POLYOMINO WEAK ACHIEVEMENT GAMES ON 3-DIMENSIONAL RECTANGULAR BOARDS

Nándor Sieben, Elaina Deabay

10/13/2004

Abstract

Polyomino achievement games on 3-dimensional rectangular boards are studied. All except two polyominoes with less than six cells are weak winners. The smallest known paving losers contain six cells. There are finitely many winning polyominoes.

1. Introduction

Achievement games for polyominoes have been introduced by Frank Harary [Gar, Ha1, Ha2, HH, Ha6]. They are generalizations of the well known game Tic-Tac-Toe. In these games the target shape can be some predetermined set of polyominoes. The type of the board can vary as well. It can be a tiling of the plane by triangles [HH, BH3] or hexagons [BH2]. The game board can be a Platonic solid [BH1] or the hyperbolic plane [Bod]. A comprehensive investigation of these possibilities can be found in [Bod]. Many more abstract generalizations were studied in [EH, AH, Ha3, Ha4, Ha5]. The research of polyomino achievement games on 3-dimensional boards was started in [HW]. In this paper we continue this study.

A *board* is a set of cells. An *n-dimensional rectangular board* is a board whose cells are the translations of an n-dimensional unit hypercube $[0,1]^n$ by vectors of $\mathbf{Z}^n$. Informally, an n-dimensional rectangular board is the simple tessellation of the n-dimensional Euclidean space with hypercubes, that is, an infinite n-dimensional chessboard.

An *n-dimensional polyomino* or an *n-polyomino* is a finite set of cells of the n-dimensional rectangular board such that both the polyomino and its complement are connected through $(n-1)$-dimensional faces of the cells. A polyomino is also called an *animal.* A 2-polyomino is simply called a polyomino and a 3-polyomino is sometimes called a *polycube.* We only consider polyominoes up to congruence, that is, the location of the polyomino on the board is not important.

In a *polyomino achievement game* two players alternately mark previously unmarked cells of the board using their own colors. The player who marks a set of cells congruent to a given polyomino wins the game. In a *weak achievement* game the second player only tries to prevent the first player from achieving the given polyomino. Weak achievement games are also called *A-achievement games.* In a weak achievement game the first player is called the *maker* and the second player is called the *breaker.*

A polyomino is called a (*weak*) *winner* if the first player can always win the (weak) achievement game with the given polyomino. Otherwise the polyomino is called a *loser.* If a polyomino is a winner then it is also a weak winner. A weak loser is also a loser. A polyomino that is a subset of a winning polyomino is also a winner. A polyomino that is a superset of a losing polyomino, is also a loser. So to find all the winning polyominoes it suffices to find the winning polyominoes with maximal size. Such polyominoes are also called *elementary winners.* A minimal loser polyomino is also called an *elementary loser.*

A polyomino is called a (*weak*) *economical winner* if the first player can (weakly) achieve it using as many marks as the number of cells in the polyomino.

The (*weak*) *handicap number* [HHS, HS1] of a polyomino is the number of extra cells the first player need to mark, before the game starts, to win in the (weak) achievement game. Note that a winning polyomino has handicap number 0.

Polycube achievement games were studied in [HW]. It only considers polycubes containing up to four cells. In this paper we extend the results of [HW]. We show that every polyomino with fewer than five cells is a weak winner. We give winning strategies for all but two polyominoes with five cells. The smallest known

2000 *Mathematics Subject Classification:* Primary 05B50

Keywords: polyomino, hypercube, animal, achievement game, hypergraph game

losers have six cells. We also show that there are only finitely many winners and we give upper bounds for the number of winners containing a certain amount of cells.

We thank Heiko Harborth and András Pluhár for their help.

2. Winning strategies

In this section we describe what a winning strategy is in a polyomino weak achievement game.

Definition 2.1. A *situation* s is a pair (C_s, N_s) where the *core* C_s is a set containing cells marked by the first player and the *neighborhood* N_s is a set of unmarked cells. Note that $C_s \cap N_s = \emptyset$. A situation is called *winning* if it can be won by the first player after any mark of the second player. Any situation congruent to a winning situation is also winning.

If $a \in C_s$ then the situation $(C_s \setminus \{a\}, N_s \cup \{a\})$ is called a *deletion* of s. Let $S = \{s_i \mid i \in I\}$ be a set of situations. If $C = \cup_{s \in S} C_s$ and $N = \cup_{s \in S} N_s$ are disjoint then the *join* of S is the situation (C, N). If $\cap_{s \in S} N_s = \emptyset$ then we say that the join is *good*.

Any good join of deletions of winning situations is winning. To see this, note that no matter how the second player marks a cell of the neighborhood of the join, the first player is always able to achieve one of the original winning situations.

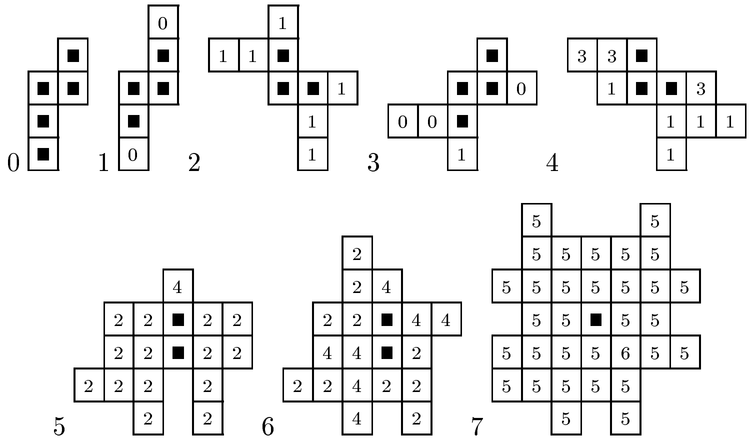
Definition 2.2. A *winning strategy* for the first player to achieve the polyomino A is a finite sequence $s_0, \ldots, s_n$ of situations such that C_{s_0} is congruent to A, $N_{s_0} = \emptyset$, C_{s_n} is a singleton set and every s_i is a good join of situations which are congruent to deletions of situations in the sequence with index less than i. A winning strategy on a finite board requires that $C_{s_n} \cup N_{s_n}$ fits into the board.

A winning strategy works on any finite board that is large enough to hold s_0. The size of the minimal board on which a polyomino is a (weak) winner is called the (weak) board number [HW]. A winning strategy gives an upper bound for the board number of the polyomino.

Definition 2.3. The *length of a winning strategy* is the number of marks the maker has to make following the strategy to win for certain. The *economy number of a winning strategy* is the difference of the length of the strategy and the number of cells in the polyomino. The *economy number of a polyomino* is the minimum of the economy numbers of the winning strategies of the polyomino. The *complexity of a winning strategy* is the number of situations in the strategy.

Note that an economical winner has economy number 0.

Example 2.4. The following is a diagram of a winning strategy for the 2-polyomino called Z.



The strategy has complexity 8. This is an improvement over the strategy in [BH2] which has the same length but complexity 14. We are going to see later that the corresponding strategy on the 3-dimensional board has complexity 5. The solid blocks indicate the marks of the maker. The cells with numbers in them denote unmarked cells. An actual game starts from situation s_7 and ends at situation s_0. If the breaker marks an unmarked cell with number i in it, then the maker is able to mark an unmarked cell to achieve situation i. The following figure shows all the possible flows of the game following this strategy:

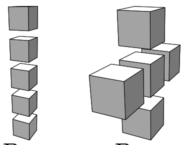
$$\begin{tikzcd}
s_7 \arrow[r] \arrow[dr] & s_6 \arrow[r] \arrow[dr] & s_4 \arrow[r] \arrow[drr] & s_3 \arrow[dr] \arrow[drr] & & \\
& s_5 \arrow[r] \arrow[ur] & s_2 \arrow[rr] & & s_1 \arrow[r] & s_0
\end{tikzcd}$$

Note that the length of the longest directed path from s_7 to s_0 is one less than the length of the strategy. So the strategy has length 6 and economy number 1.

3. Polycubes with fewer than five cells

In Appendix A, we include the best known strategy for all but two polyominoes with five cells. Every situation is represented by a 3-dimensional picture and by horizontal slices. The slices are taken from the bottom to the top. Again the solid blocks indicate the marks of the maker. The cells with numbers in them denote unmarked cells. Empty cells are there to show the alignment of the slices.

We have found a winning strategy for all polyominoes with five cells except for P_{11} and P_{33}. We could not even improve the handicap numbers of the 2-dimensional strategies for these polyominoes.



P_{11} P_{33}

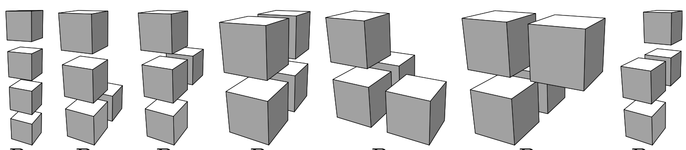
The following table compares the best strategies on 2-dimensional and 3-dimensional boards for flat polyominoes. We include the handicap number h of the polyomino and the economy number e and complexity c of the best known strategy. The 2-dimensional values are based on [BH2, HHS] and Example 2.4.

		P_{11}	$P_{12} = L$	$P_{13} = Y$	P_{14}	P_{15}	P_{16}	P_{21}	$P_{24} = Z$	P_{28}	P_{29}	P_{31}	P_{33}
3D	h	?	0	0	0	0	0	0	0	0	0	0	?
	e		1	1	0	3	2	1	0	0	0	1	
	c		11	8	6	10	8	13	5	6	6	6	
2D	h	2 or 3	0	0	2	1	1	1	0	1	1	1	4
	e		2	1					1				
	c		33	17					8				

Note that there are several 2-dimensional losers which are 3-dimensional winners. Also note that as expected, the economy number and the complexity of a winning strategy often improves in the 3-dimensional case.

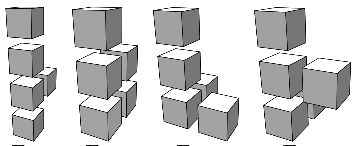
Proposition 3.1. *All 3-polyominoes with fewer than five cells are weak winners.*

Proof. There are seven polyominoes with four cells:



P_4 P_5 P_6 P_7 P_8 P_9 P_{10}

All of these are subsets of the following winning polyominoes with five cells.



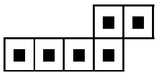
P_{13} P_{14} P_{17} P_{18}

We have P_4, P_5, $P_6 \subseteq P_{13}$ and P_7, $P_{10} \subseteq P_{14}$ and $P_8 \subseteq P_{17}$ and $P_9 \subseteq P_{18}$. Hence all polyominoes with fewer than five cells are subsets of winning polyominoes and hence also winners. $\square$

4. Losers

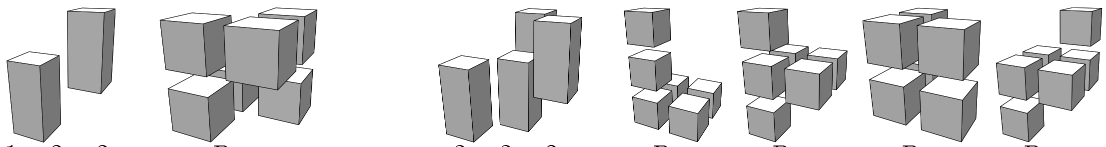
The most successful method to show that a polyomino is a loser in the weak achievement game is to pave the board with pairs of cells such that every position of the polyomino on the board contains a full pair. The existence of such a paving allows the breaker to win by marking the pair of the cell marked by the maker at each move. A polyomino is called a *paving loser* if such tiling exists, otherwise it is called a *paving winner* [HS3].

All 2-dimensional polyominoes are known to be winners or losers except for Snaky. Furthermore all known 2-dimensional losers are paving losers while Snaky is a paving winner [HS2, HS3].



Snaky

In the 3-dimensional case paving gives less satisfactory results. As we saw in the previous section, there are two polyominoes with five cells for which we were unable to find a winning strategy. Unfortunately we were not able to find any pavings that prove any of these polyominoes losers. The smallest known paving losers have six cells. They are elementary losers. The following figure shows these polyominoes and the fundamental regions of their pavings:



$1 \times 2 \times 2$ P_{107} $2 \times 2 \times 2$ P_{69} P_{84} P_{99} P_{134}

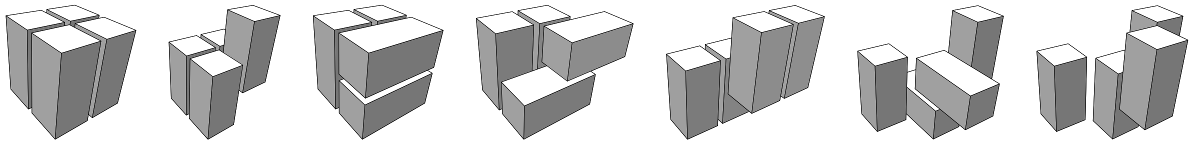
The dimensions of the fundamental regions are written below the picture of the pairs. To get a paving from a fundamental region with dimensions $a \times b \times c$, we fill the board with translations by vectors in the set $\{(am, bn, cl \mid m, n, l \in \mathbf{Z})\}$.

We tried to carry out a search for a set of pavings that makes all polyominoes with a large enough size a loser. This method was used for hexagonal polyominoes in [BH2]. The search was not successful due to a search space larger than that in the hexagonal case but we believe that a state of the art computing cluster could deal with the problem.

The following tables show our partial results. First we created fundamental regions of pavings with a certain size. The first table shows the number of tiles found:

Size	$1 \times 2 \times 8$	$1 \times 3 \times 6$	$1 \times 4 \times 5$	$1 \times 4 \times 6$	$2 \times 2 \times 3$
Number of pavings	61	24	103	154	63
Size	$2 \times 2 \times 4$	$2 \times 2 \times 5$	$2 \times 3 \times 3$	$2 \times 3 \times 4$	$2 \times 2 \times 2$
Number of pavings	178	2477	577	11695	840

We mostly used pavings where the pairs share a face because this seemed to work best. This is due to the fact that the polyominoes are connected through faces. The following figure shows the seven such pavings with size $2 \times 2 \times 2$:



For size $2 \times 2 \times 2$ we also allowed larger separations between the pairs. We found 840 such pavings. For larger fundamental regions this was not practical since there were too many such pavings.

The following table shows the upper bounds for the number of winning polyominoes with a certain size.

Size	1	2	3	4	5	6	7	8	9	10	11
Possible winners	1	1	2	7	23	107	490	2311	10220	42409	162091

In the next section we use a different method to show that there are finitely many winners.

5. Hypergraph games

In this section we show that there are only finitely many winning polyominoes in the 3-dimensional, rectangular, weak achievement game.

Let $\mathcal{H} = (V, E)$ be a hypergraph, that is, $E \subseteq 2^V$. In a *hypergraph positional game* $(\mathcal{H}, p, q)$, two players alternately mark previously unmarked vertices of $\mathcal{H}$. The first player marks p and the second player marks q vertices per move. A game is called *relaxed* if the second player has the opportunity to mark at least q cells at least after each p marks of the first player. In an *achievement* game the player who marks all vertices of an edge of $\mathcal{H}$ wins the game. In a *weak achievement* game the second player only tries to prevent the first player to mark the vertices of an edge. In a weak achievement game the first player is called the *maker* and the second player is called the *breaker*.

Beck [Bec] showed that the breaker wins the weak relaxed achievement game if

$$\sum_{A \in E} (1+q)^{-\frac{|A|}{p}} < \frac{1}{1+q}.$$

This result is a generalization of the $p = 1 = q$ case studied in [ES].

A polyomino achievement game is equivalent to a hypergraph game where the vertices of $\mathcal{H}$ is the set of cells of the playing board and the edges of $\mathcal{H}$ is the set of target polyominoes in all acceptable positions on the board.

The result of Beck cannot be used immediately for our purposes since the set of edges in the corresponding hypergraph game is not finite. We need to break the infinite board into finite subboards. This idea originates from [Plu].

Proposition 5.1. *There are finitely many winning polyominoes in the 3-dimensional rectangular weak achievement game.*

Proof. Let $(\mathcal{H}, 1, 1)$ be the weak hypergraph achievement game equivalent to the original P achievement game. Let n be the number of cells in the polyomino P and let d be the size of the smallest cube that can contain P. It is clear that $d \leq n$. We can tile the board with cubes of size d by translating one such cube by the vectors in the set $\{(md, nd, ld) \mid m, n, l \in \mathbf{Z}\}$. Let us call the union of one such cube and its surrounding 26 cubes a subboard. A subboard is a cube of size $3d$. The collection of subboards cover the whole board but the subboards are not disjoint.

Since every edge of $\mathcal{H}$ is contained in at least one of the subboards, the breaker can win by preventing the maker from winning on all of the subboards. For a subboard S let $\mathcal{H}_S$ be a hypergraph isomorphic to the hypergraph whose vertices are the cells of S and whose edges are those edges of $\mathcal{H}$ that lie inside S.

We can create subgames on each $\mathcal{H}_s$ corresponding to the original game in the following way. Because of the overlap of the subboards, every mark of the maker on the original board makes a mark on 5^3 subboards. We mark this move on all the subgames. A mark of the breaker also appears on 5^3 subboards but it is not necessarily a strong move on all of those subboards. So after each mark of the maker the breaker can concentrate on one of the 5^3 subgames affected by the maker's mark, mark only in the chosen subgame, and ignore the other affected subgames. If the breaker concentrates on the subgame in which her last mark was the earliest, then on every subboard she can respond at least after the maker's 5^3-th mark. If the breaker can win in each $(\mathcal{H}_S, 5^3, 1)$ weak, relaxed subgame then she can prevent the maker from winning on all the subboards, and therefore in the original game as well.

Now we can use the result of Beck on each finite subgame. Each edge A of $\mathcal{H}_S$ has n cells. The number of edges of $\mathcal{H}_S$ is the number of different positions of P on the subboard S and so $|\mathcal{H}_S| \leq 3! \cdot 2^3 \cdot (3d)^3$. Thus the breaker can win if

$$\sum_{A \in \mathcal{H}_S} (1+1)^{-\frac{|A|}{5^3}} = |\mathcal{H}_S| \cdot 2^{-\frac{n}{5^3}} \leq 3! \cdot 2^3 \cdot (3n)^3 \cdot 2^{-\frac{n}{5^3}} < \frac{1}{2} = \frac{1}{1+1},$$

that is,

$$\log_2(3! \cdot 2^3 \cdot 3^3) + 3\log_2(n) - \frac{n}{5^3} < -1$$

which happens if $n > 6136$. Thus no polyomino with more than 6136 cells can be a weak winner. □

The proof only used finiteness arguments so it clearly can be modified to work on any finite dimensional, rectangular board and with any finite set of polyominoes as target. The bound for the maximum size for a winner given by the proof is very likely much larger than the actual maximum size. At least this is the case in the 2-dimensional case.

6. Unsolved problems

We saw that there are only finitely many winning polyominoes on every finite dimensional rectangular board. We could consider an infinite dimensional rectangular board. Are there still finitely many winners? Are there any losers at all on the infinite dimensional board?

Every rectangular board can be embedded into a higher dimensional rectangular board, and so every polyomino can be considered as a higher dimensional polyomino. Increasing the dimension of the board makes it easier to achieve a given polyomino. For example the 2-polyomino called Fatty containing four cells in a square is a loser on the 2-dimensional board but is a winner on the 3-dimensional board. Is it true that every polyomino is a winner on a board that has a large enough dimension?

If a 2-polyomino has a strategy with handicap number 1 then this strategy can be extended to get a winning strategy on the 3-dimensional board. We can create three deletions of the last situation of the strategy and join them mutually perpendicularly to get a situation with a single marked cell. For an example see the winning strategy for P_{12} where three copies of deletions of s_9 are joined to get s_{10}. Can we always reduce the handicap number of a strategy using a similar procedure by playing the game on a board with a larger dimension, even if the handicap number is larger than 1?

References

[AH] M. Anderson, F. Harary, *Achievement and avoidance games for generating abelian groups*, Internat. J. Game Theory 16 (1987), no. 4, 321–325.

[Bec] J. Beck, *Remarks on positional games I.*, Acta Math. Acad. Sci. Hungar. **40** (1982), no. 1-2, 65–71.

[Bod] J. Bode, *Strategien für Aufbauspiele mit mosaik-polyominos*, Doctoral dissertation, Technishen Universität Braunschweig.

[BH1] J. Bode, H. Harborth, *Achievement games on Platonic solids*, Bull. Inst. Combin. Appl. **23** (1998), 23–32.

[BH2] J. Bode, H. Harborth, *Hexagonal polyomino achievement*, Discrete Math. **212** (2000), np. 1–2, 5–18.

[BH3] J. Bode, H. Harborth, *Triangle polyomino set achievement*, Congr. Numer. **148** (2001), 97–101.

[EH] M. Erickson, F. Harary, *Generalized Ramsey theory. XV. achievement and avoidance games for bipartite graphs*, Lecture Notes in Math. **1073**, 212–216.

[ES] P. Erdős, J. L. Selfridge, *On a combinatorial game*, J. Combinatorial Theory Ser. A **14** (1973), 298–301.

[Gar] M. Gardner, *Mathematical games*, Sci. Amer. **240** (1979), 18–26.

[Ha1] F. Harary, *Achieving the Skinny animal*, Eureka **42** (1982), 8–14.

[Ha2] F. Harary, *Achievement and avoidance games for graphs*, Ann. Discrete Math. **13** (1982), 111–120.

[Ha3] F. Harary, *Achievement and avoidance games designed from theorems*, Rend. Sem. Mat. Fis. Milano 51 (1981), 163–172 (1983).

[Ha4] F. Harary, *Achievement and avoidance games on finite configurations*, Recreational Math. **16** (1983–84), 182–187.

[Ha5] F. Harary, *Achievement and avoidance games on finite configurations with one color*, Recreational Math. **17** (1984–85), 253–260.

[Ha6] F. Harary, *Is Snaky a winner?*, Geombinatorics **2** (1993), 79–82.

[HH] F. Harary, H. Harborth, *Achievement and avoidance games with triangular animals*, Recreational Math. **18** (1985–86), 110–115.

[HHS] F. Harary, H. Harborth, M. Seeman, *Handicap achievement for polyominoes*, Congr. Numer. **145** (2000), 65–80.

[HW] F. Harary, M. Weisbach, *Polycube achievement games*, J. Recreational Mathematics **15** (1982-83), 241–246.

[HS1] H. Harborth, M Seemann, *Handicap achievement for squares*, To appear

[HS2] H. Harborth, M Seemann, *Snaky is an edge-to-edge loser*, Geombinatorics **5** (1996), no. 4, 132–136.

[HS3] H. Harborth, M Seemann, *Snaky is a paving winner*, Bull. Inst. Combin. Appl. **19** (1997), 71–78.

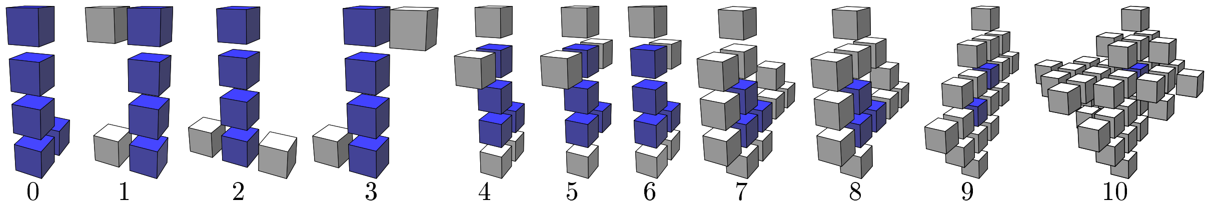
[Plu] A. Pluhár, *Generalized Harary games*, Acta Cybernet. **13** (1997), no. 1, 77–83.

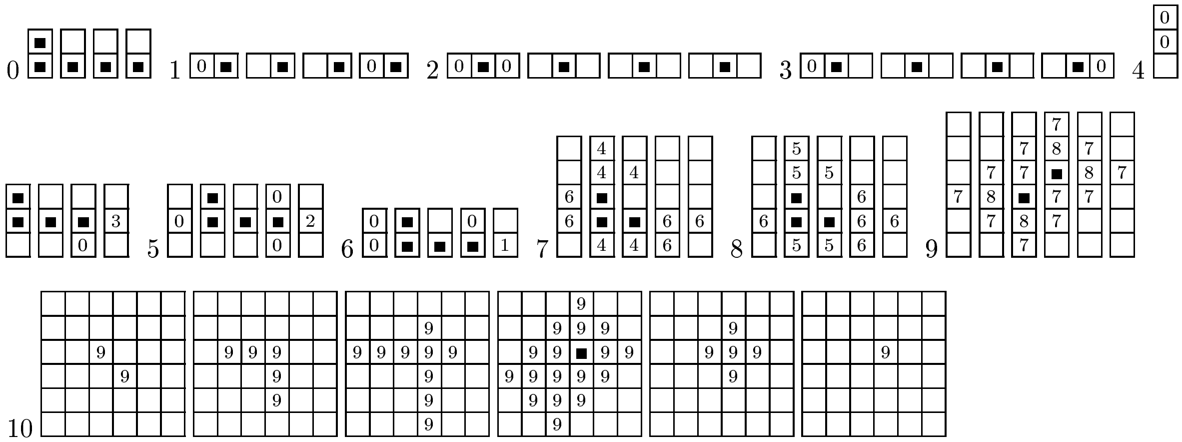
Email: `MDeabay@aol.com, nandor.sieben@nau.edu`

DEPARTMENT OF MATHEMATICS, NORTHERN ARIZONA UNIVERSITY, FLAGSTAFF, AZ 86011-5717

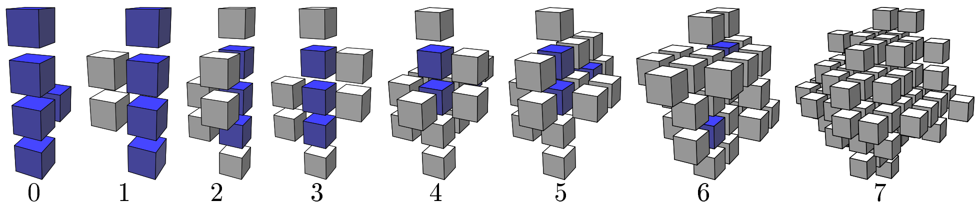
7. Appendix A.

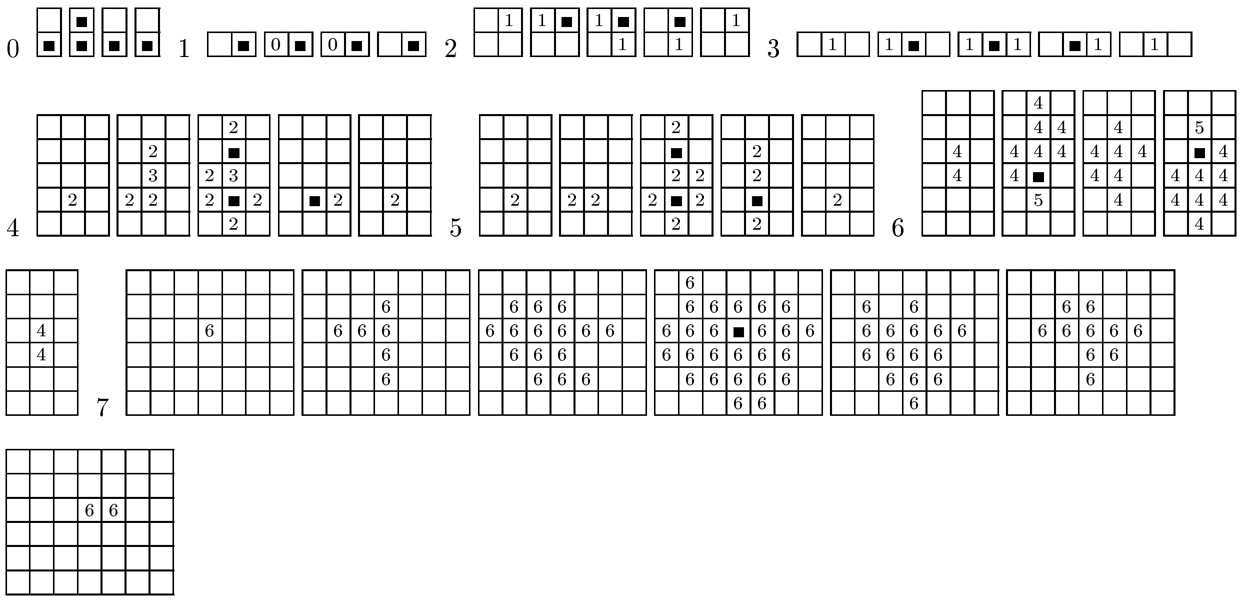
Strategy for Polyomino 12



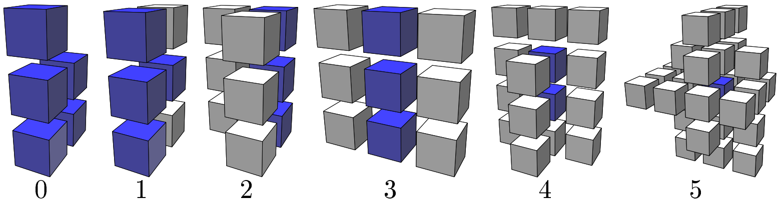


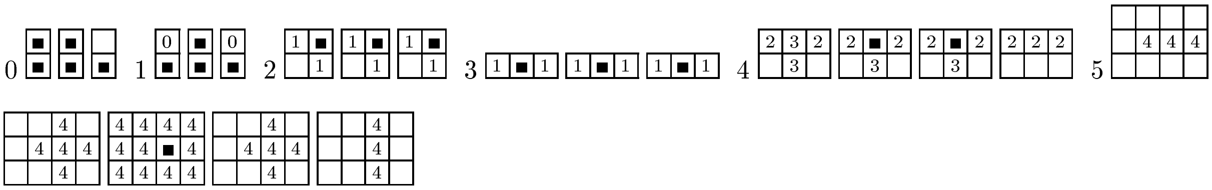
Strategy for Polyomino 13



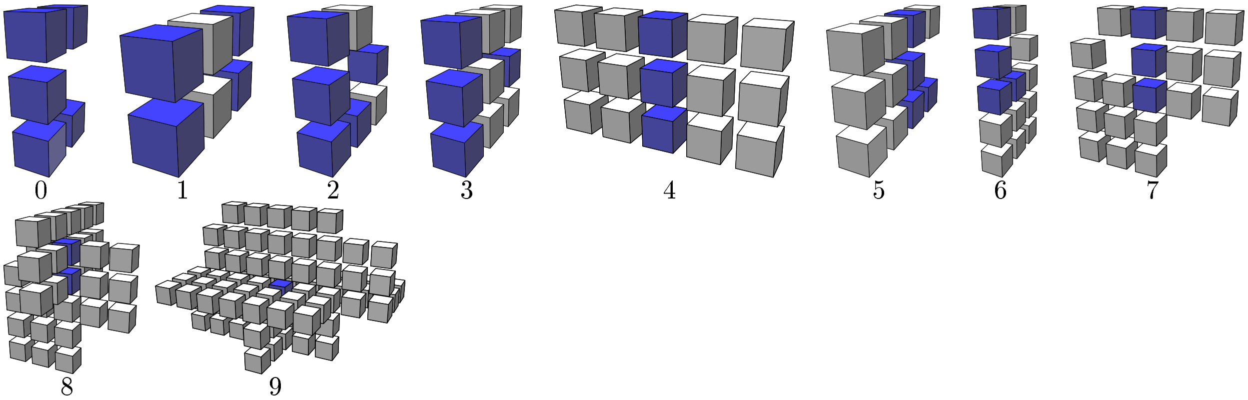


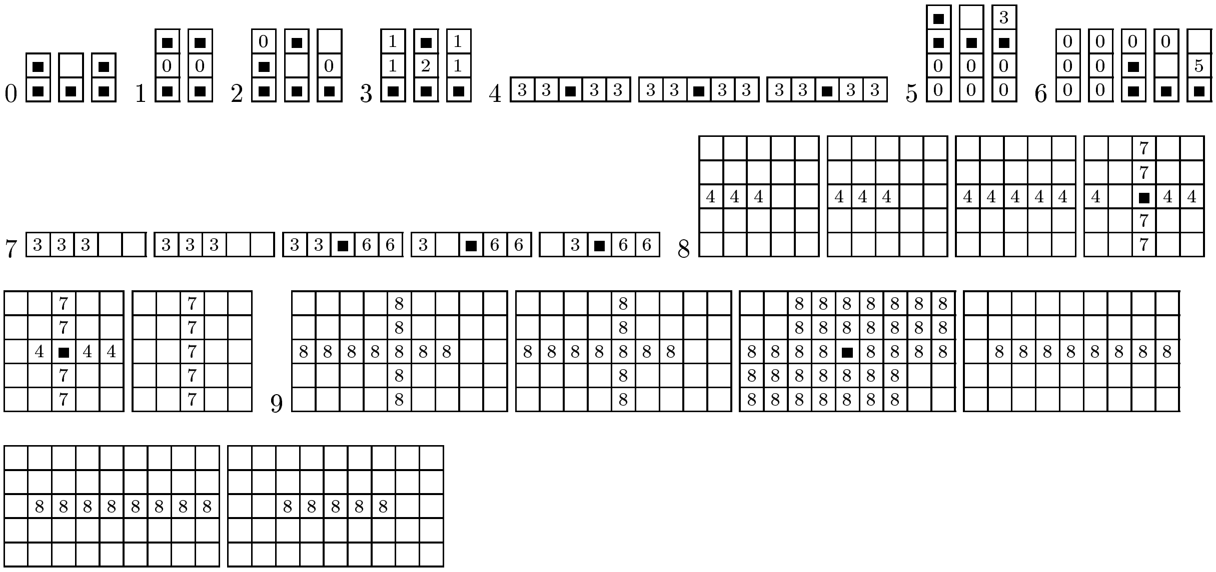
Strategy for Polyomino 14



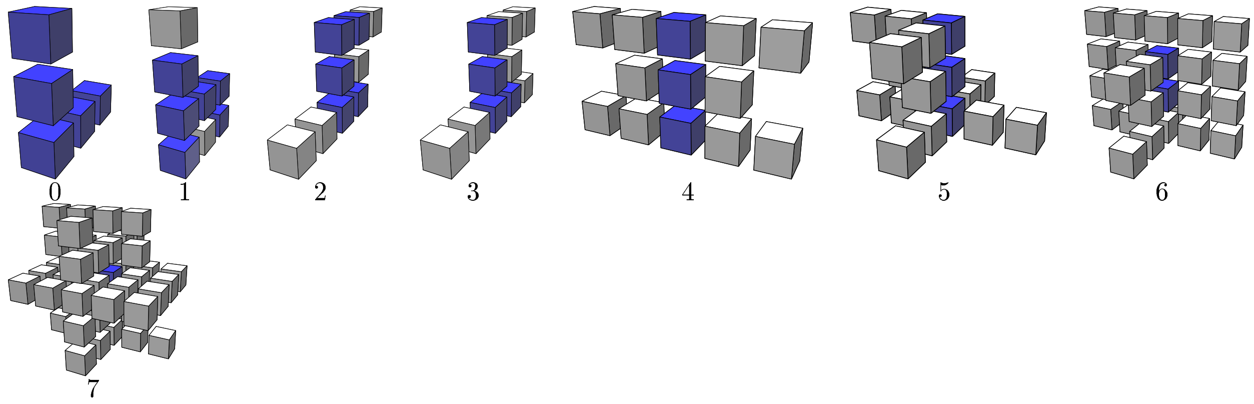


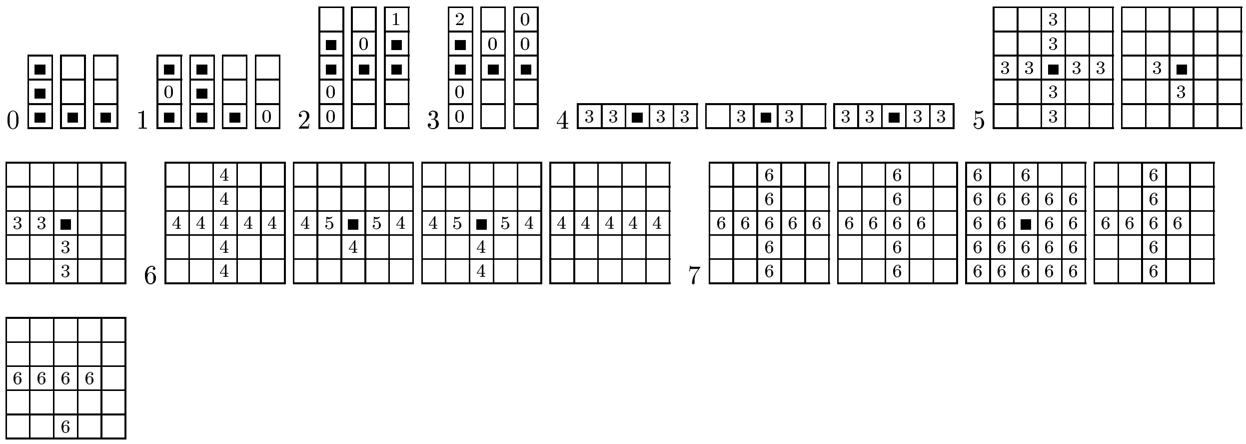
Strategy for Polyomino 15



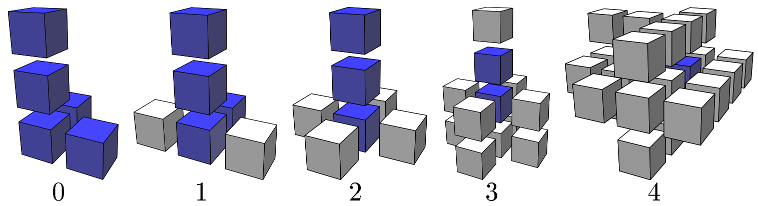


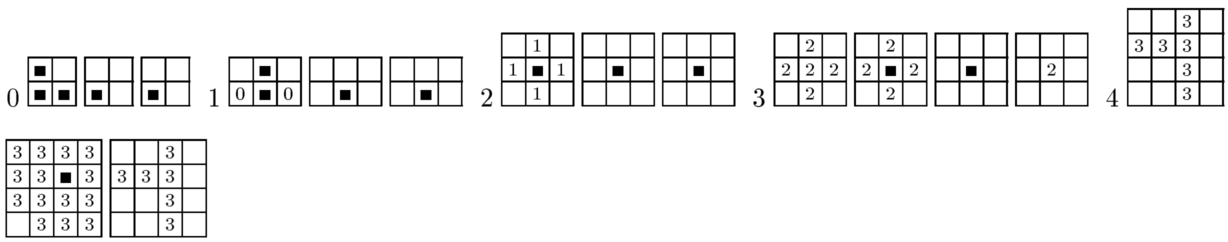
Strategy for Polyomino 16



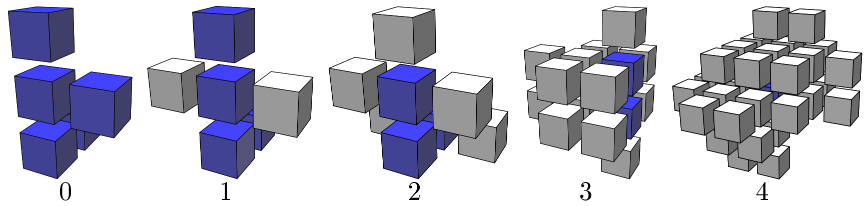


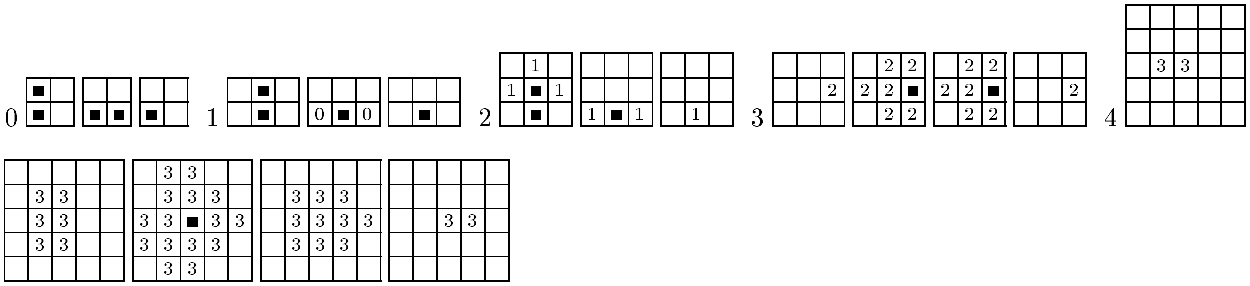
Strategy for Polyomino 17



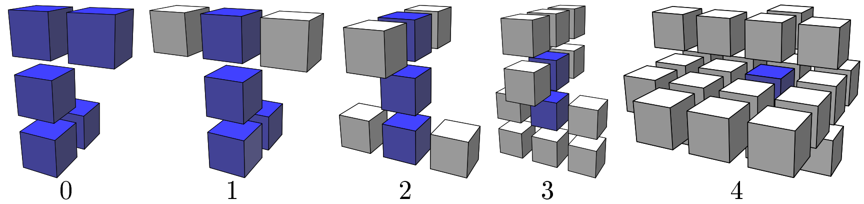


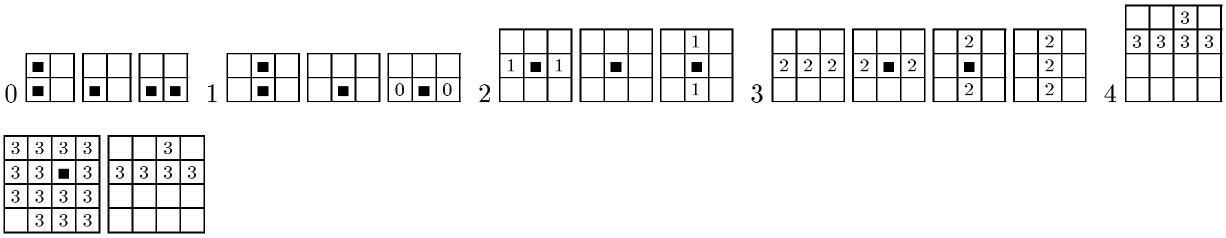
Strategy for Polyomino 18



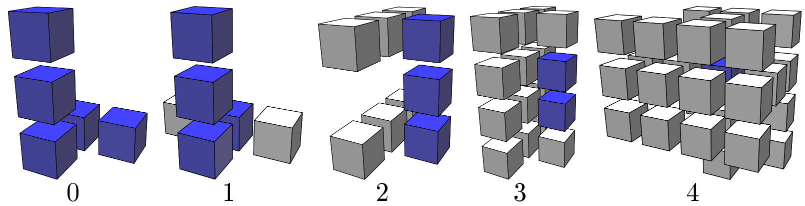


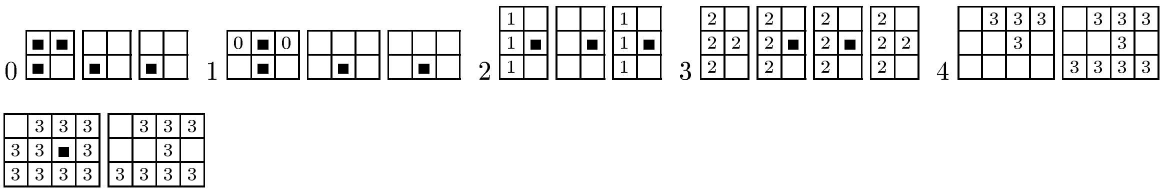
Strategy for Polyomino 19



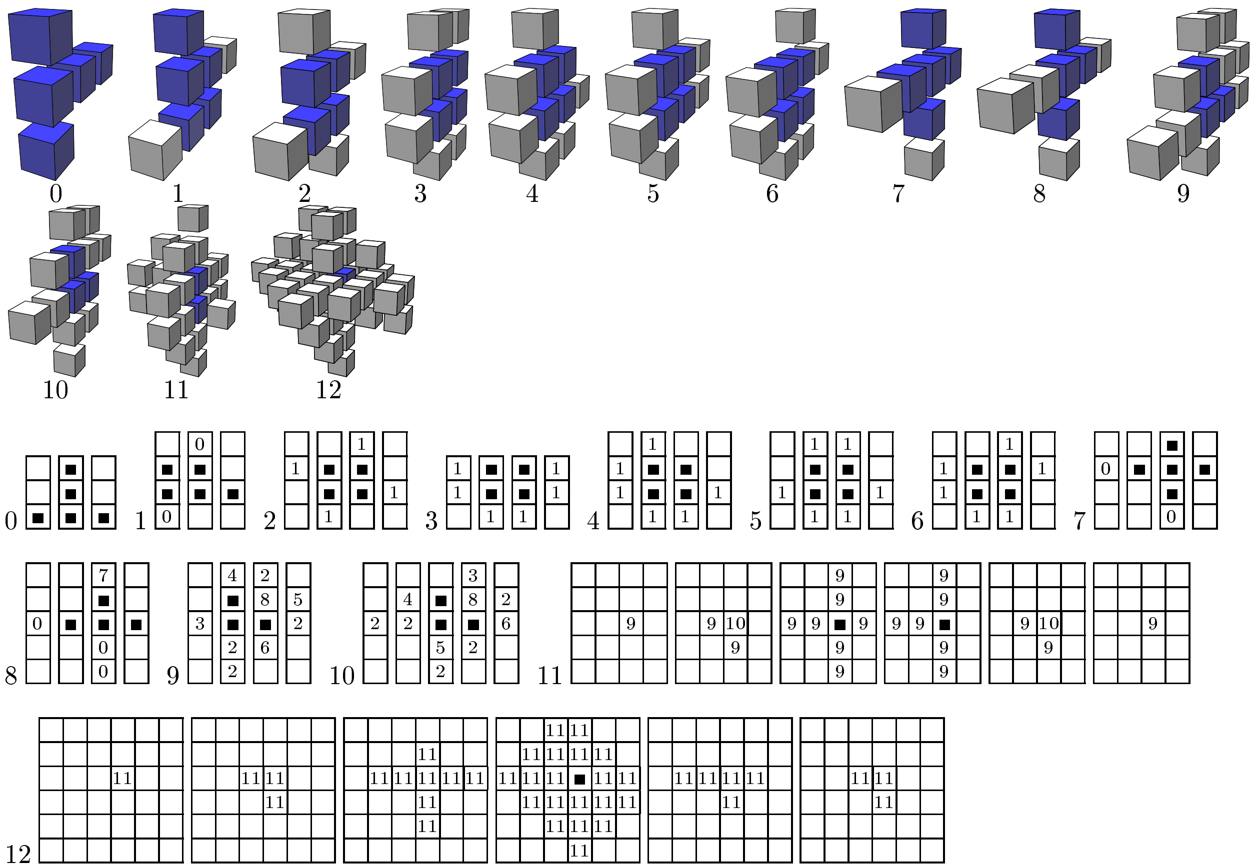


Strategy for Polyomino 20

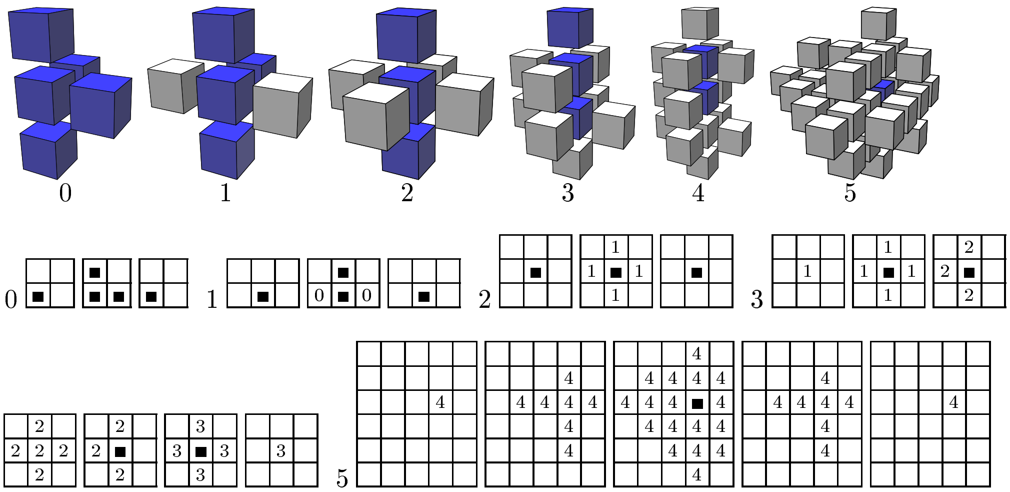




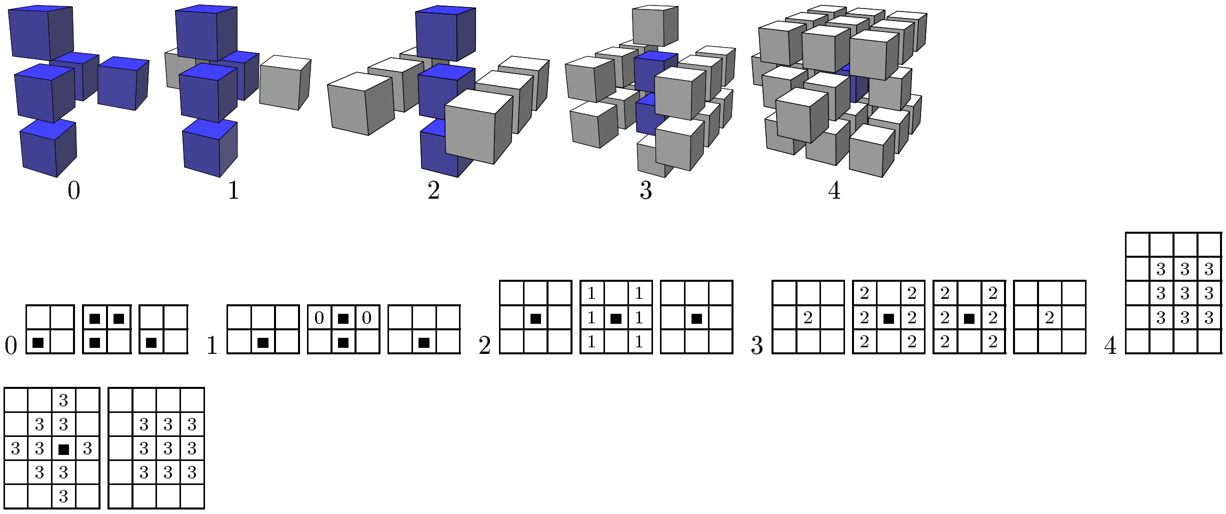
Strategy for Polyomino 21



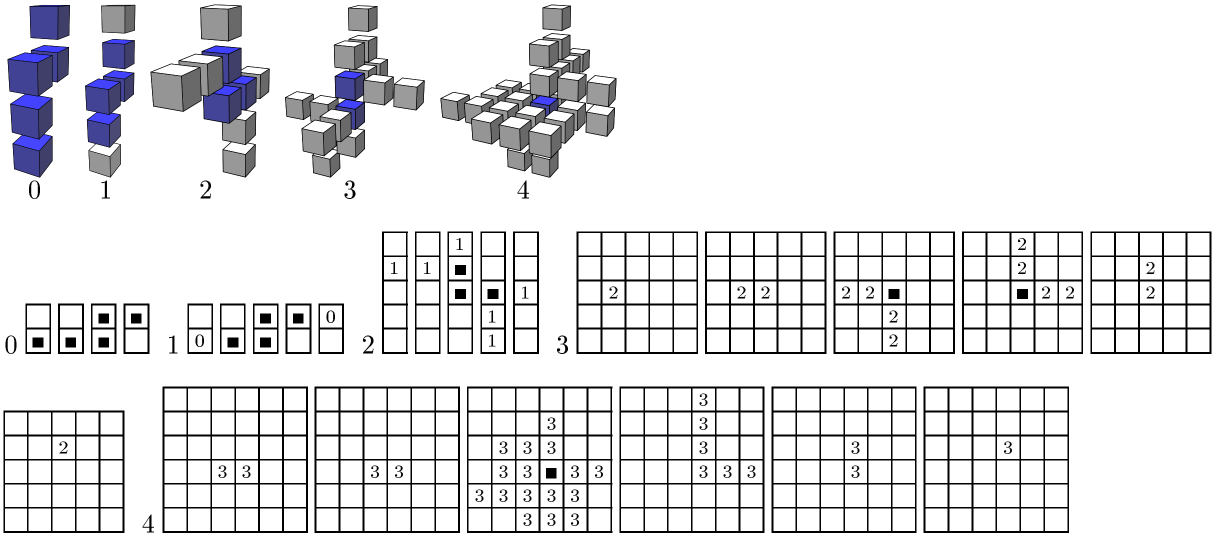
Strategy for Polyomino 22



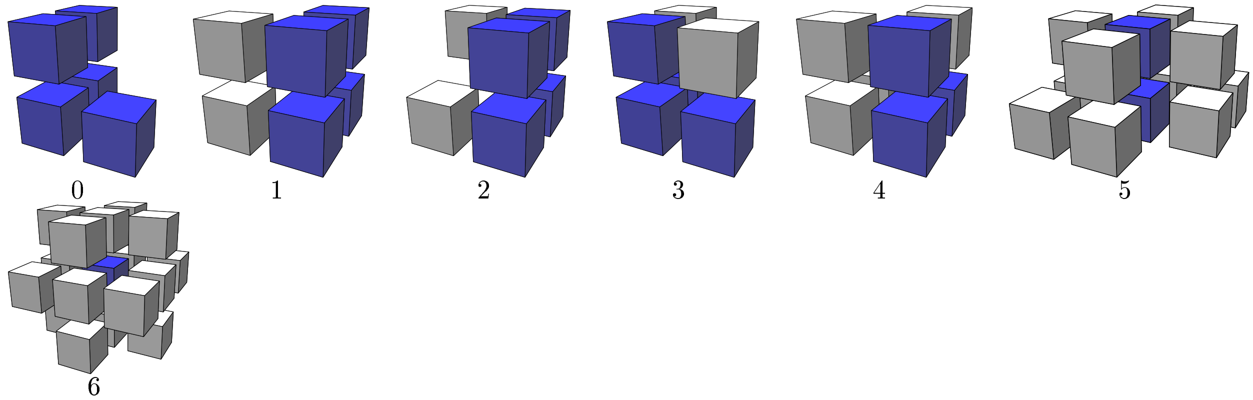
Strategy for Polyomino 23

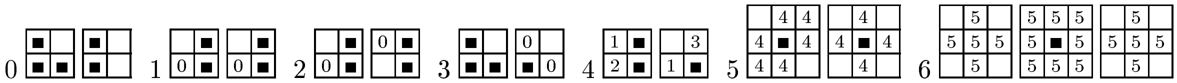


Strategy for Polyomino 24

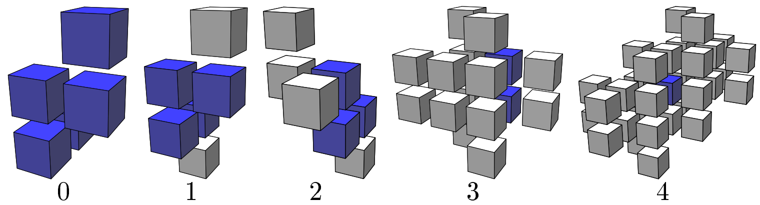


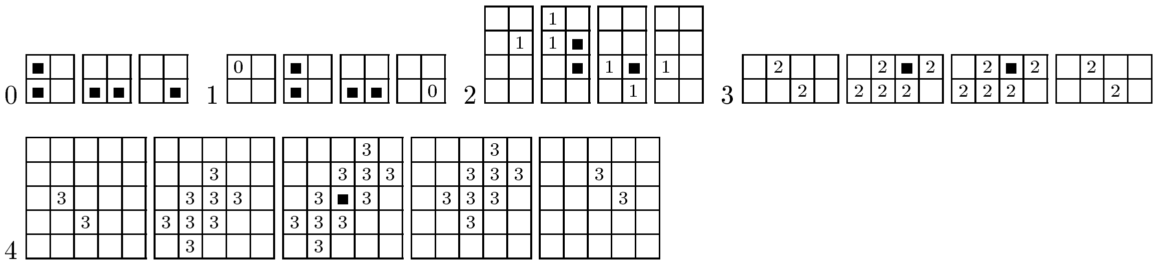
Strategy for Polyomino 25



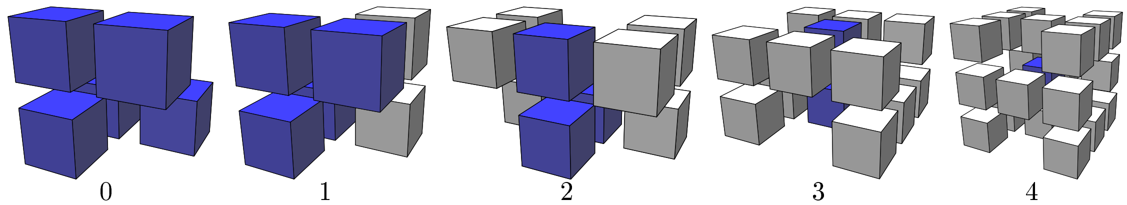


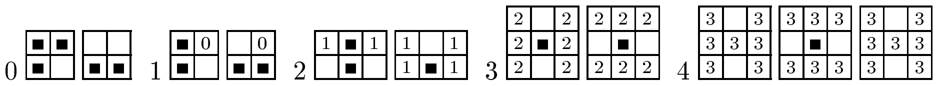
Strategy for Polyomino 26



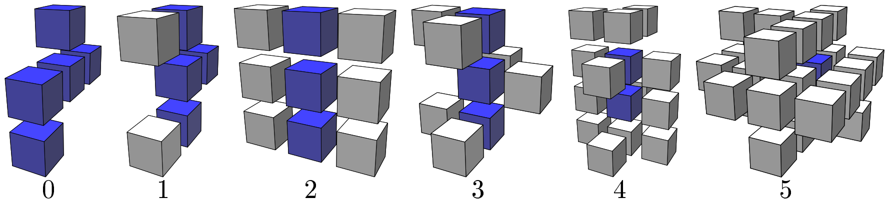


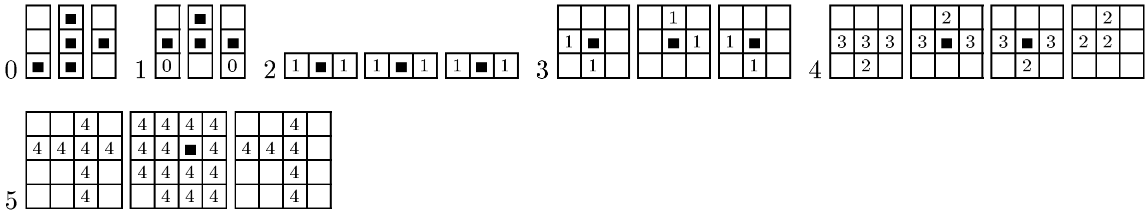
Strategy for Polyomino 27



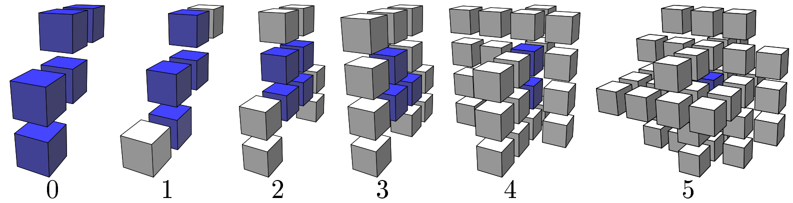


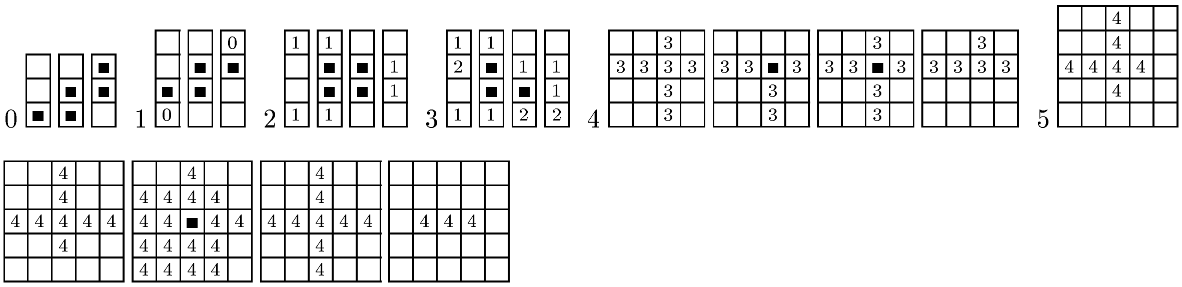
Strategy for Polyomino 28



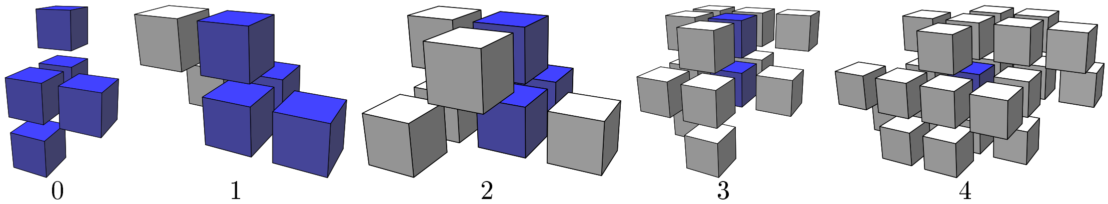


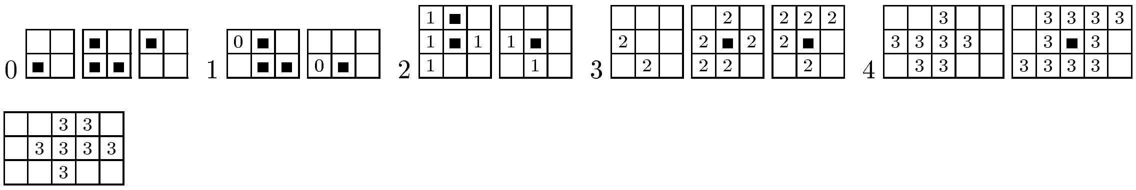
Strategy for Polyomino 29



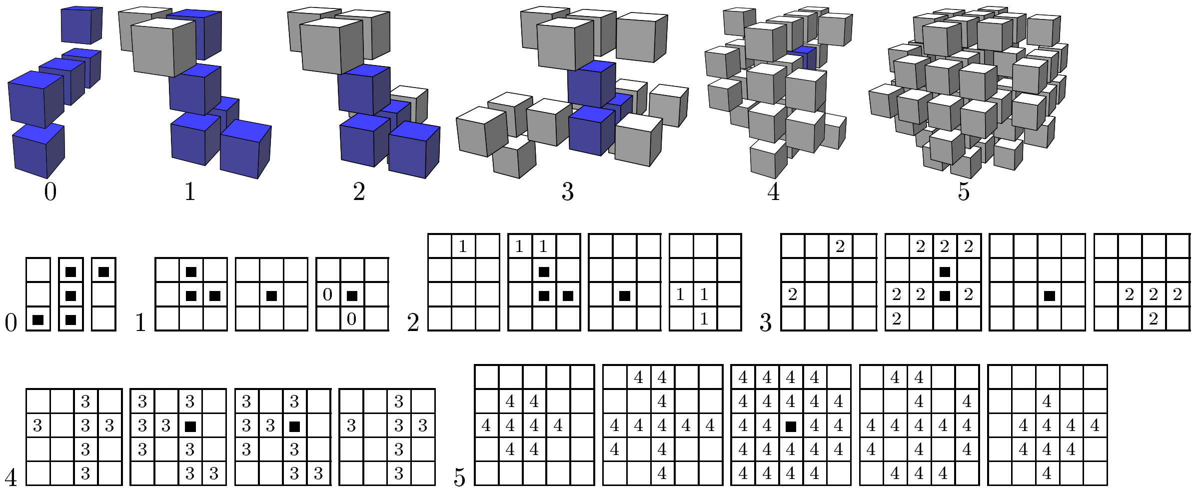


Strategy for Polyomino 30





Strategy for Polyomino 31



Strategy for Polyomino 32