A New Result Regarding Hexagons in Pascal's Triangle

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In the literature associalted with Pascal's triangle and binomial coefficients there have been numerous papers concerned with proofs regarding regular hexagons with two and four points on a side. However, there have be relatively few regarding regular hexagons with an odd number of points per side and none involving hexagons of arbitrary size. The results of this paper, while restricted to a special case, are the first to give results for hexagons of arbitrarily large size.

The original theorem in this area was the Star of David Theorem discovered by Gould and proved in several papers. This concerns hexagons made up of six points around a point $\binom{n}{r}$ in the interior of Pascal's triangle, and typically would be

$$\begin{array}{cccc} & \binom{n-1}{r-1} \quad \binom{n-1}{r} & \\ \binom{n}{r-1} & & \binom{n}{r+1} \\ & \binom{n+1}{r} \quad \binom{n+1}{r+1} & \end{array}$$

If we index from the upper left corner of the hexagon then the points a_1, a_3 and a_5 with odd indices would be

$$a_1 = \binom{n-1}{r-1}, \quad a_3 = \binom{n}{r+1}, \quad a_5 = \binom{n+1}{r}$$

and the points a_2, a_4 and a_6 with even indices would be

$$a_2 = \binom{n-1}{r}, \quad a_4 = \binom{n+1}{r+1}, \quad a_6 = \binom{n}{r-1}.$$

The Star of David Theorem asserts that the greatest common divisor of these two sets is the same:

$$\mathrm{GCD}(a_1, a_3, a_5) = \mathrm{GCD}(a_2, a_4, a_6).$$

Although there are only proofs of theorems analogous to this for regular hexagons with four points on a side, numerical analysis suggests that it is true for regular hexagons of any size as long as they have an even number of points on a side. The same analytical techniques show it is not true for regular hexagons with an odd number of points on a side, but reveal interesting characteristics of these hexagons. With the aid of a modicum of new notation we can explain how these results arise and prove our results in a clear and general manner.

Theorem: if $p = 6k+1$ is prime, and a regular hexagon with $s = 4k+1$ points on a side is situated such that in the binomial coefficient $\binom{n}{r}$ positioned in the middle of the top side we have $n \equiv r \equiv 0 \pmod{p}$, then $\nu_p(\mathbf{GCDeven}) = \nu_p(\mathbf{GCDodd}) + 1$

We use the notation $\nu_p(a) = \alpha$ if p^α divides a but $p^{\alpha+1}$ does not divide a. We index the points, clockwise from the upper left, a_1, a_2, $a_3, \cdots, a_{24k}$, use GCDodd for the greatest common divisor of $\{a_1,\, a_3,\, a_5, \cdots, a_{24k-1}\}$ and GCDeven for the greatest common divisor of $\{a_2,\, a_4,\, a_6, \cdots, a_{24k}\}$.

As a first example consider a regular hexagon with five points on a side. Now $5 = 4{\cdot}1{+}1$, and hence $k = 1$ and $p = 6 \cdot 1 + 1 = 7$ is prime. (Note that in this case $4k + 1 = 5$ happens to be prime, but $4k+1$ is not necessarily prime in general.) If the binomial coefficient at the top middle is $\binom{35}{21}$ then the odd indexed points (written with this point as the first) are:

$$\binom{35}{21}\ \binom{35}{23}\ \binom{37}{25}\ \binom{39}{27}\binom{41}{27}\ \binom{43}{27}\ \binom{43}{25}\ \binom{43}{23}\binom{41}{21}\ \binom{39}{19}\ \binom{37}{19}\ \binom{35}{19}$$

The even indexed points, with the point to the right of the middle point listed first, are

$$\binom{35}{22}\ \binom{36}{24}\ \binom{38}{26}\ \binom{40}{27}\binom{42}{27}\ \binom{43}{26}\ \binom{43}{24}\ \binom{42}{22}\binom{40}{20}\ \binom{38}{19}\ \binom{36}{19}\ \binom{35}{20}$$

A result of Glaisher (see [Singmaster 73], p xx) states that the highest power of a particular prime p which will divide a binomial coefficient is the number of borrows required in the p-ary subtraction $n-r$, where $\binom{n}{r}$ is the binomial coefficient in question. In this case we seek the number of borrows in the 7-ary sutbraction, and to illustrate these values we present the hexagon with points written in 7-ary digital notation.

$$\begin{array}{ccccccccccccc}
 & & & & \binom{50}{25} & \binom{50}{26} & \binom{50}{30} & \binom{50}{31} & \binom{50}{32} & & & & \\
 & & & \binom{51}{25} & & & & & & \binom{51}{33} & & & \\
 & & \binom{52}{25} & & & & & & & & \binom{52}{34} & & \\
 & \binom{53}{25} & & & & & & & & & & \binom{53}{35} & \\
\binom{54}{25} & & & & & & & & & & & & \binom{54}{36} \\
 & \binom{55}{26} & & & & & & & & & & \binom{55}{36} & \\
 & & \binom{56}{30} & & & & & & & & \binom{56}{36} & & \\
 & & & \binom{60}{31} & & & & & & \binom{60}{36} & & & \\
 & & & & \binom{61}{32} & \binom{61}{33} & \binom{61}{34} & \binom{61}{35} & \binom{61}{36} & & & &
\end{array}$$

The **odd indexed points** starting at the top center will be, (in 7-ary notation)

$$\binom{50}{30}\ \binom{50}{32}\ \binom{52}{34}\ \binom{54}{36}\ \binom{56}{36}\ \binom{61}{36}\ \binom{61}{34}\ \binom{61}{32}\ \binom{56}{30}\ \binom{54}{25}\ \binom{52}{25}\ \binom{50}{25}$$

Note that it will be necessary to make one borrow in each of the subtractions $n-r$ with the exception of $\binom{50}{30}$ $\binom{56}{36}$ and $\binom{56}{30}$, each of which require no borrows. Thus 7 divides all but the binomial coefficients in this sequence except for these three elements. On the other hand the **even indexed points** (in 7-ary notation) will be

$$\binom{50}{31}\ \binom{50}{33}\ \binom{53}{35}\ \binom{55}{36}\ \binom{60}{36}\ \binom{61}{35}\ \binom{61}{33}\ \binom{62}{31}\ \binom{55}{26}\ \binom{53}{25}\ \binom{51}{25}\ \binom{50}{26}$$

Note that the 7-ary subtractions in this sequence all require one borrow, and hence 7 divides all of these binomial coefficients. Hence, as the theorem predicts, $\nu_7(\mathrm{GCDeven}) = \nu_7(\mathrm{GCDodd}) + 1$.

To prove that this is the case for all hexagons with 5 points on a side situated so that the top middle point $\binom{n}{r}$ where $n \equiv r \equiv 0 \pmod 7$, consider this binomial coefficient written in 7-ary notation which we represent with $\binom{\mu 0}{\sigma 0}$, where the μ represents all the 7-ary digits to the left of the last digit in n and similarly with σ and r. Thus, written out, we have

$$n = \mu 0 = 0 + \mu_1 \cdot 7 + \mu_2 \cdot 7^2 + \mu_3 \cdot 7^3 + \mu_4 \cdot 7^4 + \mu_5 \cdot 7^5 + \ldots$$

and similarly with r and σ. For the purpose of counting borrows, we may consider the μ and σ to be 7-ary numbers themselves, i.e.

$$\begin{aligned}\mu &= \mu_1 \cdot 1 + \mu_2 \cdot 7 + \mu_3 \cdot 7^2 + \ldots \\ \sigma &= \sigma_1 \cdot 1 + \sigma_2 \cdot 7 + \sigma_3 \cdot 7^2 + \ldots\end{aligned}$$

A borrow from the first column of μ in the 7-ary subtraction $n-r$ corresponds to a reduction by one of the 7-ary number μ. Defining $\dot{\mu} = \mu + 1$ and $\bar{\sigma} = \sigma - 1$, we can then describe an arbitrary regular 5 points/side hexagon with top center point in 7-ary notation by $\binom{\mu 0}{\sigma 0}$.

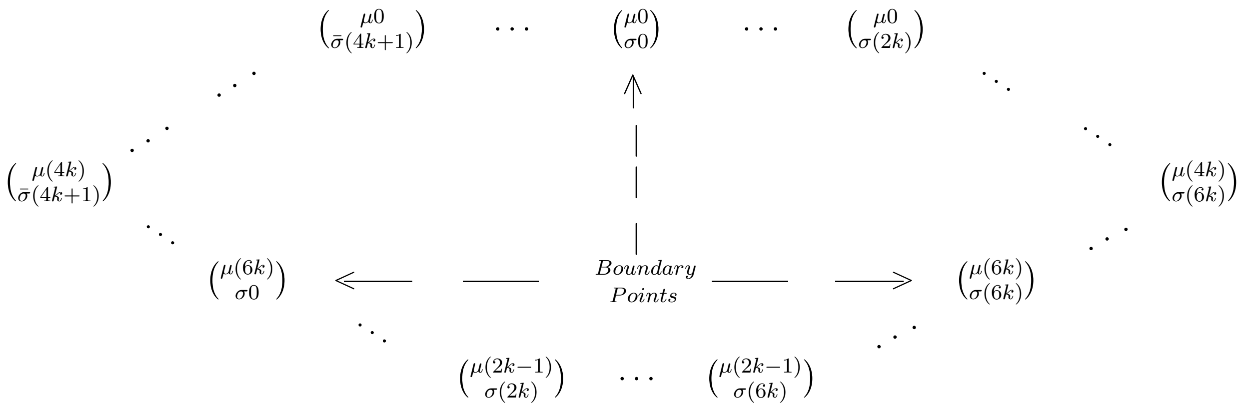
$$\begin{array}{ccccccccccccc}
 & & & & \binom{\mu 0}{\bar\sigma 5} & \binom{\mu 0}{\bar\sigma 6} & \binom{\mu 0}{\sigma 0} & \binom{\mu 0}{\sigma 1} & \binom{\mu 0}{\sigma 2} & & & & \\
 & & & \binom{\mu 1}{\bar\sigma 5} & & & \uparrow & & & \binom{\mu 1}{\sigma 3} & & & \\
 & & \binom{\mu 2}{\bar\sigma 5} & & & & & & & & \binom{\mu 2}{\sigma 4} & & \\
 & \binom{\mu 3}{\bar\sigma 5} & & & & & & & & & & \binom{\mu 3}{\sigma 5} & \\
\binom{\mu 4}{\bar\sigma 5} & & & & & & & & & & & & \binom{\mu 4}{\sigma 6} \\
 & \binom{\mu 5}{\bar\sigma 6} & & & & & & & & & & \binom{\mu 5}{\sigma 6} & \\
 & & -> \binom{\mu 6}{\sigma 0} & & & & & & & & \binom{\mu 6}{\sigma 6} <- & & \\
 & & & \binom{\dot\mu 0}{\sigma 1} & & & & & & \binom{\dot\mu 0}{\sigma 6} & & & \\
 & & & & \binom{\dot\mu 1}{\sigma 2} & \binom{\dot\mu 1}{\sigma 3} & \binom{\dot\mu 1}{\sigma 4} & \binom{\dot\mu 1}{\sigma 5} & \binom{\dot\mu 1}{\sigma 6} & & & &
\end{array}$$

Note that the only binomial coefficients for which the value in the ones column in n is greater than or equal to that in r occur as the middle points of the top, lower left, and lower right sides of the hexagon. These *boundary points* may be written, with $w \geq x$, as $\binom{\mu w}{\sigma x}$. These boundary points separate the hexagon into three distinct classes of points of the three forms, with $y > x$:

$$\binom{\mu x}{\sigma y}, \binom{\dot\mu x}{\sigma y} \text{ and } \binom{\mu x}{\bar\sigma y}.$$

For the three boundary points the number of borrows (and thus the highest power of seven which divides the binomial coefficient) is determined by the subtraction $\mu - \sigma$, since the ones column does not require a borrow. The other three borrow classes all require one borrow for the ones column. After performing this borrow the total number of borrows required will be the number required for $\bar\mu - \sigma + 1$ in the case of $\binom{\mu x}{\sigma y}$, $\mu - \sigma + 1$ for $\binom{\dot\mu x}{\sigma y}$, or $\bar\mu - \bar\sigma + 1$ for $\binom{\mu x}{\bar\sigma y}$. Depending on the values of μ and σ at least one of these three *borrow classes* requires exactly *one* more borrow than $\mu - \sigma$, but all require at least one more borrow. They are thus divisible by at least one higher power of 7 than the three boundary points, while one of these three borrow classes is divisible by exactly one higher power of seven. Since both the odd and the even indexed points contain example of the three outer borrow classes, while only the odd indexed points contain the boundary points, we have $\nu_7(\mathrm{GCDeven}) = \nu_7(\mathrm{GCDodd}) + 1$.

The proof for the general case, i.e. a hexagon with $s = 4k + 1$ points on a side and $p = 6k + 1$ a prime, is strictly analogous. The three boundary points are $\binom{\mu 0}{\sigma 0}$, $\binom{\mu(6k)}{\sigma 0}$, and $\binom{\mu(6k)}{\sigma(6k)}$. Note that here $(6k)$ represents a single digit in th p-ary expansion of n and r. All points are written in p-ary notation, with $p = 6k + 1$).



Thus the value $\sigma(6k)$ conveys that the ones column in r is $6k$, and one less than p. Since the number of points per side is odd, all corner points are odd–indexed. the boundary points are likewise odd–indexed, being an even number of points away from the corner points. Points beyond the three boundary points fall into one of the same three borrow classes as in the 5 points per side hexagon, but written in p-ary notation rather than 7-ary, and it follows that $\nu_p(\text{GCDodd}) = \nu_p(\text{GCDodd}) + 1$.

References

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