# RECTANGULAR POLYOMINO SET WEAK (1,2)-ACHIEVEMENT GAMES

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Abstract. In a polyomino set (1,2)-achievement game the maker and the breaker alternately mark one and two previously unmarked cells respectively. The maker's goal is to mark a set of cells congruent to one of a given set of polyominoes. The breaker tries to prevent the maker from achieving his goal. The families of polyominoes for which the maker has a winning strategy is determined up to size 4. In set achievement games, it is natural to study infinitely large polyominoes. This enables the construction of super winners that characterize all winning families up to a certain size.

## 1. Introduction

A *rectangular board* is the set of *cells* that are the translations of the unit square $[0,1] \times [0,1]$ by vectors of $\mathbf{Z}^2$. Informally, a rectangular board is the infinite chessboard. Two cells are called *adjacent* if they share a common edge. A *polyomino* (or animal) is a subset of the rectangular board in which the cells are connected through adjacent cells. Note that we allow infinitely many cells in a polyomino. We only consider polyominoes up to congruence, that is, the location of the polyomino on the board is not important. The number of cells of a polyomino is called the *size* of the polyomino.

In a *polyomino set* $(p,q)$*-achievement game* two players alternately mark $p$ and $q$ previously unmarked cells of the board using their own colors. If $p$ or $q$ is not 1 then the game is often called *biased*. The player who marks a polyomino congruent to one of a given set of finite polyominoes wins the game. In a *weak set achievement* game the second player (*the breaker*) only tries to prevent the first player (*the maker*) from achieving one of the polyominoes. A set of finite polyominoes is called a *winning set* if the maker has a winning strategy to achieve this set. Otherwise the set is called a *losing set*. Polyomino achievement games were introduced by Harary [6, 8, 7, 9]. Winning strategies on rectangular boards can be found in [3, 13]. Biased games are studied in [2] in a more general setting. Biased games are needed [11] to apply the theory of weight functions [1, 5] to unbiased games on infinite boards.

In this paper we study rectangular weak set $(1,2)$-achievement games. Triangular unbiased set achievement games were studied in [4]. Our purpose is to further develop the theory of set achievement games. We have chosen the rectangular game because the rectangular board is the most intuitive. The unbiased rectangular game is very complex. To handle this difficulty we have chosen a biased version to limit the number of winning sets. The $(1,2)$ game is still rich enough to uncover many of the unexpected properties of set games. This approach also has its challenges, since the $(1,2)$ game needs new tools for finding winning strategies.

## 2. Preliminaries

Figure 2.1 shows some polyominoes we are going to use. In this figure, the polyominoes are in standard position. Roughly speaking, a polyomino is in standard position if its cells are as much

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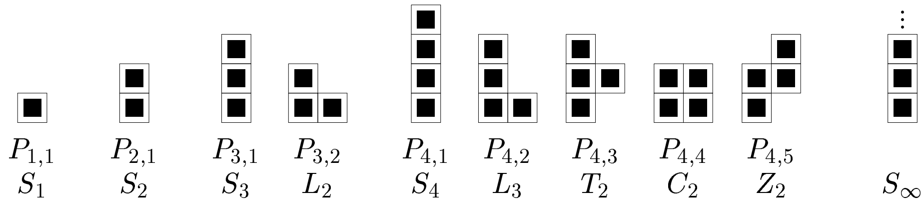



FIGURE 2.1. All polyominoes up to size 4 together with infinite skinny.

FIGURE 2.2. Two polyominoes which are ancestors of each other.

to the left and to the bottom as possible. The exact definition involves the lexicographic order of the list of coordinates of the cells of the polyomino pushed against the coordinate axes in the first quadrant. The naming convention comes from the ordering of the polyominoes by size and by lexicographic order of their standard position.

We use special names for several important class of polyominoes. These names are also given in the figure. $S_n = P_{n,1}$ stands for the skinny polyomino of size $n$. $C_n$, $L_n$, $T_n$ and $Z_n$ are chosen because the shape of those polyominoes is similar to the shape of letters. Note that only one end of $S_\infty$ is infinitely long.

**Definition 2.1.** A set of polyominoes is called *bounded* if it contains only finite polyominoes. It is called *unbounded* if it contains at least one infinite polyomino.

Note that an infinite set of finite polyominoes is still called bounded even though the size of a polyomino in the set can be arbitrarily large.

**Definition 2.2.** We say the polyomino $P$ is an *ancestor* of the polyomino $Q$ if $Q$ can be constructed from $P$ by adding some (possibly none) extra cells. We use the notation $P \sqsubseteq Q$. A set $\mathcal{F}$ of polyominoes is called a *family* if no element of $\mathcal{F}$ is the ancestor of another element of $\mathcal{F}$.

It is easy to see that the ancestor relation is reflexive and transitive. It is not antisymmetric, the polyominoes in Figure 2.2 are ancestors of each other. On the set of finite polyominoes the relation is antisymmetric and so is a partial order.

So far we have not defined the term winner for an unbounded set of polyominoes. An infinite polyomino cannot be marked during a finite game. We still want to talk about unbounded winners to simplify the theory, even though we do not intend to play any games with unbounded sets.

**Definition 2.3.** Let $\mathcal{T}$ be an unbounded set of polyominoes. Let $F_T$ be a finite ancestor of $T$ for all $T \in \mathcal{T}$. Then $\mathcal{F} = \{F_T \mid T \in \mathcal{T}\}$ is called a *bounded restriction* of $\mathcal{T}$. An unbounded set of polyominoes is called a *winner* if each bounded restriction of the set is a winner.

## 3. PREORDER

There are two ways to make it easier to achieve a set of polyominoes. We can make some of the polyominoes smaller or we can include more polyominoes in the set. This motivates the following definition.

**Definition 3.1.** Let $\mathcal{S}$ and $\mathcal{T}$ be sets of polyominoes. We say $\mathcal{S}$ is *simpler* than $\mathcal{T}$ if for all $Q \in \mathcal{T}$ there is a $P \in \mathcal{S}$ such that $P \sqsubseteq Q$. We use the notation $\mathcal{S} \preceq T$.

The terminology *at least* and *at most* was used in [4] for what we call simpler. Note that $\mathcal{S}$ is simpler then $\mathcal{T}$ if $\mathcal{S}$ is simpler to achieve than $\mathcal{T}$. It is easy to see that the simpler relation is reflexive and transitive and so is a preorder. It is also easy to see that a bounded restriction of an unbounded set of polyominoes is simpler than the original set. The following result shows the importance of the preorder.

**Proposition 3.2.** *Let $\mathcal{S}$ and $\mathcal{T}$ be sets of polyominoes such that $\mathcal{S} \preceq \mathcal{T}$. If $\mathcal{T}$ is a winner then so is $\mathcal{S}$. If $\mathcal{S}$ is a loser then so is $\mathcal{T}$.*

*Proof.* First assume that $\mathcal{S}$ and $\mathcal{T}$ are bounded. If $\mathcal{T}$ is a winner then during a game the maker is able to mark the cells of some $Q \in \mathcal{T}$. There is a $P \in \mathcal{S}$ such that $P \sqsubseteq Q$, so by the time the maker marks the cells of $Q$ he also marked the cells of $P$, possibly at an earlier stage.

Next assume that $\mathcal{S}$ is bounded and $\mathcal{T}$ is unbounded. For each $T \in \mathcal{T}$ define $F_T = T$ if $T$ is finite and define $F_T$ to be an element of $\mathcal{S}$ such that $F_T \sqsubseteq T$ if $T$ is infinite. Then $\mathcal{F} = \{F_T \mid T \in \mathcal{T}\}$ is a bounded restriction of $\mathcal{T}$. $\mathcal{S}$ is simpler than $\mathcal{F}$ and $\mathcal{F}$ is a winner and so $\mathcal{S}$ is also a winner.

Finally assume that $\mathcal{S}$ is unbounded. Let $\mathcal{E}$ be a bounded restriction of $\mathcal{S}$. Then $\mathcal{E} \preceq \mathcal{S} \preceq \mathcal{T}$ and so $\mathcal{E}$ is a winner which implies that $\mathcal{S}$ is a winner.

The second statement of the proposition is the contrapositive of the first statement. ☐

**Definition 3.3.** Let $\mathcal{S}$ be a bounded set of polyominoes. The set $\mathcal{L}(\mathcal{S})$ of minimal elements of $\mathcal{S}$ in the partial order is called the *legalization* of $\mathcal{S}$.

It is clear that $\mathcal{L}(\mathcal{S})$ is a family.

**Proposition 3.4.** *Let $\mathcal{S}$ be a bounded set of polyominoes. $\mathcal{S}$ is a winner if and only if $\mathcal{L}(\mathcal{S})$ is a winner.*

*Proof.* Since $\mathcal{L}(\mathcal{S})$ is a subset of $\mathcal{S}$, we must have $\mathcal{S} \preceq \mathcal{L}(\mathcal{S})$. On the other hand, consider $Q \in \mathcal{S}$. If $Q$ is minimal then $Q \in \mathcal{L}(\mathcal{S})$. If $Q$ is not minimal then there is a minimal $R \in \mathcal{S}$ such that $R \sqsubseteq Q$ and so $R \in \mathcal{L}(\mathcal{S})$. This shows that $\mathcal{S} \succeq \mathcal{L}(\mathcal{S})$. The result now follows from Proposition 3.2. ☐

Note that the existence of the minimal $R$ in the proof is not guaranteed if $\mathcal{S}$ is unbounded. There could be an infinite chain $Q_1 \sqsupseteq Q_2 \sqsupseteq \cdots$ of simpler and simpler polyominoes without a minimal polyomino. This means that we cannot talk about the legalization of an unbounded set of polyominoes.

Proposition 3.4 allows us to concentrate on families instead of sets of polyominoes in order to classify sets of finite polyominoes as winners or losers.

## 4. Winning families of all sizes

The *exterior perimeter* of a polyomino is the number of empty cells adjacent to the polyomino. The minimum exterior perimeter of the polyominoes in a finite set $\mathcal{F}$ is denoted by $\varepsilon(\mathcal{F})$. The *full family* $\mathcal{F}_s$ is the set containing all polyominoes of size $s$.

**Proposition 4.1.** *The full family $\mathcal{F}_s$ is a winner for $s \leq 4$. In fact the maker can win after $s$ marks.*

*Proof.* The maker can win after $s$ marks with the *random neighbor strategy* [10], which requires him to place his mark at a randomly chosen cell adjacent to one of his previous marks. The strategy works because $\varepsilon(\mathcal{F}_1) = 4$, $\varepsilon(\mathcal{F}_2) = 6$, $\varepsilon(\mathcal{F}_3) = 7$ and $\varepsilon(\mathcal{F}_4) = 8$ and so $\varepsilon(\mathcal{F}_s)$ is not larger than the number of cells marked by the breaker, which is $2s$ after $s$ moves. ☐

It is not hard to see that $\mathcal{F}_4$ remains a winner if we replace $\mathcal{S}_4$ by a larger skinny polyomino.

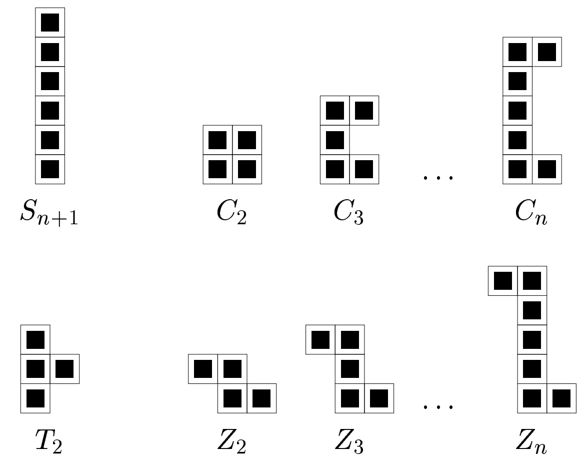



FIGURE 4.1. The winner $\mathcal{W}_n$.

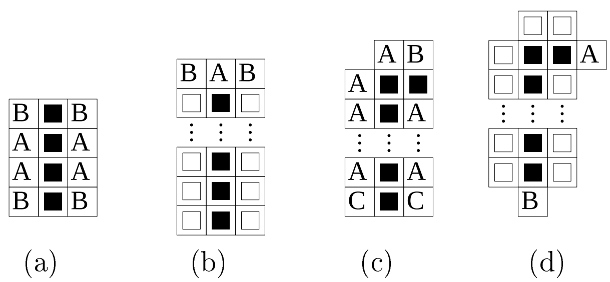



FIGURE 4.2. Situations to achieve $\mathcal{W}_n$.

**Proposition 4.2.** *The family* $\mathcal{W}_n = \{S_{n+1}, T_2, C_2, \ldots, C_n, Z_2, \ldots, Z_n\}$ *is a winner for all* $n \geq 3$.

*Proof.* Figure 4.1 shows the polyominoes in $\mathcal{W}_n$. The maker can mark one of the polyominoes in $\mathcal{F}_4 = \{S_4, L_3, T_2, C_2, Z_2\}$ after four marks by Proposition 4.1. If this polyomino is $T_2$, $C_2$ or $Z_2$ then the maker achieved $\mathcal{W}_n$ and we are done.

First consider the case when the marked polyomino is $S_4$. We show by induction that even in this case the maker is able to achieve $S_{n+1}$ and win or achieve $L_k$ for some $4 \leq k \leq n$. Consider Figure 4.2(a) that shows the situation before the fifth move of the maker. If the breaker has no marks in the cells containing the letter A, then the maker can mark one of those cells and achieve $T_2$. If the breaker has no marks in the cells containing the letter B then the maker can mark one of those cells and achieve $L_4$. So we can assume that the eight marks of the breaker are the cells with the letters A and B. This completes the base step of the induction. Now assume that we are in the situation shown in Figure 4.2(b) where the the maker already marked $S_{j-1}$ and the small empty squares show the marks of the breaker. The maker now can mark the cell containing the letter A. If the breaker does not answer by marking the two cells containing the letter B then the maker can mark one of these cells and achieve $L_j$. On the other hand if the breaker marks these two cells then we are again in the situation shown in Figure 4.2(b) but the size of the polyomino $S_j$ marked by the maker is increased by one. Hence the maker eventually achieves $S_{n+1}$ or $L_k$.

It suffices to consider the situation shown in Figure 4.2(c) where the maker marked $L_k$ after $k+1$ marks. If the breaker has no marks in the cells containing the letter A, then the maker can mark one of those cells and achieve $T_2$. If the breaker has no mark in the cell containing the letter B, then

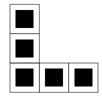


FIGURE 4.3. $P_{5,6}$.

the maker can mark that cell and achieve $Z_2$. If the breaker has no marks in the cells containing the letter C, then the maker can mark one of those cells and achieve $C_k$ or $Z_k$. So we can assume that we are in the situation shown in Figure 4.2(d). Note that the breaker can have $2k+2$ marks on the board while only $2k+1$ of those marks are shown as forced moves. Without this extra mark, the maker would have two ways to finish the game. He could mark the cell containing the letter A and mark cells to the right of his previous mark until he can make a turn up or down. He could also mark the cell containing the letter B and mark cells below his previous mark until he make a turn left or right. An inductive argument similar to the one above shows that either way he can achieve $S_{n+1}$ without a turn or he can achieve $C_j$ or $Z_j$ for some $3 \leq j \leq n$. The one extra mark of the breaker cannot ruin both of these ways to win since the cells involved are disjoint. □

**Corollary 4.3.** *The unbounded family*

$$\mathcal{W} = \{S_\infty, T_2\} \cup \{C_n \mid n \geq 2\} \cup \{Z_n \mid n \geq 2\}$$

*is a winner.*

*Proof.* The bounded restrictions of $\mathcal{W}$ are all simpler than $\mathcal{W}_n$ for some $n$. □

**Corollary 4.4.** *The families* $\{P_{2,1}\}$, $\{P_{n,1}, P_{3,2}\}$ *for* $n \geq 3$ *and* $\{P_{3,1}, P_{4,4}, P_{4,5}\}$ *are winners.*

*Proof.* The first and the third family is simpler than $\mathcal{W}_3$. The second family is simpler than $\mathcal{W}_{n-1}$. □

Note that $\mathcal{W}_2$ is not a family but $\mathcal{L}(\mathcal{W}_2) = \{S_3, C_2, Z_2\} = \{P_{3,1}, P_{4,4}, P_{4,5}\}$ is a winning family and so $\mathcal{W}_2$ is a winning set.

**Corollary 4.5.** *There is a winning family of size* $s$ *for all* $s \in \mathbf{N}$ *except for* $s = 4$.

*Proof.* The families in Corollary 4.4 are of size 1, 2 and 3. The family in Proposition 4.1 has size 5. It is clear that $\mathcal{W}'_n = \mathcal{W}_n \cup \{P_{5,6}\}$ is a family for $n \geq 3$ (see Figure 4.3). $\mathcal{W}'_n$ is a winner since it is simpler than $\mathcal{W}_n$. Since $|\mathcal{W}_n| = 2n$ and $|\mathcal{W}'_n| = 2n+1$, we have a winning family of size $s$ for all $s \geq 6$. □

## 5. Losing families

**Definition 5.1.** A 2-*paving* of the board is an irreflexive relation on the set of cells where each cell is related to at most two other cells.

**Example 5.2.** Figure 5.1 visualizes some 2-pavings. Related cells are connected by a tile. The dark cells show a fundamental set of tiles. All the tiles are translations of the dark tiles by a linear combination of the two given vectors with integer coefficients. A 2-paving determines the following strategy for the breaker. In each turn, the breaker marks the unmarked cells related to the cell last marked by the maker. If there are fewer than two such cells then she uses her remaining marks randomly.

**Definition 5.3.** The strategy described above is called the *paving strategy* based on a 2-paving.

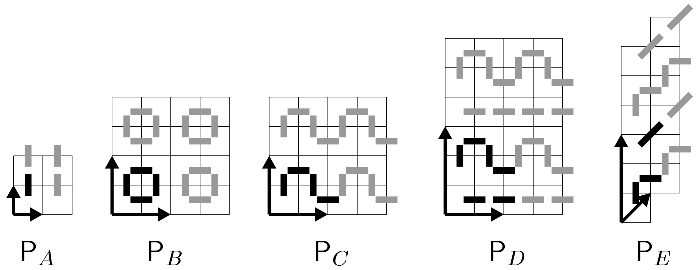



FIGURE 5.1. 2-pavings. Each picture shows four copies of the fundamental set of tiles.

| | $\mathsf{P}_A$ | $\mathsf{P}_B$ | $\mathsf{P}_C$ | $\mathsf{P}_D$ | $\mathsf{P}_E$ |
|---|---|---|---|---|---|
| $S_3$ | | • | • | | |
| $L_2$ | • | | | | |
| $S_4$ | | • | • | • | • |
| $L_3$ | • | • | • | | • |
| $T_2$ | • | • | • | • | |
| $C_2$ | • | | • | • | • |
| $Z_2$ | • | • | | • | • |

FIGURE 5.2. Polyominoes and their killer 2-pavings. $S_1$ and $S_2$ are not listed since those polyominoes are winners.

**Proposition 5.4.** *If the breaker follows the paving strategy then the maker cannot mark two related cells during a game.*

*Proof.* Suppose that it is the maker's turn and there is an empty cell $c$ related to the cell $d$ marked by the maker. But then cell $c$ was empty after the maker marked cell $d$. So the breaker should have been able to use one of her two marks on cell $c$ since cell $d$ is not related to more than two other cells. This is a contradiction. □

This result allows the breaker to win against certain sets of polyominoes.

**Definition 5.5.** If $\mathsf{P}$ is a 2-paving such that every placement of the polyomino $Q$ on the board contains a pair of related cells then we say that $Q$ is *killed* by $\mathsf{P}$. If every element of a set $\mathcal{S}$ of polyominoes is killed by a 2-paving $\mathsf{P}$ then we say that $\mathcal{S}$ is killed by $\mathsf{P}$.

Note that if $P \sqsubseteq Q$ and $P$ is killed by a 2-paving, then $Q$ is also killed by the same 2-paving. The following is an easy consequence of Proposition 5.4.

**Proposition 5.6.** *A set of polyominoes killed by a* 2*-paving is a losing set, the breaker can win with the paving strategy.*

**Example 5.7.** Figure 5.2 shows the polyominoes up to size 4 with their killer 2-pavings. The table helps decide if a family is a loser. For example $\{S_3, C_2\}$ is a loser because it is killed by $\mathsf{P}_C$.

It is easy but tedious to check that a given 2-paving in fact kills a polyomino. We used a computer program to verify our hand calculations.

We used another computer program to find useful killer 2-pavings. This program uses backtracking to pick more and more related cells to find a 2-paving that kills a set of polyominoes on a finite region of the board. The program places every polyomino inside the finite region in every position

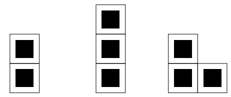



FIGURE 6.1. Characterizing families for size 1. Killer 2-pavings are listed for losing families.

that does not have a pair of related cells yet. If one of these placements does not have two cells that can be made related then the program backtracks. Otherwise the program picks the placement that has the least number of cells that can be made related and tries to consider every such pairing. The program stops if the set cannot be killed by a 2-paving or if a killer 2-paving is found. If a set cannot be killed by a 2-paving on a finite region then of course it cannot be killed on the infinite board either. In this case the set is called a *paving winner*. The 2-pavings found by the program are often chaotic close to the boundary of the finite region, but in most cases a pattern or sometimes several patterns can be discovered in some portion of a sufficiently large region.

**Proposition 5.8.** *There is a losing family of size $s$ for all $s \in \mathbf{N} \cup \{\infty\}$.*

*Proof.* The families $\{C_2, \ldots, C_{s+1}\}$ and $\{C_2, C_3, \ldots\}$ are killed by $\mathsf{P}_A$. □

**Proposition 5.9.** *If $\mathcal{F}$ is a winning family then $S_n \in \mathcal{F}$ for some $n$.*

*Proof.* If $S_n$ is not in $\mathcal{F}$ for any $n$ then $\{L_2\} \preceq \mathcal{F}$. Hence $\mathcal{F}$ is a loser since $L_2$ is killed by $\mathsf{P}_A$. □

**Proposition 5.10.** *A set $\mathcal{S}$ containing polyominoes of size 5 or larger is a loser.*

*Proof.* It is easy to see that $\mathcal{F} := \{S_3, Z_2\} \preceq \mathcal{S}$ and $\mathcal{F}$ is killed by $\mathsf{P}_B$. □

## 6. Classification of families

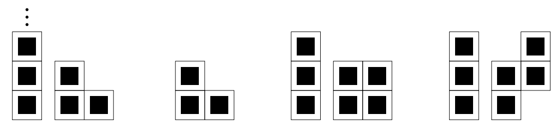
In this section we find all winning families up to size 4. For each such size $s$ we present a characterizing winning family $\mathcal{Y}_s$. Then we show that a family $\mathcal{F}$ of size $s$ is a winner if and only if it is simpler then $\mathcal{Y}_s$. To do this we use a characterizing collection $\mathcal{N}_{s,1}, \ldots, \mathcal{N}_{s,k_s}$ of losing families and we show that if $\mathcal{F}$ is not simpler than $\mathcal{Y}_s$ then there is a losing family in $\mathcal{N}_{s,i}$ that is simpler than $\mathcal{F}$. For size 4 families we do not have a characterizing winner since there are no size 4 winning families. These characterizing families are shown in Figures 6.1–6.6. Each $\mathcal{Y}_i$ is simpler than $\mathcal{W}$ of Corollary 4.3 and so a winner. To show that the characterizing losing families are in fact losers, we provide killer 2-pavings in the figures.

**Proposition 6.1.** $\mathcal{Y}_1 = \{S_2\}$, $\mathcal{N}_{1,1} = \{S_3\}$ *and* $\mathcal{N}_{1,2} = \{L_2\}$ *is a characterizing collection of winners and losers for size 1 families.*

*Proof.* By [12], the only size 1 winners are $\{S_1\}$ and $\{S_2\}$. Both of these are simpler than $\mathcal{Y}_1$. Every other polyomino $P$ has at least 3 cells and so either $S_3$ or $L_2$ must be simpler then $P$. □

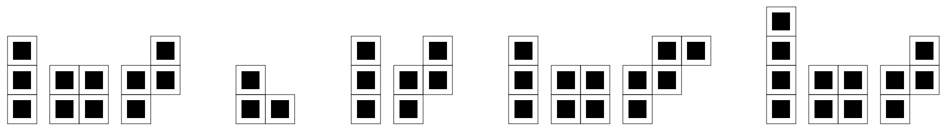
**Proposition 6.2.** $\mathcal{Y}_2 = \{S_\infty, L_2\}$, $\mathcal{N}_{2,1} = \{L_2\}$, $\mathcal{N}_{2,2} = \{S_3, C_2\}$ *and* $\mathcal{N}_{2,3} = \{S_3, Z_2\}$ *is a characterizing collection of winners and losers for size 2 families.*

*Proof.* Let $\mathcal{F}$ be a family of size 2. If $S_n$ is not in $\mathcal{F}$ then $\mathcal{N}_{2,1} \preceq \mathcal{F}$ by the proof of Proposition 5.9. So we can assume that $\mathcal{F} = \{S_n, Q\}$ for some $n \geq 3$. Note that if $n \leq 2$ then $\mathcal{F}$ cannot be a family.



$\mathcal{Y}_2$ $\{S_\infty, L_2\}$ | $\mathcal{N}_{2,1}$ $\{L_2\}$ $\mathsf{P}_A$ | $\mathcal{N}_{2,2}$ $\{S_3, C_2\}$ $\mathsf{P}_C$ | $\mathcal{N}_{2,3}$ $\{S_3, Z_2\}$ $\mathsf{P}_B$

FIGURE 6.2. Characterizing families for size 2. Killer 2-pavings are listed for losing families.



$\mathcal{Y}_3$ $\{S_3, C_2, Z_2\}$ | $\mathcal{N}_{3,1}$ $\{L_2\}$ $\mathsf{P}_A$ | $\mathcal{N}_{3,2}$ $\{S_3, Z_2\}$ $\mathsf{P}_B$ | $\mathcal{N}_{3,3}$ $\{S_3, C_2, P_{5,10}\}$ $\mathsf{P}_C$ | $\mathcal{N}_{3,4}$ $\{S_4, C_2, Z_2\}$ $\mathsf{P}_D$

FIGURE 6.3. Characterizing families for size 3. Killer 2-pavings are listed for losing families.

First assume that $|Q| \le 4$. Then $Q \in \{L_2, L_3, T_2, C_2, Z_2\}$ since $S_i$ is related to $S_n$. If $Q = L_2$ then $\mathcal{F} \preceq \mathcal{Y}$. If $Q \in \{L_3, T_2\}$ then $\mathcal{N}_{2,2}, \mathcal{N}_{2,3} \preceq \mathcal{F}$. If $Q = C_2$ then $\mathcal{N}_{2,2} \preceq \mathcal{F}$. If $Q = Z_2$ then $\mathcal{N}_{2,3} \preceq \mathcal{F}$.

Next assume that $|Q| \ge 5$. Then $Q$ is not skinny and so there is an $R \in \{L_2, L_3, T_2, C_2, Z_2\}$ such that $R \sqsubseteq Q$. Hence $\{S_n, R\} \preceq \mathcal{F}$ and so $\mathcal{F}$ is characterized since $\{S_n, R\}$ is characterized as we saw in the previous case. □

**Corollary 6.3.** *The only winning size* 2 *families are* $\{S_\infty, L_2\}$ *and* $\{S_n, L_2\}$ *for* $n \ge 3$.

**Proposition 6.4.** $\mathcal{Y}_3 = \{S_3, C_2, Z_2\}$, $\mathcal{N}_{3,1} = \{L_2\}$, $\mathcal{N}_{3,2} = \{S_3, Z_2\}$, $\mathcal{N}_{3,3} = \{S_3, C_2, P_{5,10}\}$ *and* $\mathcal{N}_{3,4} = \{S_4, C_2, Z_2\}$ *is a characterizing collection of winners and losers for size* 3 *families.*

*Proof.* Let $\mathcal{F}$ be a family of size 3. If $S_n$ is not in $\mathcal{F}$ then $\mathcal{N}_{3,1} \preceq \mathcal{F}$. So assume $\mathcal{F} = \{S_n, Q, R\}$ for some $n \ge 3$. We do not have $L_2 \in \mathcal{F}$ because every polyomino is related to $S_n$ or $L_2$. Thus $|Q|, |R| \ge 4$.

First consider the case when $|Q| = 4 = |R|$. Then $\{Q, R\} \subseteq \{L_3, T_2, C_2, Z_2\}$. If $\{Q, R\} = \{L_3, T_2\}$ then $\mathcal{N}_{3,2} \preceq \{S_3\} \preceq \mathcal{F}$. If $Q \in \{L_3, T_2\}$ and $R = C_2$ then $\mathcal{N}_{3,3} \preceq \{S_3, C_2\} \preceq \mathcal{F}$. If $Q \in \{L_3, T_2\}$ and $R = Z_2$ then $\mathcal{N}_{3,2} \preceq \mathcal{F}$. If $\{Q, R\} = \{C_2, Z_2\}$ then $n = 3$ implies $\mathcal{F} = \mathcal{Y}_3$ and $n \ge 4$ implies $\mathcal{N}_{3,4} \preceq \mathcal{F}$.

Next consider the case when $|Q| \ge 4$ and $|R| \ge 5$. Since $Q$ and $R$ are not skinny, there is an $\mathcal{S} \subseteq \{P_{4,2}, P_{4,3}, P_{4,4}, P_{4,5}\}$ with $|\mathcal{S}| \le 2$ such that $\mathcal{S} \preceq \{Q, R\}$. Then $\mathcal{E} := \mathcal{L}(\{S_n\} \cup \mathcal{S}) \preceq \{S_n\} \cup \mathcal{S} \preceq F$ and $1 \le |\mathcal{E}| \le 3$.

If $|\mathcal{E}| = 1$ then $\mathcal{N}_{3,2} \preceq \mathcal{E} \preceq \mathcal{F}$. If $|\mathcal{E}| = 2$ then $\mathcal{E}$ is a loser by Corollary 6.3, since $\mathcal{E}$ has a polyomino with size 4. Hence $\mathcal{N}_{2,1}$, $\mathcal{N}_{2,2}$ or $\mathcal{N}_{2,3}$ is simpler than $\mathcal{E}$. We have $\mathcal{N}_{3,1} = \mathcal{N}_{2,1}$, $\mathcal{N}_{3,3} \preceq \mathcal{N}_{2,2}$ and $\mathcal{N}_{3,2} = \mathcal{N}_{2,3}$ which implies $\mathcal{N}_{3,i} \preceq \mathcal{N}_{2,j} \preceq \mathcal{E} \preceq \mathcal{F}$ for some $i$ and $j$ as desired.

Assume $|\mathcal{E}| = 3$. If $\mathcal{E} \ne \mathcal{Y}_3$ then $\mathcal{N}_{3,i} \preceq \mathcal{E} \preceq \mathcal{F}$ for some $i$ by the first part of the proof. So it remains to consider the case when $\mathcal{E} = \mathcal{Y}_3$. Then we must have an ancestor $Q'$ of $Q$ and an ancestor $R'$ of $R$ such that $|Q'| = 4$ and $|R'| = 5$. Figure 6.4 shows the size 5 descendants of $C_2$ and $Z_2$. From this we can see that either we have $Q' = Z_2$ and $R' = P_{5,4}$ or we have $Q' = C_2$ and $R' \in \{P_{5,4}, P_{5,8}, P_{5,9}, P_{5,10}\}$. In the first case $\mathcal{N}_{3,2} \preceq \{S_n, Z_2, P_{5,4}\} \preceq \mathcal{F}$. In the second case one of

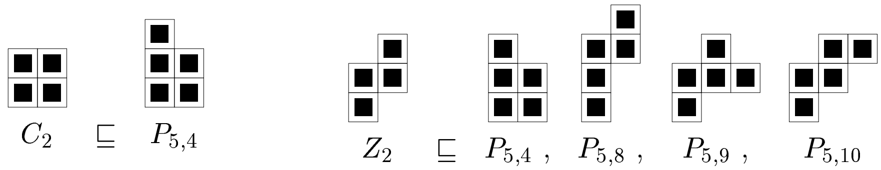



FIGURE 6.4. Descendants of $C_2$ and $Z_2$ with size 5 .

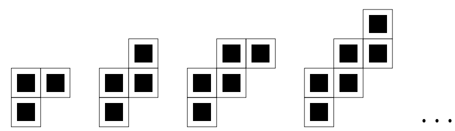



FIGURE 6.5. Squiggle polyominoes.

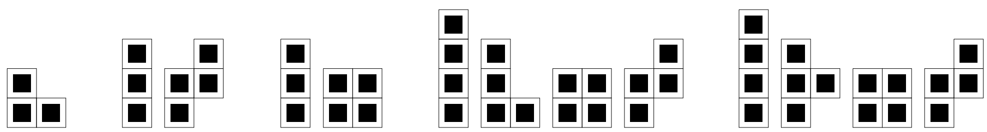



FIGURE 6.6. Characterizing families for size 4. Killer 2-pavings are listed for losing families. No winning family is required.

the following holds:

$$
\begin{aligned}
\mathcal{N}_{3,3}, \mathcal{N}_{3,4} &\preceq \{S_n, C_2, P_{5,4}\} \preceq \mathcal{F} \\
\mathcal{N}_{3,4} &\preceq \{S_n, C_2, P_{5,8}\} \preceq \mathcal{F} \qquad (n \geq 4 \text{ since } S_3 \sqsubseteq P_{5,8}) \\
\mathcal{N}_{3,4} &\preceq \{S_n, C_2, P_{5,9}\} \preceq \mathcal{F} \qquad (n \geq 4 \text{ since } S_3 \sqsubseteq P_{5,9}) \\
\mathcal{N}_{3,3} &\preceq \{S_n, C_2, P_{5,10}\} \preceq \mathcal{F}.
\end{aligned}
$$

□

We need a preliminary result before we can deal with size 4 families. The polyominoes shown in Figure 6.5 are called squiggle polyominoes.

**Proposition 6.5.** *A family $\mathcal{F}$ of size 4 or more does not have any polyominoes of size 3 or less.*

*Proof.* It is clear that the full family $\mathcal{F}_s$ cannot be extended to a larger family. Hence neither $S_1$ nor $S_2$ can be a member of $\mathcal{F}$. We cannot have both $S_3$ and $L_2$ in $\mathcal{F}$ either.

If $L_2 \in \mathcal{F}$ then all the other polyominoes in $\mathcal{F}$ must be skinny since the non-skinny polyominoes are related to $L_2$. Only one skinny polyomino is allowed so this limits the size of $\mathcal{F}$ to 2.

Suppose that $S_3 \in \mathcal{F}$. The only polyominoes not related to $S_3$ are $C_2$ and the squiggle polyominoes. Any two squiggle polyominoes are related so $\mathcal{F}$ cannot contain more than one. This limits the size of $\mathcal{F}$ to 3. □

**Proposition 6.6.** *There are no winning families with size 4. $\mathcal{N}_{4,1} = \{L_2\}$, $\mathcal{N}_{4,2} = \{S_2, Z_2\}$, $\mathcal{N}_{4,3} = \{S_2, C_2, P_{5,10}\}$, $\mathcal{N}_{4,4} = \{S_4, L_3, C_2, Z_2\}$ and $\mathcal{N}_{4,5} = \{S_4, T_2, C_2, Z_2\}$ is a characterizing collection of losers for size 4 families.*

*Proof.* Let $\mathcal{F}$ be a family of size 4. If $S_n$ is not in $\mathcal{F}$ then $\mathcal{N}_{3,1} \preceq \mathcal{F}$. So assume $\mathcal{F} = \{S_n, P, Q, R\}$. for some $n \geq 3$. By Proposition 6.5 we can assume that $n, |P|, |Q|, |R| \geq 4$. There is an $\mathcal{S} \subseteq \{P_{4,2}, \ldots, P_{4,5}\}$ with $|\mathcal{S}| \leq 3$ such that $\mathcal{S} \preceq \{P, Q, R\}$. Then $\mathcal{E} := \mathcal{L}(\{S_n\} \cup \mathcal{S}) \preceq \{S_n\} \cup \mathcal{S} \preceq \mathcal{F}$ and $1 \leq |\mathcal{E}| \leq 4$.

If $|\mathcal{E}| = 1$ then $\mathcal{N}_{4,2}, \mathcal{N}_{4,3} \preceq \mathcal{E} \preceq \mathcal{F}$. If $|\mathcal{E}| = 2$ then one of the following holds:

$$\mathcal{N}_{4,2}, \mathcal{N}_{4,3}, \mathcal{N}_{4,4} \preceq \{S_n, L_3\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,2}, \mathcal{N}_{4,3}, \mathcal{N}_{4,5} \preceq \{S_n, T_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,3}, \mathcal{N}_{4,4}, \mathcal{N}_{4,5} \preceq \{S_n, C_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,2}, \mathcal{N}_{4,4}, \mathcal{N}_{4,5} \preceq \{S_n, Z_2\} = \mathcal{E} \preceq \mathcal{F}.$$

If $|\mathcal{E}| = 3$ then one of the following

$$\mathcal{N}_{4,2}, \mathcal{N}_{4,3} \preceq \{S_n, L_3, P_{4,3}\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,3}, \mathcal{N}_{4,4} \preceq \{S_n, L_3, C_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,2}, \mathcal{N}_{4,4} \preceq \{S_n, L_3, Z_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,3}, \mathcal{N}_{4,5} \preceq \{S_n, T_2, C_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,2}, \mathcal{N}_{4,5} \preceq \{S_n, T_2, Z_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,4}, \mathcal{N}_{4,5} \preceq \{S_n, C_2, Z_2\} = \mathcal{E} \preceq \mathcal{F}$$

holds. Finally if $|\mathcal{E}| = 4$ then one of the following holds:

$$\mathcal{N}_{4,3} \preceq \{S_n, L_3, T_2, C_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,2} \preceq \{S_n, L_3, T_2, Z_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,4} \preceq \{S_n, L_3, C_2, Z_2\} = \mathcal{E} \preceq \mathcal{F}$$
$$\mathcal{N}_{4,5} \preceq \{S_n, T_2, C_2, Z_2\} = \mathcal{E} \preceq \mathcal{F}.$$

□

**Definition 6.7.** A family $\mathcal{Y}$ of polyominoes is called an *$n$-super winner* if each winning family with size at most $n$ is simpler than $\mathcal{Y}$.

**Example 6.8.** $\mathcal{Y}_s$ is an $s$-super winner for $s \in \{1, 2\}$. $\mathcal{W}$ in Corollary 4.3 is a 4-super winner.

The main result of our paper is the following.

**Theorem 6.9.** *A family of polyominoes containing fewer than* 5 *polyominoes is a winner if and only if it is simpler than* $\mathcal{W}$.

## 7. Further questions

There are several questions to be answered about set games.

(1) The families $\mathcal{Y}_2$ and $\mathcal{W}$ are infinite winners. Both of these are unbounded. Is there an infinite winning family that is bounded?
(2) Even though there are no winning families with size 4, we could say that $\mathcal{Y}_4 = \{S_1\}$ is a characterizing winner for size 4 families. So there is a characterizing winning family for sizes from 1 to 4. Is there a characterizing winner for each size?
(3) Is there an $s$-super winner for each $s$? Is there a super winner that is $s$-super for each $s$?
(4) Is there a useful notion of a super loser?
(5) Are there any characterizing or super winners in the unbiased or differently biased set games played on triangular, hexagonal and higher dimensional rectangular boards?

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