3D POLYOMINO ACHIEVEMENT GAMES

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ABSTRACT

A-achievement games for 3D rectangular polyominoes are studied. All but five polyominoes with at least five cells are determined to be winners. Winning strategies for the winning polyominoes are given. Some losing polyominoes, containing as few as six cells, are determined using tilings.

1. INTRODUCTION

Frank Harary was the first to explore achievement games using polyominoes or *animals* [G, H]. Harborth, and Bode continued Harary's research and were able to classify animals of many varieties. The case using a board tiled with triangles was examined, and all winning animals were found [HH]. Also examined in the triangular case was the possibility of winning sets, where any triangular shape in a given set must be made, before the set could be considered winning [BH3]. The 2D square case was considered, and all but one animal, snaky, was classified as winning or losing [HS3]. Also researched was the case of the hexagonally tiled board. Although not all animals were examined individually, they were able to prove that no animal larger than 18 tiles could be winning, thus the number of winning animals in a hexagonal case is finite [BH2]. Considering the above research, the 3D case appeared the most interesting unexplored region of this study.

In the preliminaries, a definition is given for a polyomino, or animal, and also a description on how these animals are used to play the game. Requirements to make an animal a winner or a loser are explained as well as details on the 2D square cases. The results section shows the 3D animals that were found to be winners, as well as the animals that we were not able to determine. A method to find losers in 3D is also explained. Appendix A includes all the winning strategies that were found for 3D animals. Appendix B illustrates handicap strategies for the remaining five block animals. Appendix C shows the known 3D losers, up to six cells, along with the tilings that prove them to be losers.

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2. PRELIMINARIES

Definition 2.1. A 2D *animal* is created by connecting squares at any edge, to form a solid shape. No animal can contain a blank space surrounded by squares, and no two squares can be connected only by a corner.

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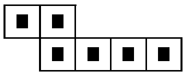
Examples of Animals

Examples of Non-Animals

With these animals, a game is played. Using an infinite 2D rectangular grid, each player alternately fills in a tile. The first player (player A) has an objective to form a predetermined animal, or any of its possible transformations. An animal can be rotated, mirrored, or shifted in any way. The second player (player B) is simply trying to foil the first. This method is called A-achievement. There is also a version of this game where both players attempt to form an animal. That is called A-B achievement, and is not considered in this paper. Using the A-achievement method, there are certain animals that can always be formed. Such animals have been dubbed *winners*. All 2D animals have been classified, except one, a six block formation called *snaky*



Snaky is a mystery. It cannot be classified as a winner or a loser. One method of finding losers is tiling a grid into pairs, or *dominoes*. To determine if an animal is a loser, an infinite grid is divided into dominoes of two squares each in some pattern. The pattern can vary from animal to animal. An animal is then taken, and transformed all the possible ways (rotation, reflection, and shift). If there is no transformation of the animal that can be placed onto the grid without containing an entire domino, then that animal is a *loser*. It would be impossible to create this animal if the second player always chose the counterpart square to the first player. That is, the second player could always mark the other half of the domino contained within the animal, making it impossible for the first player to achieve his objective. There is no tiling that proves snaky to be a loser [HS3].

x	x	o	o	x	x
o	o	x	x	o	o
x	x	o	o	x	x
o	o	x	x	o	o
x	x	o	o	x	x

x	x	—	*	x	x
o	o	—	*	o	o
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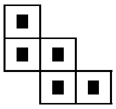
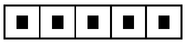
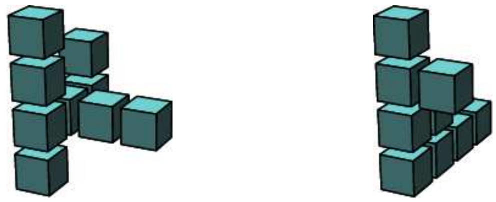
o	o	—	*	o	o



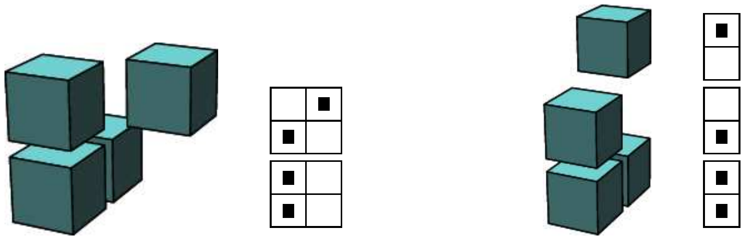


Some basic losers and their tilings

Definition 2.2. A *3D animal* is a shape created by connecting cubes. Each cube must be connected to another by a face.



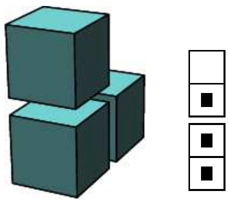
Examples of Animals



Examples of Non-Animals

The 3D picture and the slice picture illustrate the same shape, in different ways. The picture on the right is composed of slices, made at the differing values of z. The vertical space between the boxes indicates a movement in the z direction. To form the animal these slices should be imagined on top of each other. Boxes containing a square indicate where blocks are. Boxes that are empty are simply blank spaces, used only to help visualization.

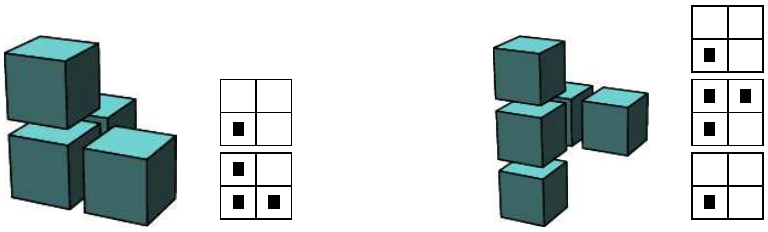
The players compete the same way, now coloring full cubes, instead of just squares. There are 34 shapes that can be made from using five or less cubes. Some can be considered winners automatically, simply by comparing to a 2D winner. That is, 3D shapes that contain only two directions are essentially *flat*. A flat 3D shape is analogous to a 2D shape, and therefore can be determined as a winner automatically, assuming the shape is a winner in 2D.



A Flat 3D Shape

A 2D loser cannot immediately be transformed into a 3D loser, as now there is a larger grid to consider, and therefore could be a transformation that would allow this 2D loser to become a 3D winner. However, a flat 3D loser is always a 2D loser.

There are also some shapes that are not flat, and therefore have no two-dimensional counterparts.



3D Animals That Are Not Flat

In addition to classifying animals as definite losers, they can also be identified as definite winners as well. To determine a winner, one can find a corresponding *winning strategy*.

Definition 2.3. A *core* is a set containing cells marked by the first player.

Definition 2.4. A *neighborhood* is a set of unmarked cells.

Definition 2.5. A *situation* s is a pair (C_s, N_s) where C_s is a core, and N_s is a neighborhood. Note that $C_s \cap N_s = \emptyset$.

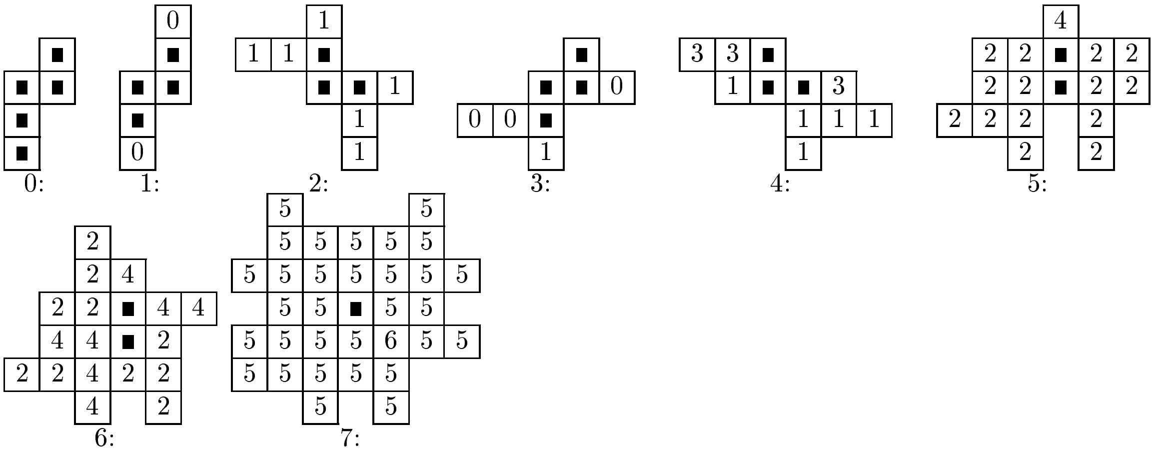
Definition 2.6. If $a \in C_S$ then the situation $\text{del}_a(s) = (C_s \setminus \{a\}, N_s \cup \{a\})$ is the *deletion* of s at a.

Definition 2.7. Let $S = \{s_i \mid i \in I\}$ be a set of situations. If $C = \cup_{s\in S} C_s$ and $N = \cup_{s\in S} N_s$ are disjoint then the *join* of S is the situation $\text{join}(S) = (C, N)$. If $\cap_{s\in S} N_s = \emptyset$ then $\text{join}(S)$ is called a *good join*.

Any good join of deletions of winning situations is winning. This is because no matter how the second player marks a cell of the neighborhood of the join, the first player is always able to achieve one of the original winning situations.

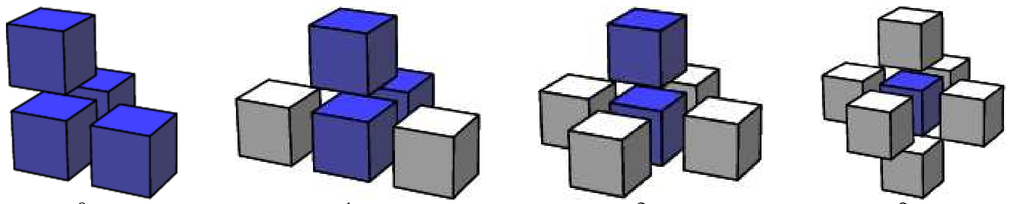
Definition 2.8. A *winning strategy* for the first player to achieve the shape A on a board is a finite sequence $s_0, \ldots, s_n$ such that $C_{s_0} = A$, $N_{s_0} = \emptyset$, C_{s_n} is a singleton set and every s_i is a good join of situations which are congruent to deletions of situations in the sequence with index less than i. A winning strategy on a finite board requires that $C_{s_n} \cup N_{s_n}$ fits into the board.

The following is a winning strategy for animal Z:



The animal above had a winning strategy that had been found previously [BH2], however, the above winning strategy is shorter that the one found earlier.

The following is an example of a 3D winning strategy



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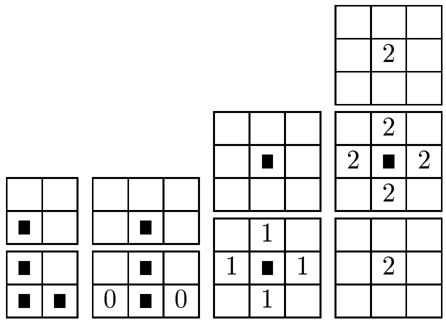

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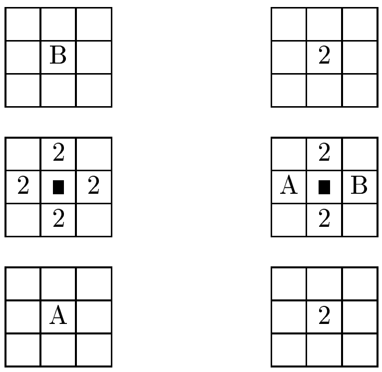
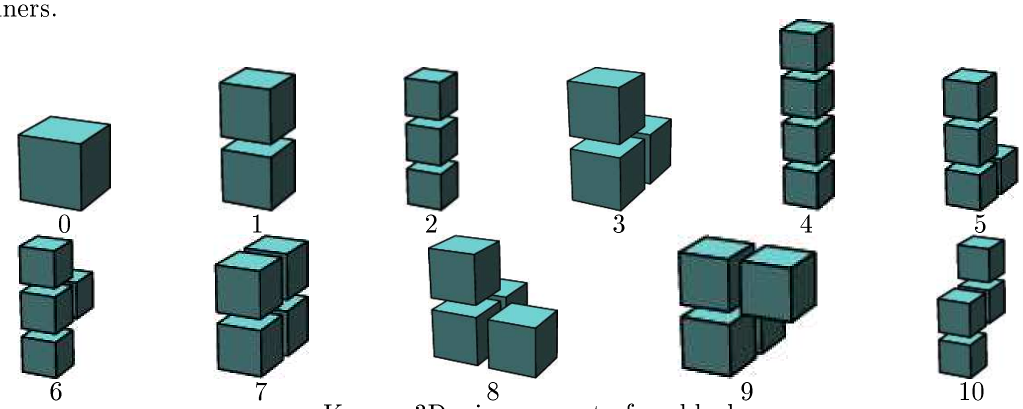
The dark colored cells represent core cells, the light colored cells are the neighborhood cells. The numbers explain how to attain an earlier position in the sequence. In this case, state 1 is created by combining two copies of state 0. A core cell is erased from each state 0, one is transformed, and then the two are combined. The only restrictions in the combination of parts is that the overall intersection of the neighborhoods of the combined pieces must be empty. The winning strategy can be used to assure the first player a win. For example, in the above winning strategy, if the second player chooses to mark any of the neighborhood cells in position 3, the first player can still form position 2.



Two examples how A should respond to different moves of B to form position 2

3. Results

All but five of the 34 3D animals composed of five or less blocks, have been identified as definite winners.



Known 3D winners up to four blocks

Definition 3.1. An *elementary winner* is a winner that is not contained within any other known winner.

Animals 0 through 10 are not elementary winners, because for each there is at least one winner composed of five blocks that contains the four block winner. At this time, no elementary winners have been found. Appendix A includes the winning strategies for the known winners, containing five blocks, excluding animal 12. Animal 12 is a winner, as it translates to a flat 2D winner [BH2], however, no winning strategy simpler than the 2D strategy had been found yet in 3D.

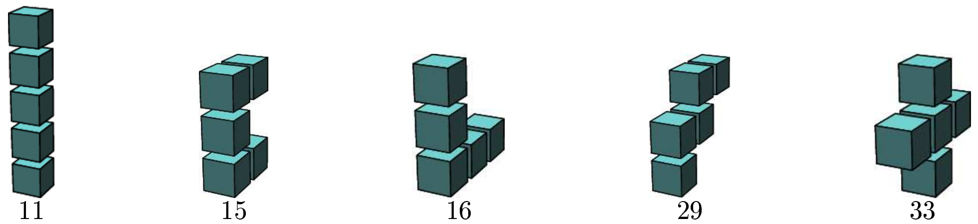
The winning strategies are shown in 3D, then followed by slices. The numbers in these slices indicate the prior position(s) in the strategy that the current position is derived from.

To obtain these numbers the prior position is placed on the current position, and all the empty blocks are filled with the number of the prior position. That position is then rotated and transformed in all the possible ways, continuing to fill in the numbers. This is done to show that any block in the neighborhood can be removed, and the prior position can still be obtained.

These winning strategies were found using a computer program. The program found winning strategies by taking an animal, and deleting a core cell. Then, all the transformations of the deleted animal were found. These were stored, and each transformation was attempted to be combined with another transformation. As each core cell was deleted in turn, the transformations were then combined, to find states that would join together, reducing the core size, and allowing for an empty intersection in the neighborhood.

Non-winning animals can become winning if the first player is given a *handicap*.

Definition 3.2. A *handicap* is the number of cells that the first player is allowed to mark before the game begins. A winner has handicap zero.



Undetermined Animals with Five Blocks

Of the non-winning five block animals, 15 and 29 have the lowest handicaps, each having a handicap of at most one. Animals 16 and 11 each have a handicap number of at most three. Animal 33 has a handicap of at most four, which indicates that all but one of the squares that form the shape would have to be filled before the game is begun, thus as soon as the first player marks one block, that player would win. The handicap strategies of these animals are given in appendix B.

Definition 3.3. An *elementary loser* is a loser that does not contain any other known loser.

As explained earlier in reference to 2D animals, a tiling can be used to prove that an animal is a loser. The same method can be used in a 3D case. To find these tilings, a computer program was written, which found all the tilings in 3D space. The 3D space had a fundamental region, which was changed for different applications. Each space outside the fundamental region that was considered, could only be filled if the corresponding spot in the fundamental region was not filled. This allowed for a *ragged edge* that allowed many more tilings to be found. Also, most of the tilings that were found to be useful at eliminating animals had a ragged edge. As each tiling was found, each animal (and all of its transformations) are then run through the tiling to check if a domino in contained within the animal. If any domino is contained within every transformation of one animal, then that animal is a loser.

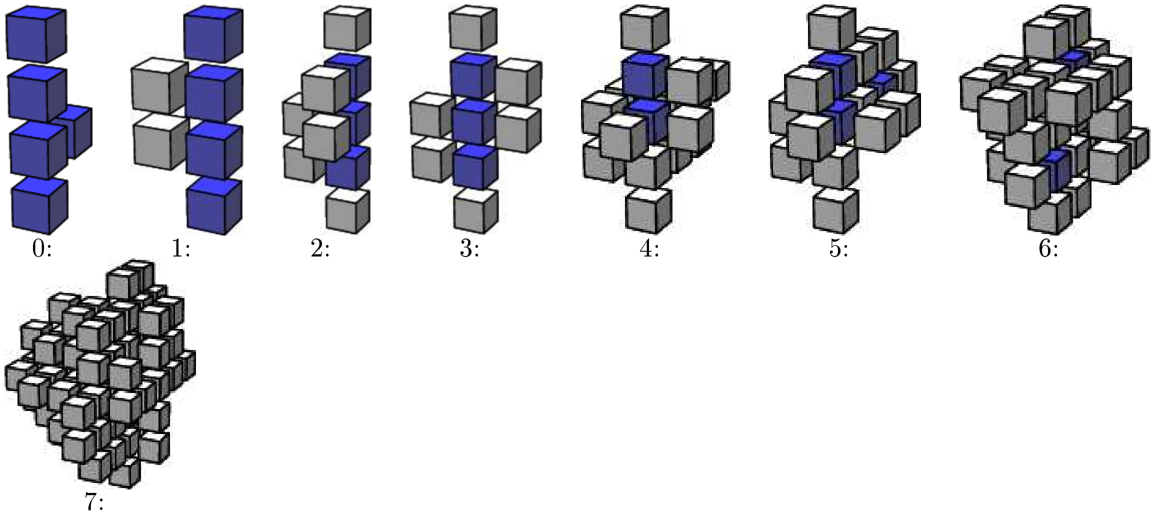
All the animals that were found to be losers are elementary losers. In many cases there were several tilings that would prove the animal to be a loser, however, for the purpose of illustration, the simplest is shown. The smallest losers that were found contain six blocks. These losers and their tilings are listed in appendix C.

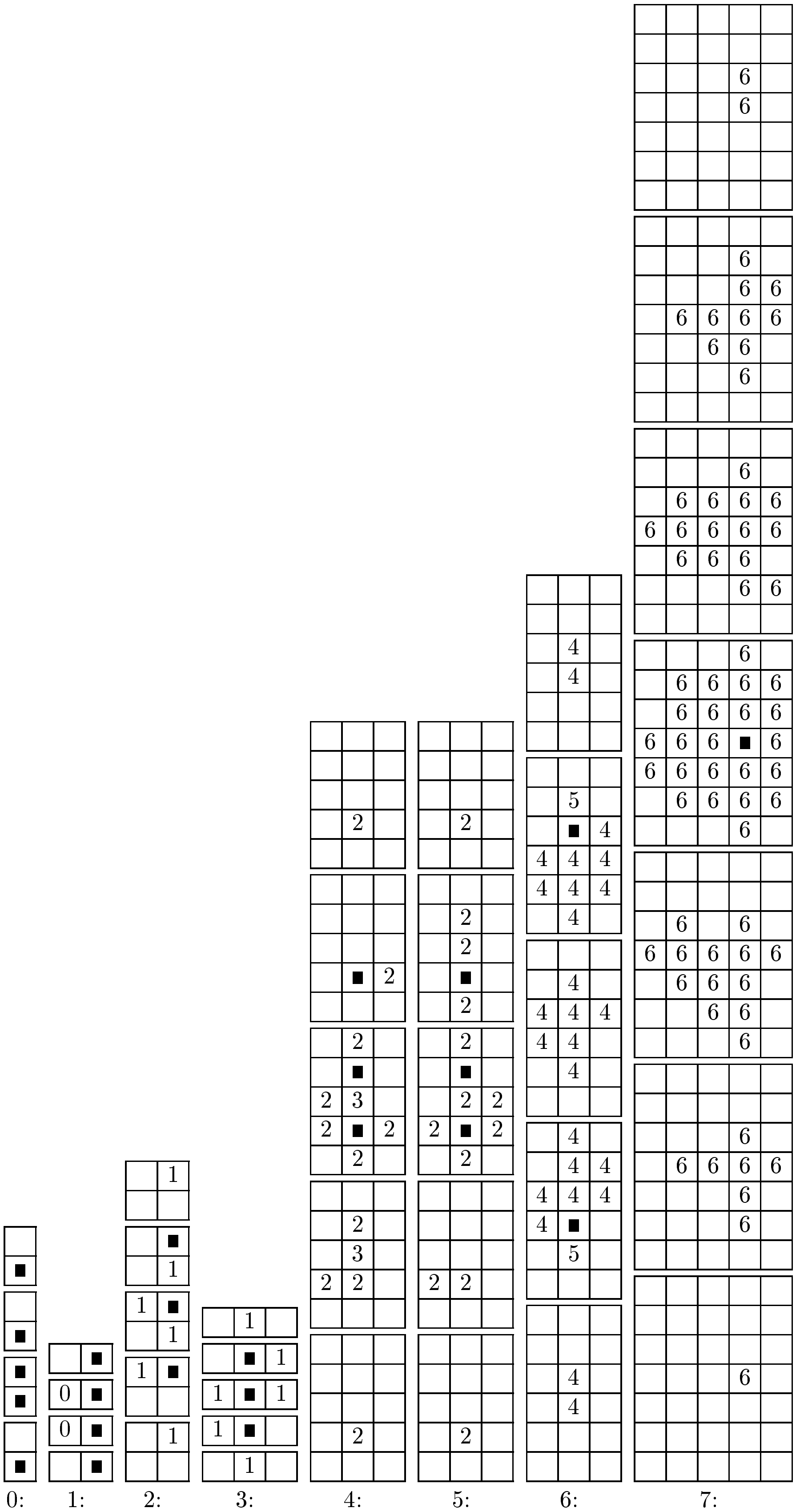
4. Future Research

The investigation of 3D animals is a new field, so there is much research to be done. The animals here are still small, winners containing only five blocks, and losers containing only six. Larger animals still need to be investigated. Other areas of interest could include set achievement with 3D animals, as well as fining a finite number of possible winners, as done previously in the hexagon case [BH2].

APPENDIX A
Winning Strategies of Five Block 3D Animals

Winning Strategy for Animal 13

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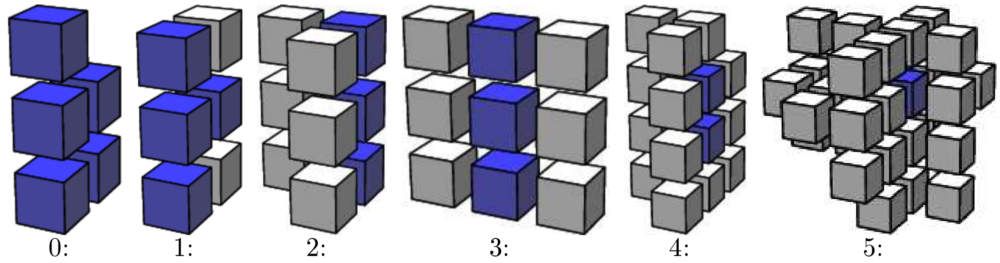
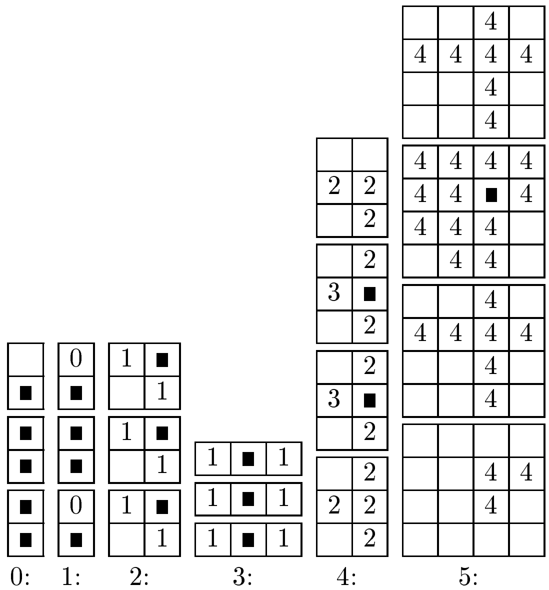
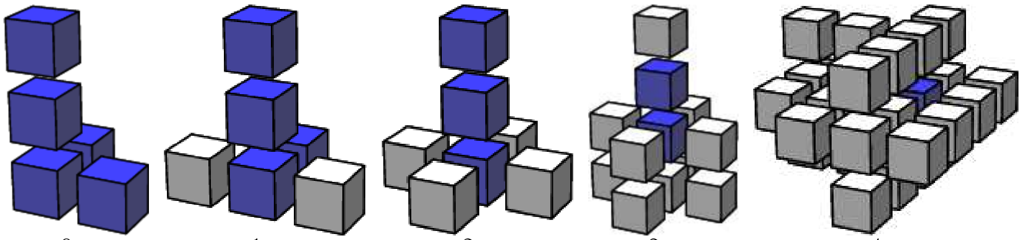
Winning Strategy for Animal 14





Winning Strategy for Animal 17



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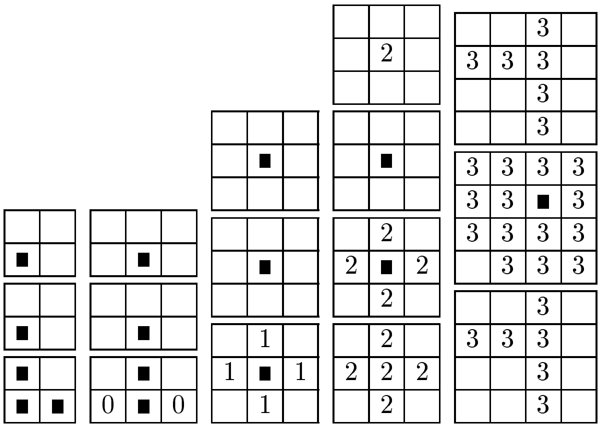

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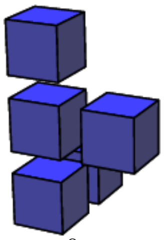
Winning Strategy for Animal 18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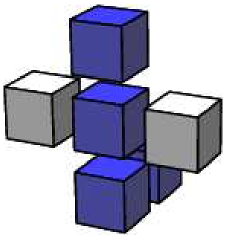
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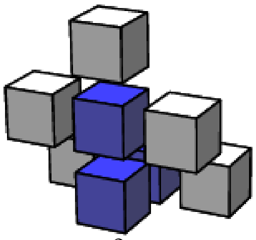
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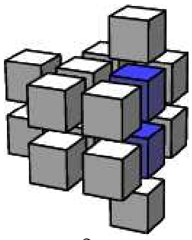
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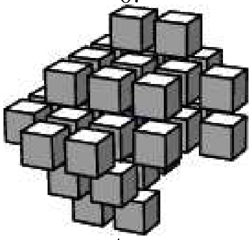
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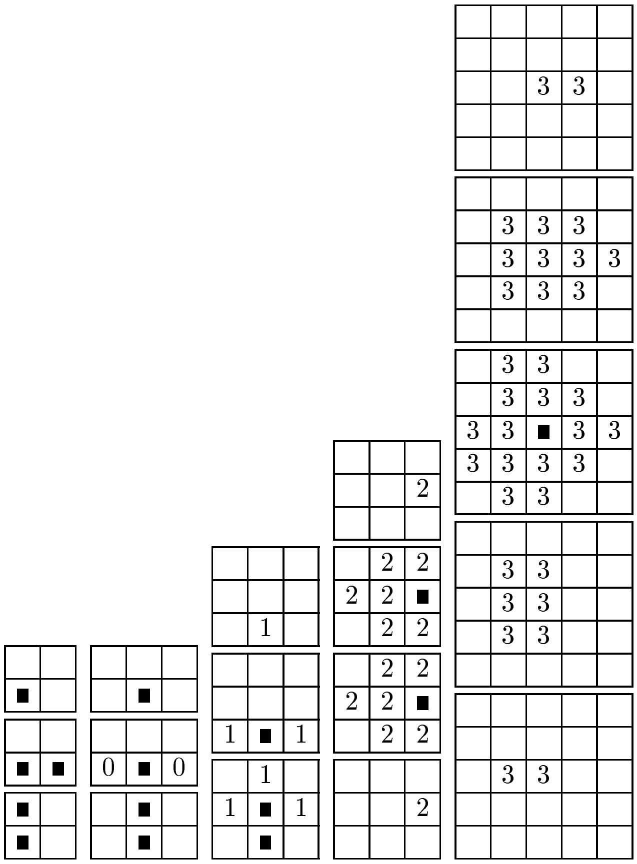

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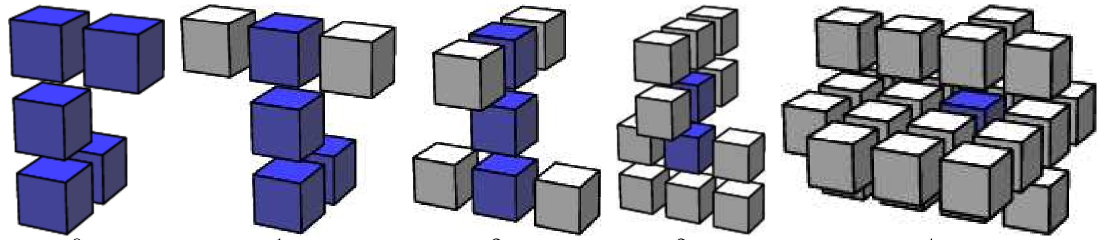
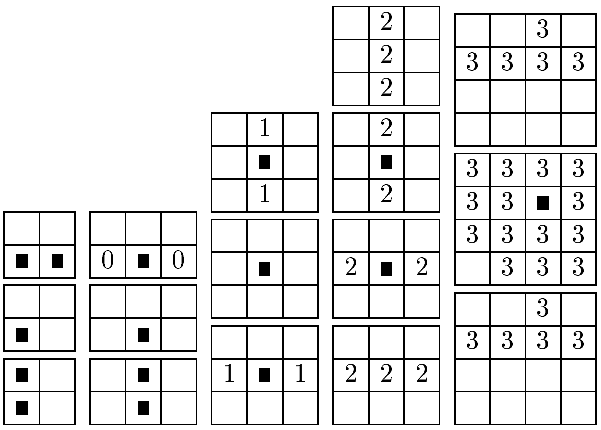
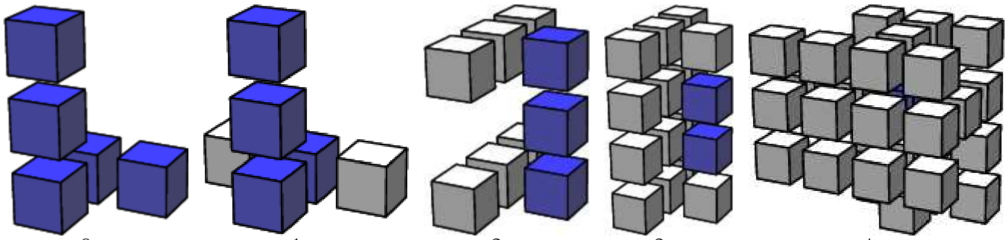
Winning Strategy for Animal 19





Winning Strategy for Animal 20



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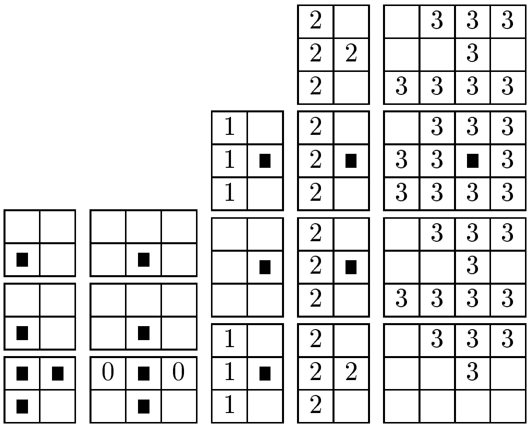

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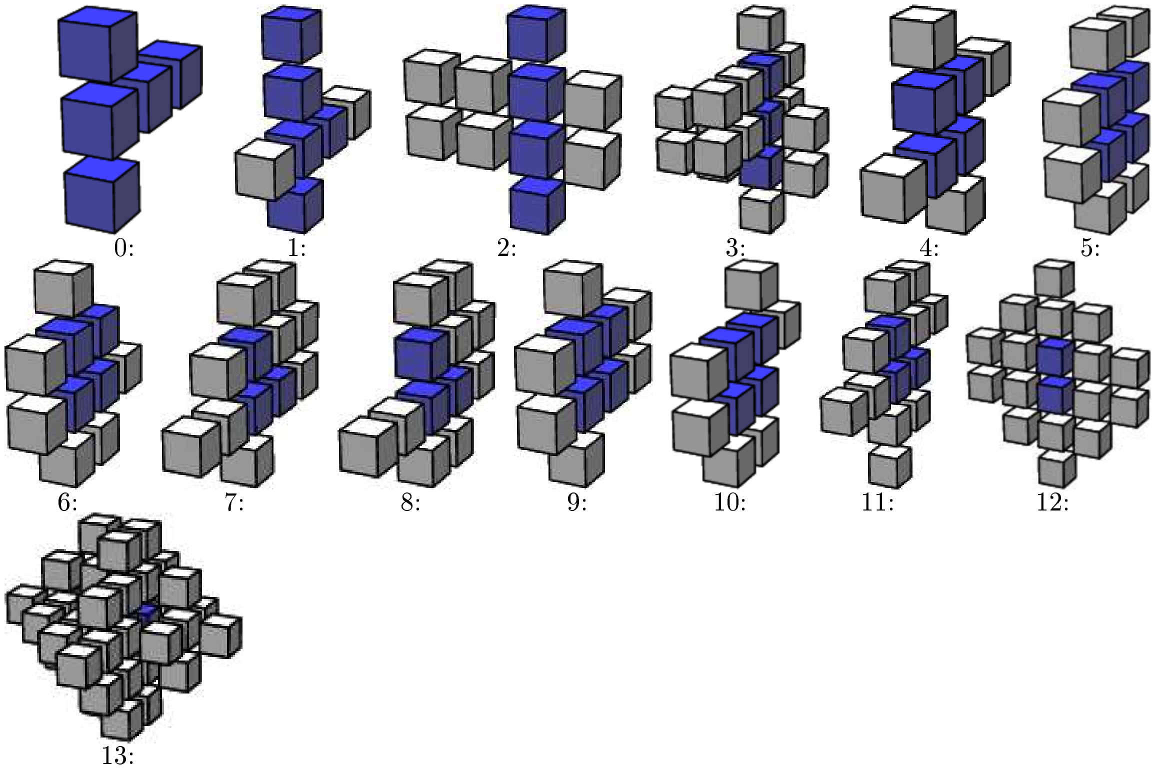
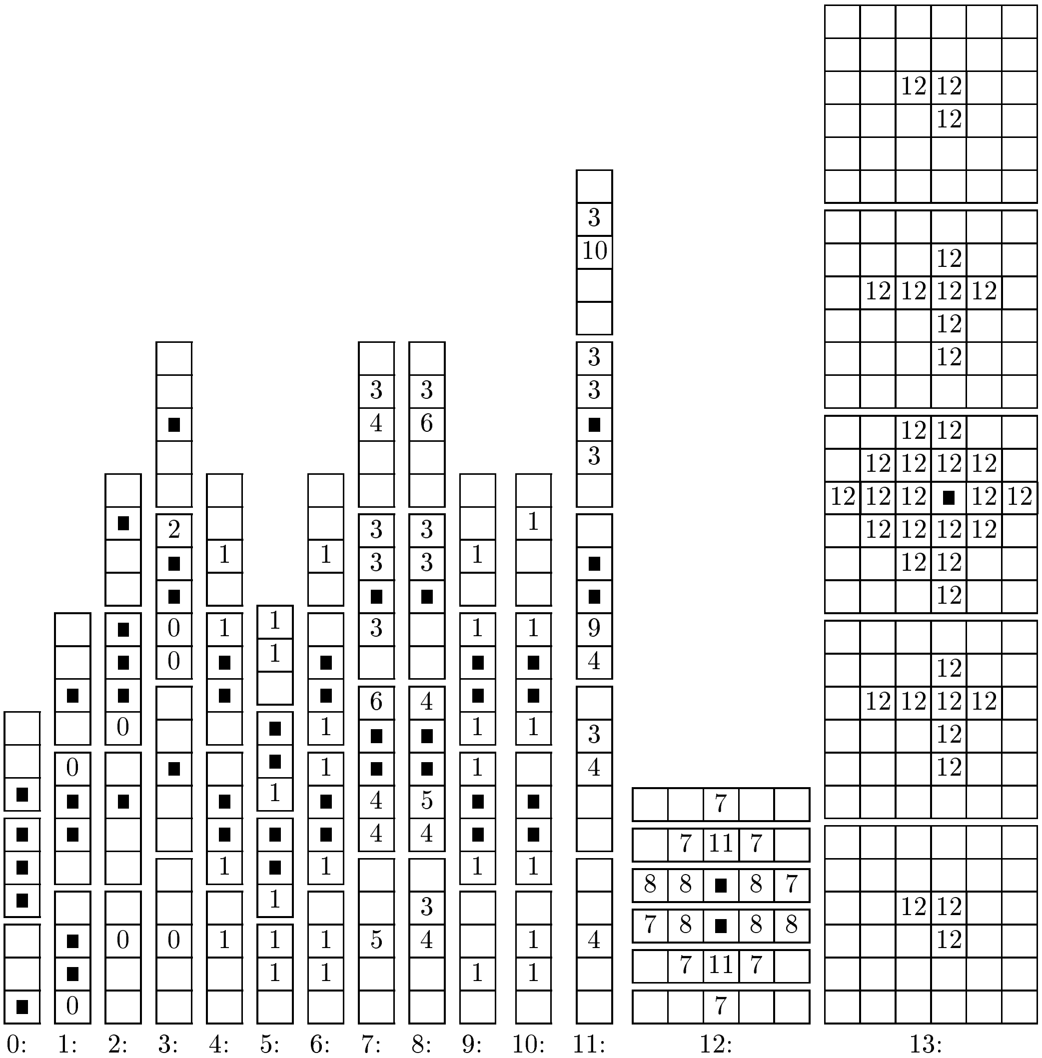
Strategy for Animal 21

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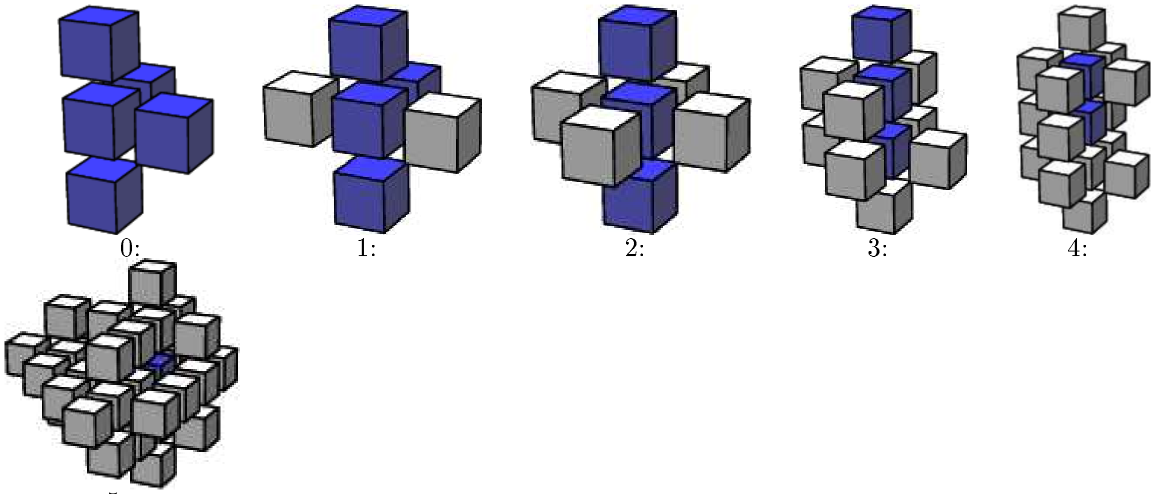
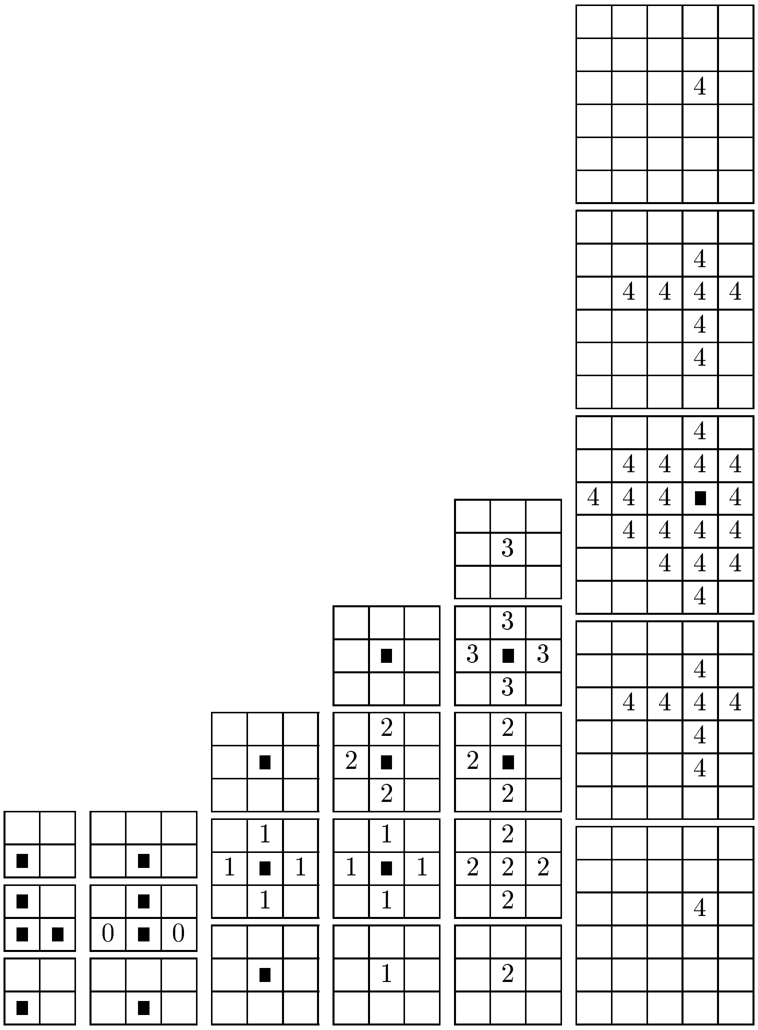
Winning Strategy for Animal 22





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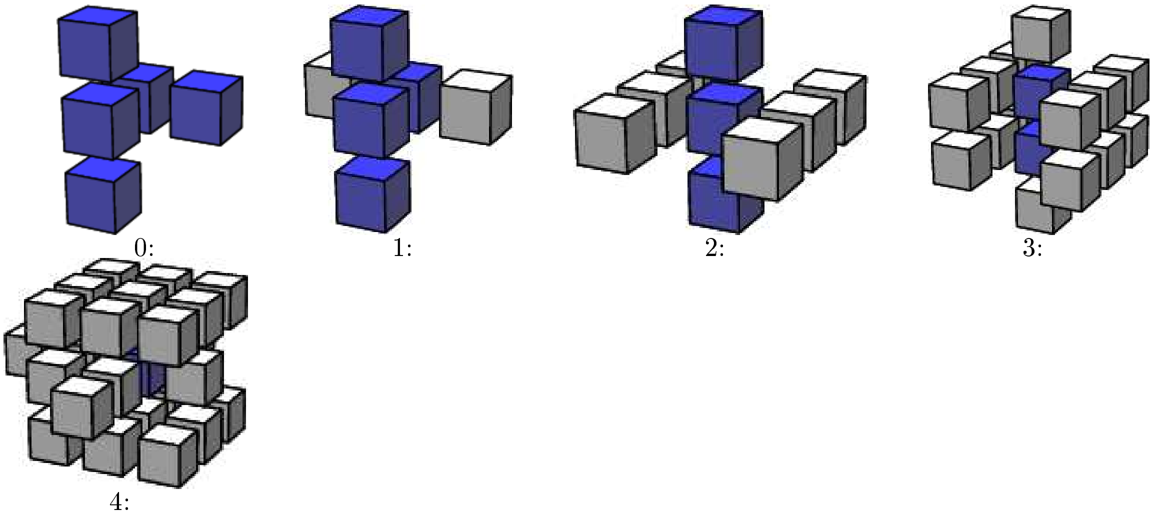
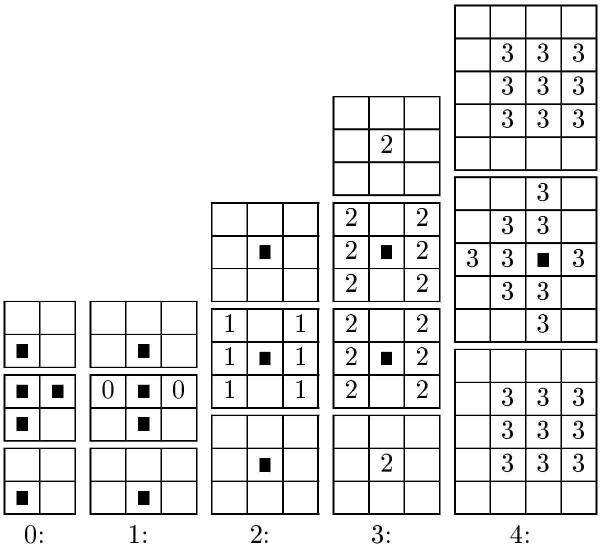
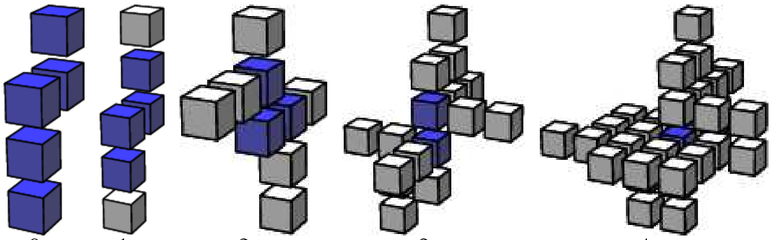
Winning Strategy for Animal 23





Winning Strategy for Animal 24



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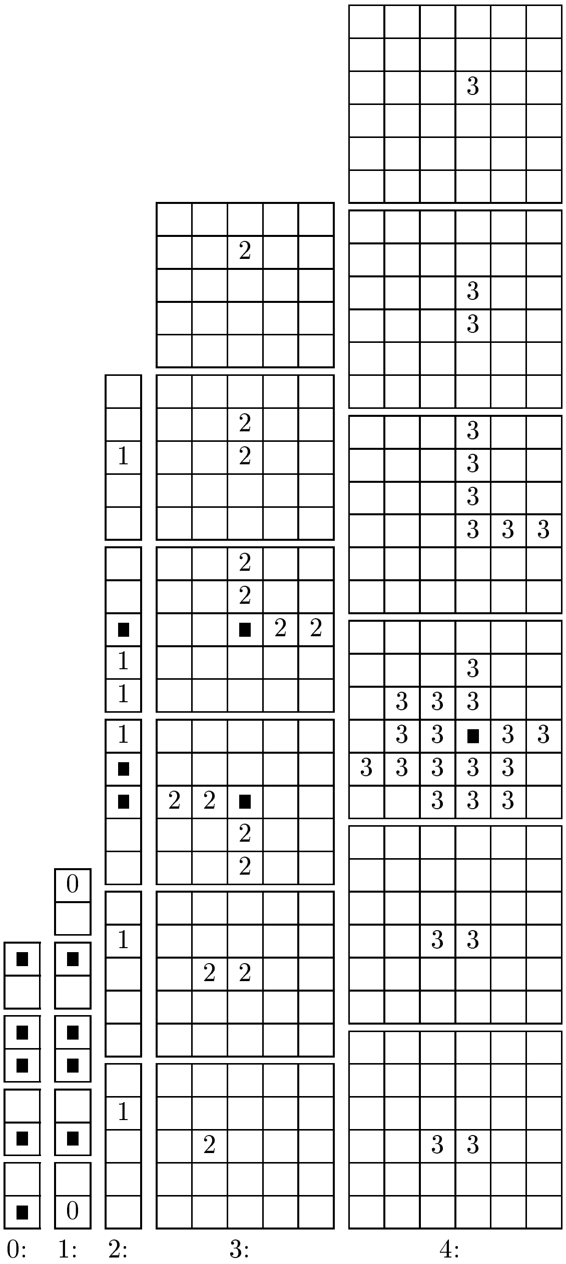
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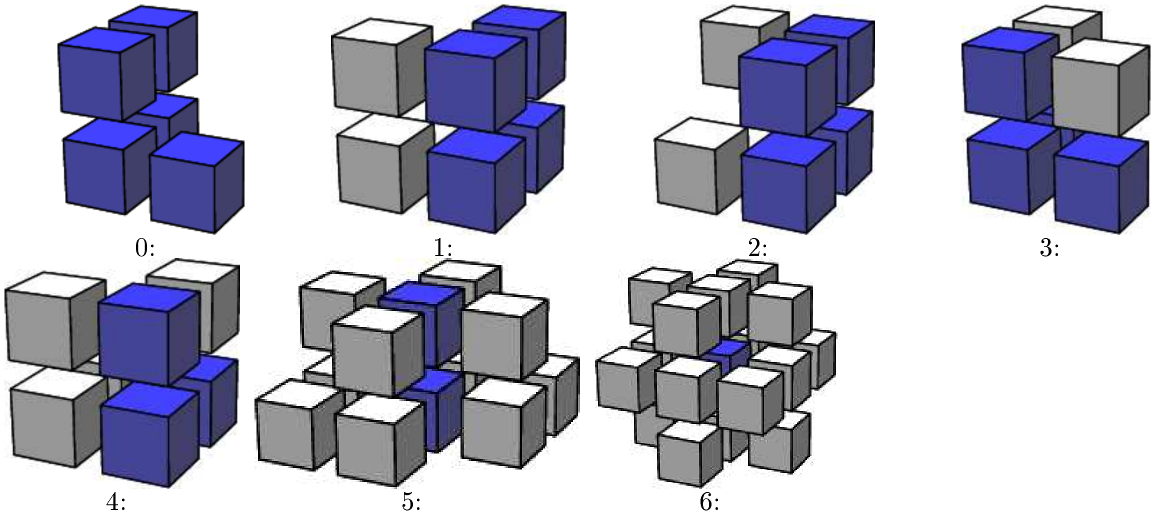
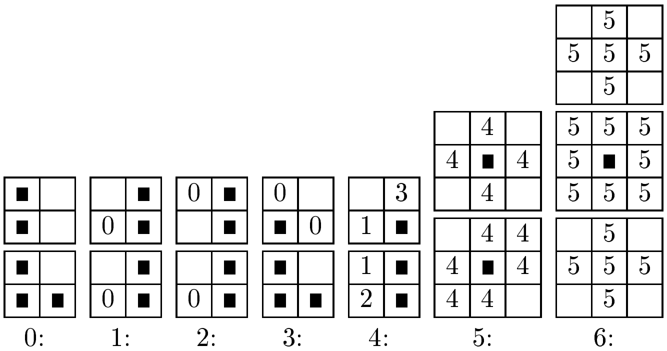
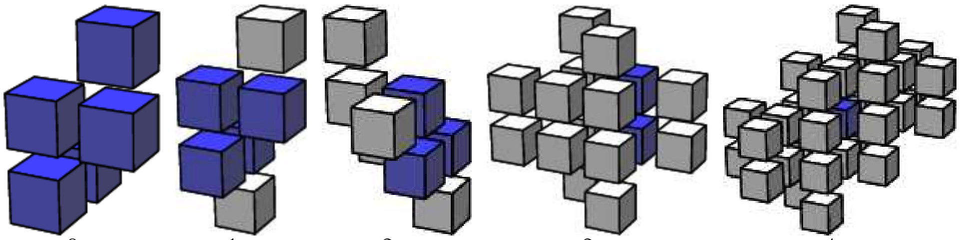
Winning Strategy for Animal 25





Winning Strategy for Animal 26



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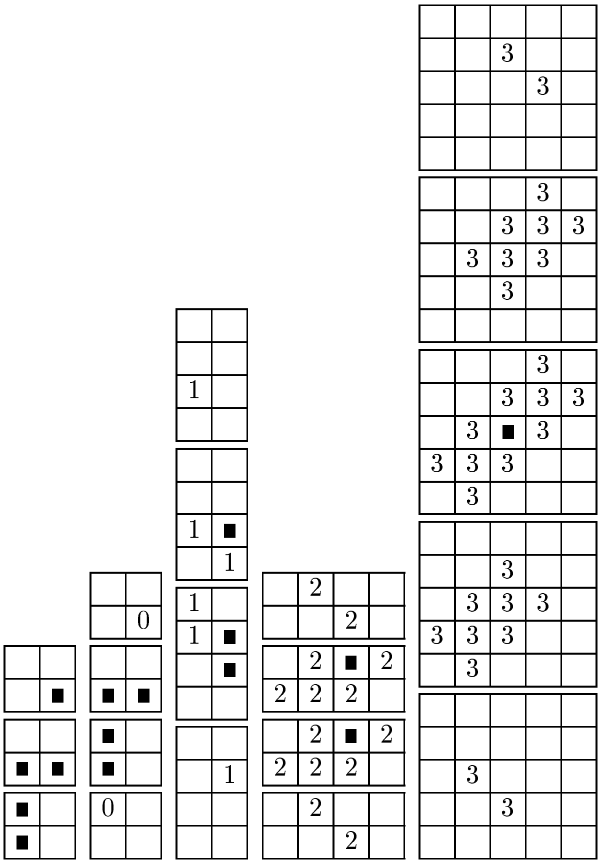

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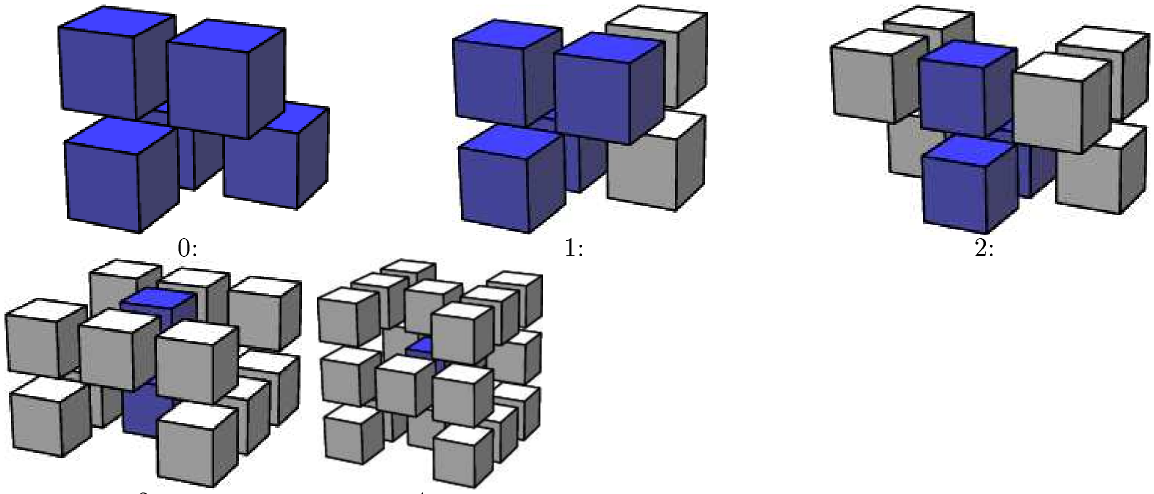
Winning Strategy for Animal 27



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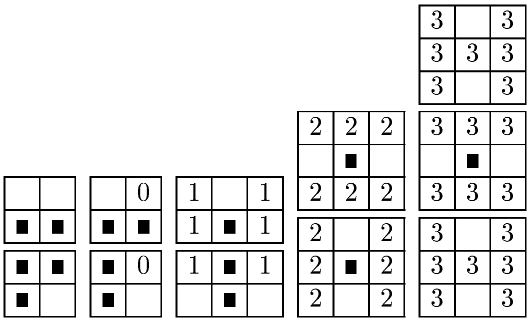

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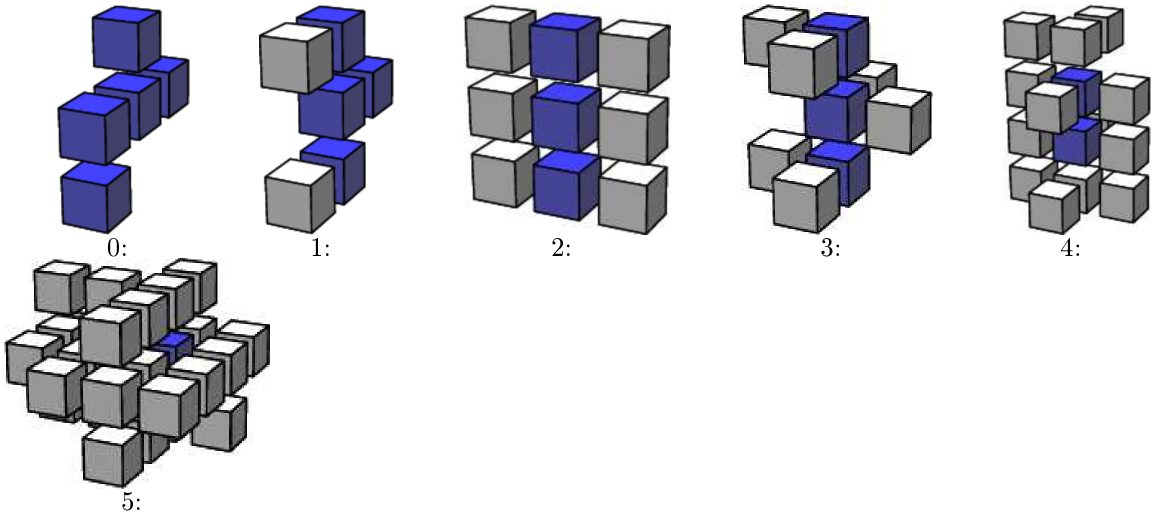
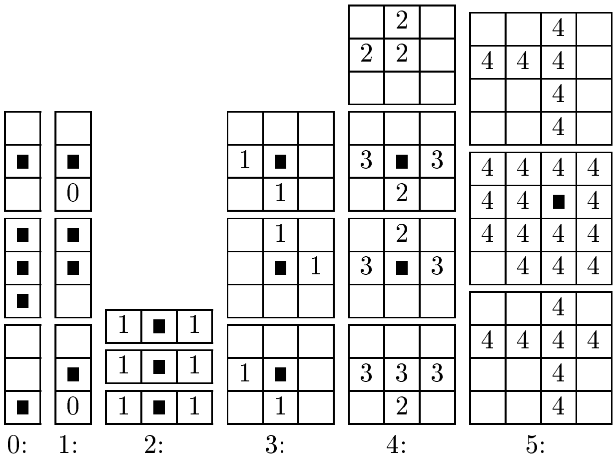
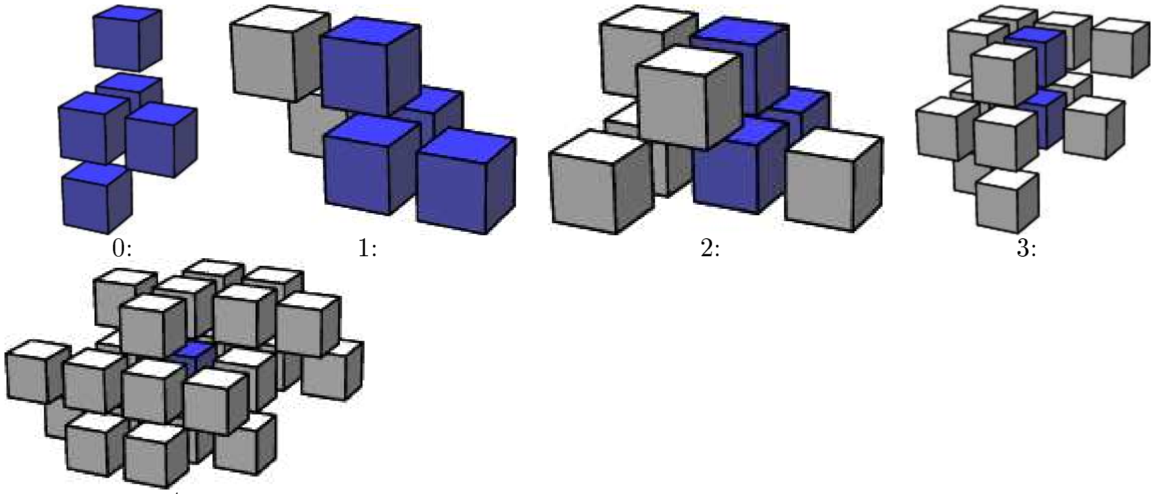
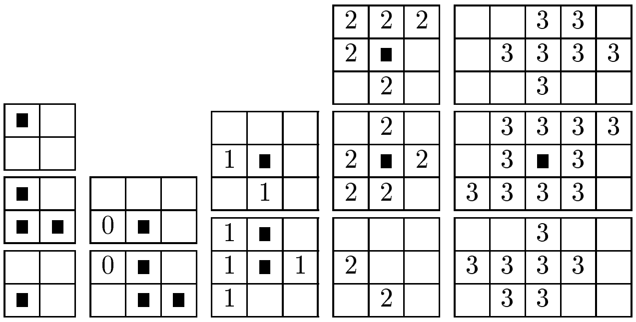
Winning Strategy for Animal 28



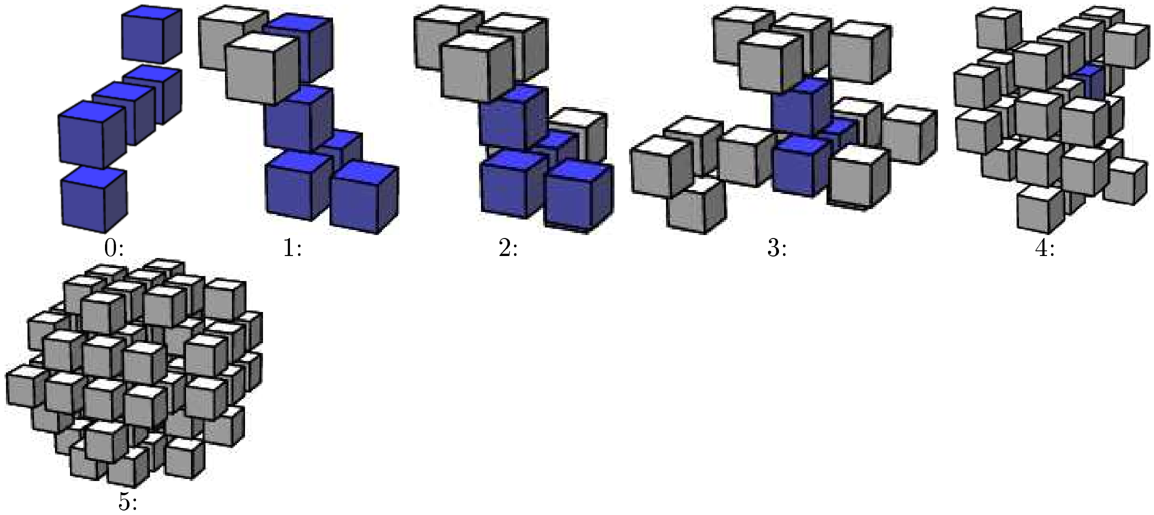


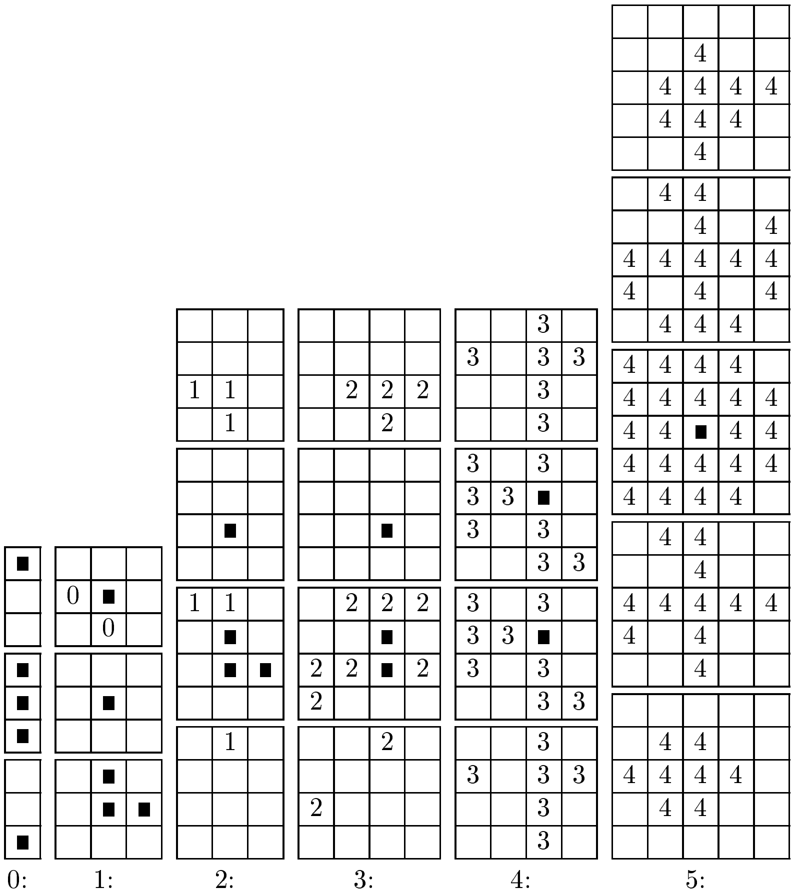
Winning Strategy for Animal 30





Winning Strategy for Animal 31

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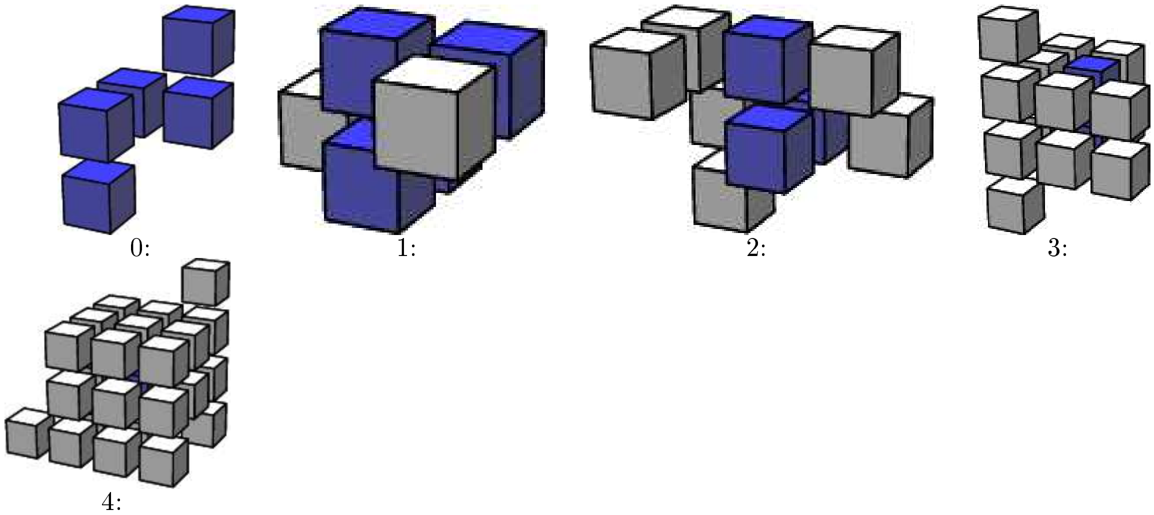
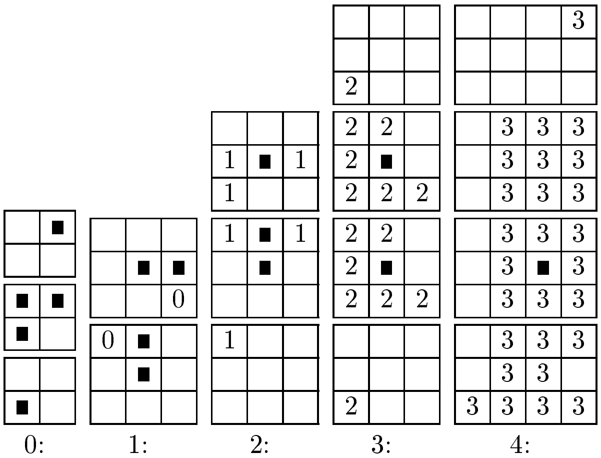
Winning Strategy for Animal 32

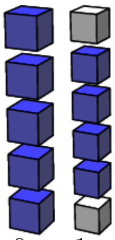




APPENDIX B

Handicap Strategies of Five Block Animals

Strategy for Animal 11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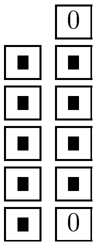
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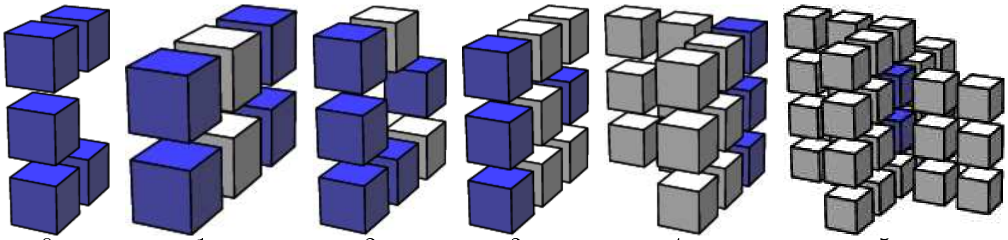
Strategy for Animal 15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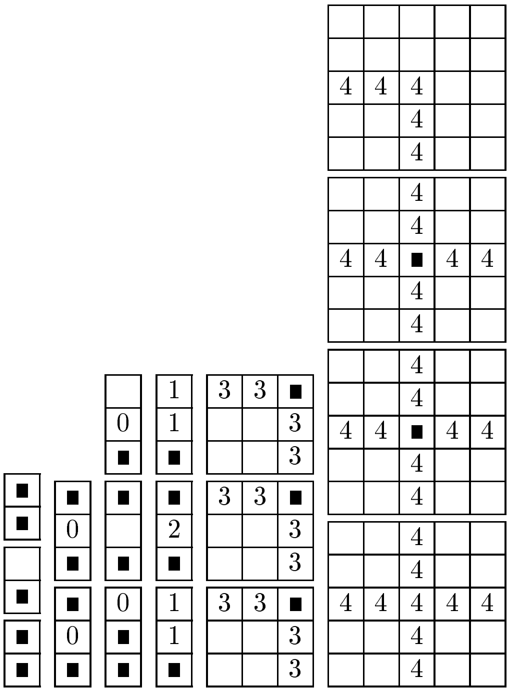
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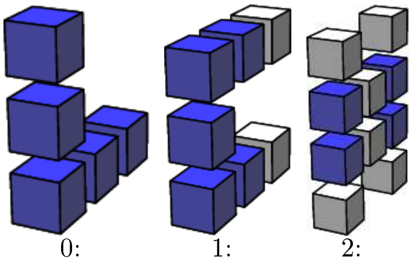
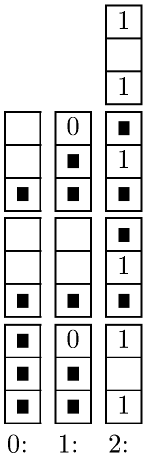
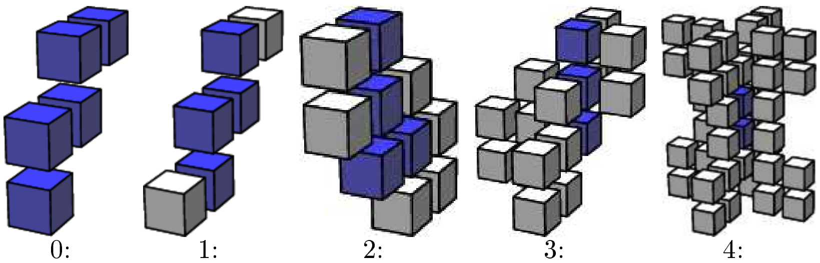
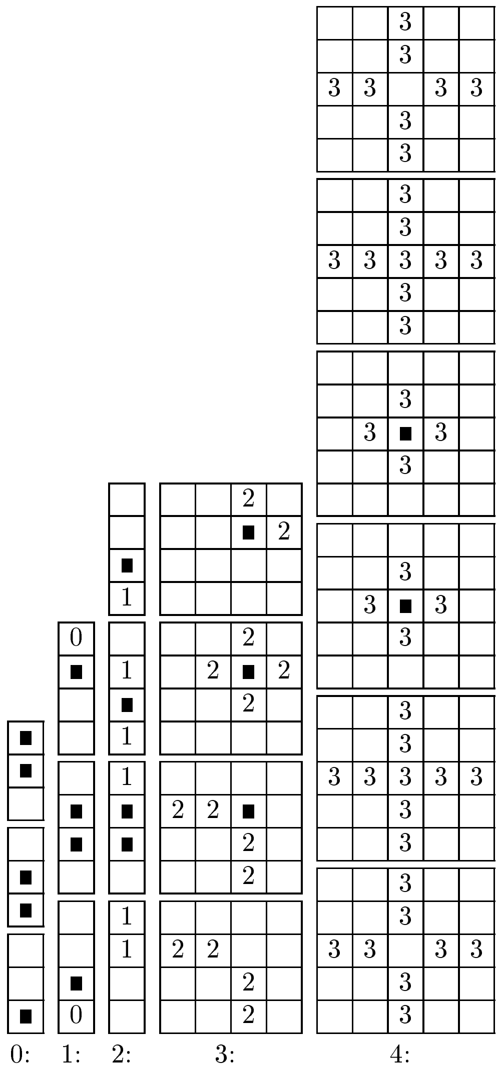
Strategy for Animal 16





Strategy for Animal 29

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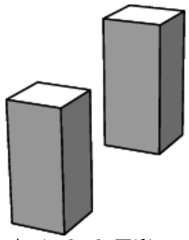
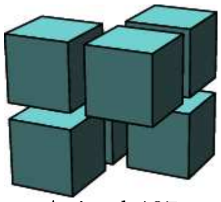
APPENDIX C

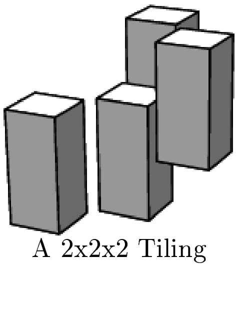
3D losers, and their Tilings



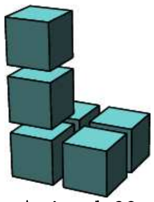
A 1x2x2 Tiling



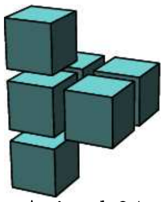
Animal 107



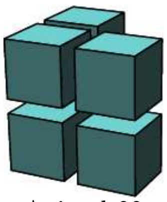
A 2x2x2 Tiling



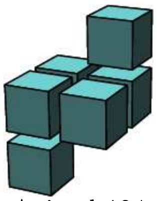
Animal 69



Animal 84



Animal 99



Animal 134

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