) = 0 \iff i, j,$ and k in a K3 in G $\} \cap \{x \in C^{3n} | det(x_i, x_j, x_k) \neq 0 \iff i, j, k$ are not in a K3 in G $\}$

This leads us to the following lemma:

Lemma 6.2. *If there exists a continuous map, $\phi : \mathbb{C}^l \to \mathbb{C}^{3n}$ where $l \leq n$ such that if $W = \{y \in \mathbb{C}^l | det(\phi_i[(y)], \phi_j[(y)], \phi_k[(y)]) \neq 0\}$ and $\phi(W)$ is onto $R(M')$ then $R(M')$ is path-connected.*

Proof. If there is such a continuous map since W is a Zariski open set on C^{3l}, we know it is path-connected, hence its image under a continuous map is also path connected. So if the image of W under ϕ is onto $R(M')$ then $R(M')$ is path connected. □

Hence, in example 1.6 where we find the realization space of the rank-3 truncation of the complete graph on 5 vertices the parameters a, b, and c are in $\mathbb{C}^3$ and $W = \{(a, b, c) \in \mathbb{C}^3 | ab - b + c \neq 0$ and $a - 1 - c \neq 0\}$ then $\phi : \mathbb{C}^3 \to \mathbb{C}^{30}$ is a map defined by B, and is such that $\phi(W) = R(M')$.

In general via Caley algebra we have three tools for explicitly defining a map such as ϕ, namely if an edge is at the intersection of two K3's then $x_m = (x_i \vee x_j) \wedge (x_k \vee x_l) = -det(x_i, x_j, x_k) \cdot x_l + det(x_i, x_j, x_l) \cdot x_k$, if an edge is at the intersection of one K3 then it lies along a the same affine line that x_i and x_j lie along i.e. they are linearly dependent so let $x_m = x_i + \alpha x_j$ for some complex scalar α. Finally if an edge does not lie on a K3, then we determine its coordinates in general position, i.e. $x_m = (1, a, b)$ for some complex scalars a and b. This observation leads us to the following lemma:

Lemma 6.3. *Let G be a graph and M its associated matroid, then R(M') is path connected if there exists some ordering of the edges of M, $\{e_1, \ldots, e_n\}$, such that $T(H_{i+1}) - T(H_i) \leq 2$ $\forall i \in \{1, \ldots, n-1\}$. Here H_i is the subgraph of G induced by $\{e_1, \ldots, e_i\}$ and $T(L)$ for some graph L denotes the number of triangles in graph L.*

Proof. We prove this by induction on i. Clearly in the base case H_1, the subgraph induced by a single edge, any non-zero vector in $\mathbb{C}^3$ is equivalent to any other non-zero vector under $\sim$ hence without loss of generality we may let the first vector be $(1, 0, 0)$.

Likewise assume we choose the next edges such that the e_1, e_2, e_3 do not form a K3, then the first three vectors form a basis of three dimensional complex space and hence they are equivalent under $\sim$ to any other basis so again we may let the first three entries in $R(M')$ be the cannonical basis without loss of generality if such an edge does not exist, we can choose our graph such that the first three edges do not form a K3. Then G is a K3, and hence, trivially, G has a path-connected realization space, where ϕ is defined as $\phi(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & a & 0 \end{bmatrix}$.

Otherwise the first three edges (and hence rows of $R(M')$ are the cannonical basis and ϕ maps to the canonical basis. Now consider H_i for $i \geq 4$. We let $\phi_i : \mathbb{C}^k \to \mathbb{C}^{3Ki}$ be such that $\phi^i(W_i)$ is onto $R(M(H_i)')$ where $R(M(H_i)')$ is the realization space of the rank-3 truncation of the matroid of H_i and $W_i = \{y \in \mathbb{C}^k | det(\phi_j, \phi_k, \phi_l) \neq 0$ if e_j, e_k, e_l do not form a K3 $\in H_i\}$. Assuming ϕ^i exists as such then we show how we can construct ϕ^{i+1}. In the first case $T(H_i) - T(H_{i+1}) = 0$ and we can choose the ith row of $R(M')$ in general position and then $\phi^{i+1} : \mathbb{C}^{k+2} \to \mathbb{C}^{3\times i+3}$ where if $y \in \mathbb{C}^{k+2}$ and $y' \in C^k$, and y' is obtained by deleting the last two entries in y then $\phi^{i+1}_j(y) = \phi^i_j(y')$ for $j \in \{1, \dots i\}$ and $\phi^{i+1}_{i+1} = (1, a, b)$.

Now consider when $T(H_i) - T(H_{i+1}) = 1$, here e_i is dependent only with the two edges, e_l and e_k, it forms a $K-3$ with. Then $\phi^{i+1} : \mathbb{C}^{k+1} \to \mathbb{C}^{3\times i+1}$ as before $\phi^{i+1}_j = \phi^i_j$ for$j \in \{1, \dots i\}$ and $\phi^{i+1}_{i+1}(y) = x_l + a x_k$ where a is the last entry of $y \in C^{k+1}$ and x_l and x_k are the projection of y under ϕ^{i+1} on the the rows l and k respectively.

Lastly if $T(H_i) - T(H_{i+1}) = 2$ then using the Caley-algebra formula the last row of $R(M(H_i)')$ is at the unique intersection of two affine lines. Let e_k, e_l, e_{i+1} be the first triangle, let e_p, e_q, e_{i+1} be the second triangle then as before $\phi^{i+1}_j = \phi^i_j$ for $j = 1, \dots i$ and $\phi^{i+1} = -det(x_k, x_l, x_p) \cdot x_q + det(x_l, x_k, x_q) \cdot x_p$. □

We ended our research this past summer with this lemma and we were using it to try and show that the rank-3 truncation of K_4 free graphic matroids have path-connected realization spaces. The goal would be to show that such K_4 free graphs satisfy the conditions of the preceding lemma i.e. there is always an ordering of edges such that $T(H_{i+1}) - T(H_i) \leq 2$. Some progress was made but it was not conclusive. Nevertheless the above lemma can be useful as a test to verify that the rank-3 truncation of a matroid has a path-connected realization space for many graphs.

7. Diffeomorphism

Given two K3's we can connect their graphs in two ways: by taking the direct sum or the parallel connection. By taking the direct sum, we adjoin both graphs at a single vertex whereas by taking the parallel connection we join them along an edge. In the case of parallel connection, since we are adjoining two graphs along an edge we lose one edge. In this case we attach a lone edge, an "isthmus," the unique hyperplane of one variable, $x = 0$, with corresponding graph denoted by S. According to **Orlik-Solomon algebras and Tutte Polynomials**, $M(\mathcal{A}_{\Gamma_1 \oplus \Gamma_2}) \cong M(\mathcal{A}(\Gamma_1 || \Gamma_2) \oplus S)$ In fact, the isomorphism is actually a diffeomorphism. This is the class of graphs that rules out the question of having isomorphic Orlik-Solomon algebras of graphic arrangements yielding isomorphic graphs, since these graphs are certainly not isomorphic.

We can explicitly demonstrate the diffeomorphism:

For the direct sum case, $\Gamma_1 \oplus \Gamma_2$, we have variables x_1, x_2, y_1, y_2 to define our hyperplane arrangement so we are in $\mathbb{C}^4$ and each K3 attributes 2 variables. By "deconing" along one edge on each K3 and then joining these edges together we eventually get:

$$\Delta : M_1 \oplus M_2 \xrightarrow{\Delta} \mathbb{C}^* \times \overline{M_1} \times \mathbb{C}^* \times \overline{M_2} \xrightarrow{\psi} \mathbb{C}^* \times \mathbb{C}^* \times \overline{M_1||M_2} \xrightarrow{\Delta_{||}} \mathbb{C}^* \times M_1||M_2$$

$\Delta : M_1 \oplus M_2 \to \mathbb{C}^* \times \overline{M_1} \times \mathbb{C}^* \times \overline{M_2}$

$$\Delta : ((x_1, x_2), (y_1, y_2)) \to \left(x_1, \left[1 : \tfrac{x_2}{x_1}\right], y_1, \left[1 : \tfrac{y_2}{y_1}\right]\right)$$

$\psi : \mathbb{C}^* \times \overline{M_1} \times \mathbb{C}^* \times \overline{M_2} \to \mathbb{C}^* \times \mathbb{C}^* \times \overline{M_1||M_2}$

$$\psi : \left(x_1, \left[1 : \tfrac{x_2}{x_1}\right], y_1, \left[1 : \tfrac{y_2}{y_1}\right]\right) \to \left(x_1, y_1, \left[1 : \tfrac{x_2}{x_1} : \tfrac{y_2}{y_1}\right]\right)$$

$\Delta_{||} : \mathbb{C}^* \times \mathbb{C}^* \times \overline{M_1||M_2} \to \mathbb{C}^* \times M_1||M_2$

$$\Delta_{||} : \left(x_1, y_1, \left[1 : \tfrac{x_2}{x_1} : \tfrac{y_2}{y_1}\right]\right) \to \left(x_1, y_1, \tfrac{y_1 x_2}{x_1}, y_2\right)$$

$M_1 \times M_2 \to \mathbb{C}^* \times M_1||M_2$

$$\begin{aligned}
((x_1), (x_2, y_1, y_2)) &\to \left(x_1, \left[1 : \frac{x_2}{x_1}\right], y_1, \left[1 : \frac{y_2}{y_1}\right]\right) \\
&\to \left(x_1, y_1, \left[1 : \frac{x_2}{x_1} : \frac{y_2}{y_1}\right]\right) \\
&\to \left((x_1), \left(y_1, \frac{y_1 x_2}{x_1}, y_2\right)\right) = ((u_1), v_1, v_2, v_3)
\end{aligned}$$

Note that if you enter defining equations for an arrangement in the parallel connection, they pull back to equations defining an arrangement with combinatorial type like one of the direct sum.

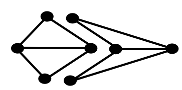
General case

$M_1 \times M_2 \to \mathbb{C}^* \times M_1||M_2$

$$\begin{aligned}
((x_1, x_2, \cdots, x_p), (y_1, y_2, \cdots, y_q)) &\to \left(x_1, \left[1 : \frac{x_2}{x_1} : \cdots : \frac{x_p}{x_1}\right], y_1, \left[1 : \frac{y_2}{y_1} : \cdots : \frac{y_q}{y_1}\right]\right) \\
&\to \left(x_1, y_1, \left[1 : \frac{x_2}{x_1} : \cdots : \frac{x_p}{x_1} : \frac{y_2}{y_1} : \cdots : \frac{y_q}{y_q}\right]\right) \\
&\to \left(x_1, \left(y_1, \frac{y_1 x_2}{x_1}, \cdots, \frac{y_1 x_p}{x_1}, y_2, \cdots, y_q\right)\right) \\
&= ((u_1), (v_1, v_2, \cdots, v_{p+q-1})
\end{aligned}$$

This is a birational map, which we know from the theory of algebraic geometry is a succession of blow-ups and blow-downs. What is going on geometrically in this map? Can we possibly study the blow-ups and blow-downs from the graph itself?

We notice that we have some sort of parallel-connection type of construction in the graph, where instead of parallel-connecting two K3s we are parallel-connecting the parallel-connection of two K3s, pictured below!



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