Connectivity of Knight's Graphs on Saw Shaped Boards

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Abstract

This paper discusses knights traveling on saw boards. Minimal boards are found.

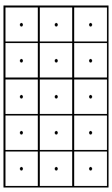
1. Introduction

The research for this paper was done at Northern Arizona University during the Research Experience for Undergraduates program under the supervision of Nándor Sieben. The author would like to thank Stephen Wilson for his ideas that helped in the process of writing this paper. In [RW], Rhodes and Wilson found that there is one minimal board on which a knight can travel between any two positions on a regular board. In this paper we look at a different type of board called a saw board. We want to prove similar things for the saw boards.

2. Boards

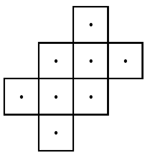
Definition 2.1. An $[x, y]$ *regular board* with $x \leq y$ is a rectangular board that is x squares wide and y squares high.

Example 2.2. A $[3, 5]$ regular board. The dots represent the positions on the board.



Definition 2.3. A $[u, v]$ *saw board* with $u \leq v$ is a rectangular board with diagonal edges that has u squares on the shorter edge and v squares on the longer edge.

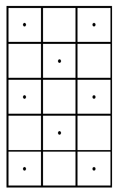
Example 2.4. A $[2, 3]$ saw board.



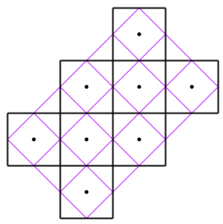
REU project at NAU supported by NSF

Definition 2.5. An $[x, y]$ *checker board* is a regular board where the only squares considered have coordinates (a, b) with a and b having the same parity.

Example 2.6. A $[3, 5]$ checker board.



A $[2u - 1, 2v - 1]$ rectangular board can be rotated $45°$ clockwise and placed on top of an $[u, v]$ saw board. The checker board positions line up with the saw board positions.



Definition 2.7. The *checker board corresponding* to a $[u, v]$ saw board is an $[x, y]$ board where $x = 2u - 1$ and $y = 2v - 1$.

Example 2.8. The checker board in Example 2.6 corresponds to the saw board in Example 2.4.

Note that the dimensions of the corresponding saw board are always odd, so we are only considering saw boards with odd dimensions.

Definition 2.9. We say that a $[c, d]$ board is *larger* than an $[x, y]$ board if $c \geq x$ and $d \geq y$. We say that a $[c, d]$ board is *strictly larger* than an $[x, y]$ board if the $[c, d]$ board is larger than the $[x, y]$ board but they are not the same. We also use the terms *smaller* and *strictly smaller* with the expected meaning.

3. KNIGHTS

In the game of chess, a knight can move one space horizontally or vertically and then two spaces in a perpendicular direction. We generalize the knights to include knights that go p spaces horizontally or vertically and then q spaces in a perpendicular direction.

Definition 3.1. If the position of a knight on a board is (a, b), then an $\langle m, n\rangle$ *move* changes the position into $(a + m, b + n)$.

Example 3.2. A $\langle 2,4\rangle$ move changes position $(1,2)$ to $(1+2,2+4)=(3,6)$.

Definition 3.3. A $\{p,q\}$ *knight* where $p<q$ is a piece that has eight possible moves: $\langle \pm p, \pm q\rangle$ and $\langle \pm q, \pm p\rangle$.

In the game of chess, we use a $\{1,2\}$ knight.

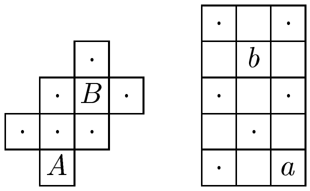
Definition 3.4. The *checker knight corresponding* to an $\{r,s\}$ saw knight is a $\{p,q\}$ knight with $p=s-r$ and $q=r+s$.

Definition 3.5. A *journey* of a knight is a collection of moves that a knight uses to travel from one position to another. It is denoted by $\{\langle a_i,b_i\rangle : 1\le i\le k\}$ where k is the total number of moves.

Proposition 3.6. *A $\{p,q\}$ checker knight corresponding to an $\{r,s\}$ saw knight travels to positions on an $[x,y]$ checker board that correspond to positions on the corresponding $[u,v]$ saw board.*

Proof. It is clear to see that an $[x,y]$ checker board, when rotated 90° clockwise and placed on top of a $[u,v]$ saw board, covers exactly the same positions.

Example 3.7. A $\{1,2\}$ knight starting at position A on the saw board below moves to the right 1 cell and up 2 cells to position B. The corresponding checker knight on the checker board is a $\{1,3\}$ knight. Starting at position a it moves 1 cell to the left and 3 cells up to position b.



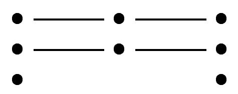
Definition 3.8. A *knight graph* is made up of vertices which are the cells of the board. Two vertices are connected with an edge exactly when the knight can move between these cells in a single move. An $\{r,s\}$ *knight graph* on a $[u,v]$ saw board will be denoted by $K^s(r,s,u,v)$. A $\{p,q\}$ *knight graph* on a $[x,y]$ checker board will be denoted $K^c(p,q,x,y)$.

Definition 3.9. A knight graph $K^s(r,s,u,v)$ *corresponds* to a knight graph $K^c(p,q,x,y)$ if $p=s-r$, $q=s+r$, $x=2u-1$, and $y=2v-1$.

The following is a consequence of Conjecture 3.6.

Corollary 3.10. *Corresponding knight graphs are isomorphic.*

Example 3.11. The knight graphs $K^s(1,2,2,3) and K^c(1,3,3,5)$ of Example 3.7 are isomorphic.



We only consider those knights which can travel between any two positions on an infinite board. The infinite saw board coincides with the infinite regular board. We know from [RW] that an $\{r, s\}$ knight can travel between any two positions on the infinite board if r and s are relatively prime and of different parity. We call such a knight an *allowed* knight. We want to find conditions on p and q so that the corresponding $\{p, q\}$ checker knight can travel between any two positions on the infinite checker board. In order to do so, we need a few lemmas. The following is a well known result.

Lemma 3.12. *If a, b, and c are integers then* $\gcd(a + cb, b) = \gcd(a, b)$.

Lemma 3.13. *If s and r are relatively prime and are not both odd then $p = s - r$ and $q = s + r$ are also relatively prime and are both odd.*

Proof. Clearly p and q are both odd. By Lemma 3.12,

$$\begin{aligned} 1 &= \gcd(s, r) \\ &= \gcd(s - r, r) \\ &= \gcd(s - r, 2r) \\ &= \gcd(s - r, (s + r) - (s - r)) \\ &= \gcd(s - r, s + r). \end{aligned}$$ □

Lemma 3.14. *If p and q are both odd and are relatively prime then $r = \dfrac{q - p}{2}$ and $s = \dfrac{q + p}{2}$ have different parity.*

Proof. Let $p = 2a + 1$ and $q = 2b + 1$ where $a, b \in \mathbf{Z}$. Then

$$\begin{aligned} r &= \frac{q - p}{2} = \frac{(2b + 1) - (2a + 1)}{2} = \frac{2b - 2a}{2} = b - a \\ s &= \frac{q + p}{2} = \frac{(2b + 1) + (2a + 1)}{2} = \frac{2b + 2a + 2}{2} = b + a + 1. \end{aligned}$$

Therefore, r and s have different parity since $b - a$ and $b + a$ have the same parity. □

Lemma 3.15. *If p and q are relatively prime and are both odd then $r = \dfrac{q - p}{2}$ and $s = \dfrac{q + p}{2}$ are relatively prime.*

Proof. Let

$$\begin{aligned} d &= \gcd\left(\frac{q - p}{2}, \frac{q + p}{2}\right) \\ &= \gcd\left(\frac{q - p}{2}, \frac{2p}{2}\right) = \gcd\left(\frac{q - p}{2}, p\right). \end{aligned}$$

Then similarly,

$$d = \gcd\left(\frac{2q}{2}, \frac{q+p}{2}\right) = \gcd\left(q, \frac{q+p}{2}\right).$$

Since $d \mid p$ and $d \mid q$ we must have $d = 1$. Therefore, r and s are relatively prime. $\square$

Lemma 3.16. *The infinite checker board is connected if and only if p and q are relatively prime and are both odd.*

Proof. A $\{p, q\}$ knight graph on the infinite checker board is isomorphic to an $\{r, s\}$ knight graph on the infinite saw board. An $\{r, s\}$ knight graph on the infinite saw board is connected if and only if r and s are relatively prime and of different parity. From Lemma 3.13, Lemma 3.14 and Lemma 3.15, p and q must be relatively prime and both odd.

4. Minimal Boards

Definition 4.1. A board is *minimal* for an allowable knight if the knight graph is connected on the board and is disconnected for any strictly smaller board.

Proposition 4.2. *A knight graph which is connected on a certain saw board that is not $[1, 1]$ is also connected on any larger saw board.*

Proof. Since a [1,Y] board is disconnected for all Y, we only consider cases where both dimensions of the smaller board are at least two. The board corresponding to the connected saw knight graph can be placed in the far left of the larger board. From there, it can be shifted southeast by one position creating an intersection with the previous board. This is repeated until the board is covered in that direction. Then, the resulting board is shifted northeast by one position once again creating an intersection with the previous board. This results in the entire larger board being covered. $\square$

Corollary 4.3. *A knight graph which is connected on a certain checker board that is not $[1, 1]$ is also connected on any larger checker board.*

Proof. The proof follows from Proposition 4.2 and Corollary 3.10. $\square$

Corollary 4.4. *A knight graph which is disconnected on a certain saw board is also disconnected on any smaller saw board.*

Proof. Suppose that a smaller saw board is connected. By Proposition 4.2, any larger saw board is also connected. This is a contradiction. $\square$

Corollary 4.5. *A knight graph which is disconnected on a certain checker board is also disconnected on and samller checker board.*

Proof. The proof follows from Corollary 4.3 and Corollary 4.4 $\square$

Proposition 4.6. *A $[u,v]$ saw board is minimal for an allowable $\{r,s\}$ knight if and only if $K^s(r,s,u,v)$ is connected and $K^s(r,s,u-1,v)$ and $K^s(r,s,u,v-1)$ are disconnected.*

Proof. The forward direction is clear. For the other direction suppose that a $[\underline{u},\underline{v}]$ board is strictly smaller than a $[u,v]$ board and $K^s(r,s,\underline{u},\underline{v})$ is connected. Then either $\underline{u} < u$ or $\underline{v} < v$. If $\underline{u} < u$ then, $K^s(r,s,u-1,v)$ is connected by Proposition 4.2 since the $[u-1,v]$ board is larger than the $[\underline{u},\underline{v}]$ board. Similarly, if $\underline{v} < v$ then, $K^s(r,s,u,v-1)$ is connected by Proposition 4.2. This is a contradiction. □

Corollary 4.7. *An $[x,y]$ checker board is minimal for an allowable $\{p,q\}$ knight if $K^c(p,q,x,y)$ is connected and $K^c(p,q,x-2,y)$ and $K^c(p,q,x,y-2)$ are disconnected.*

Proof. The proof follows from Proposition 4.6 and Corollary 3.10. □

Proposition 4.8. *If a board is minimal then no smaller or larger board can be minimal.*

Proof. Suppose $[a,b]$ is a minimal board. Any board that is strictly smaller than $[a,b]$ is disconnected and therefore cannot be a miminal board. Since $[a,b]$ is connected, any board that is strictly larger than $[a,b]$ cannot be minimal. □

5. Minimal Saw Boards

We are interested in finding the minimal saw boards on which the graph of a certain knight is connected. In Sections 7 through 13 we prove the corresponding results for the corresponding checker boards. In this section we assume that $\{r,s\}$ is an allowed knight.

Theorem 5.1. *If $s = r+1$ then an $\{r,s\}$ knight has a single minimum board with size $[s+1, r+s+1]$.*

Proof. Proposition 7.7 proves the result on the corresponding checkerboard. □

Conjecture 5.2. *If $r = 1$ and $s \neq r+1$ then an $\{r,s\}$ knight has two minimal boards with size $[s+1, r+s+1]$ and $[s, r+s+2]$.*

Conjecture 5.3. *If $r \neq 1$ and $s \bmod 2r = r-1$ then an $\{r,s\}$ knight has two minimal boards with size $[s+1, r+s+1]$ and $[s, 2r+s]$.*

Conjecture 5.4. *If $s \neq r+1$, $r \neq 1$, and $s \bmod 2r = r+1$ then an $\{r,s\}$ knight has two minimal boards with size $[s+1, r+s+1]$ and $[s, 2r+s+1]$.*

Conjecture 5.5. *Let $a = s \bmod 2r$. If $a < r$ but $a \neq r-1$ then an $\{r,s\}$ knight has a single minimum board with size $[s, r+s+1]$.*

Conjecture 5.6. *Suppose $r > 3$ and $s \bmod 2r \neq r+1$. Let $a = s \bmod 2r - r$. If $a > 0$ and $r \bmod a = \frac{a\pm1}{2}$ then there are two minimal boards with size $[s+1, r+s+1]$ and $[s, r+s+\frac{a+1}{2}]$.*

Conjecture 5.7. *Suppose $r > 3$ and $s \bmod 2r \neq r+1$. Let $a = s \bmod 2r - r$. If $a > 0$ and $r \bmod a \neq \frac{a\pm1}{2}$ then there is a single minimum board with size $[s, r+s+1]$.*

Theorem 5.8. *Every allowed $\{r, s\}$ knight satisfies the requirements of at least one of Conjectures 4.1 – 4.7.*

Proof. We consider the cases where $r = 1$, $r = 2$, $r = 3$, and $r > 3$.

First suppose $r = 1$. If $s = 2$ then the requirements of Conjecture 5.1 are satisfied. If $s \neq 2$ then the requirements of Conjecture 5.2 are satisfied.

Secondly suppose $r = 2$. If $s = 3$ then again the requirements of Conjecture 5.1 are satisfied so we can suppose that $s \neq 3$. The remainder of s modulo $2r$ cannot equal 0 or 2 since this would cause r and s to have a common factor making the knight not allowed. Therefore $s \bmod 2r = 1$ or $s \bmod 2r = 3$. In the first case, the requirements of Conjecture 5.3 are satisfied. In the latter case the requirements of Conjecture 5.4 are satisfied.

Next suppose $r = 3$. If $s = 4$ then the requirements of Conjecture 5.1 are satisfied so we can suppose that $s \neq 4$. The remainder of s modulo $2r$ cannot equal 0, 3, or 6 since this would cause r and s to have a common factor making the knight not allowed. The remainder of s modulo $2r$ also cannot equal 1 or 5 since this implies that s is odd making the knight not allowed. Therefore $s \bmod 2r = 2$ or $s \bmod 2r = 4$. In the first case, the requirements of Conjecture 5.3 are satisfied. In the latter case, the requirements of Conjecture 5.4 are satisfied.

Finally suppose $r > 3$. First suppose $s \bmod 2r < r$. If $s \bmod 2r = r - 1$ then the requirements of Conjecture 5.3 are satisfied, otherwise, the requirements of Conjecture 5.5 are satisfied. Now suppose $s \bmod 2r > r$. If $s = r+1$ then the requirements of Conjecture 5.1 are satisfied, so we can suppose that $s \neq r + 1$. If $s \bmod 2r = r + 1$ then the requirements of Conjecture 5.4 are satisfied. Otherwise, let $a = s \bmod 2r - r$. If $r \bmod a = \frac{a\pm1}{2}$ then the requirements of Conjecture 5.6 are satisfied. On the other hand, if $r \bmod a \neq \frac{a\pm1}{2}$ then the requirements of Conjecture 5.7 are satisfied. □

6. Lemmas for Checker Boards

In this section we will prove some lemmas that are needed in the next few sections.

Lemma 6.1. *Suppose the $\{p, q\}$ is allowed on an $[m, N]$ checker board. If $w - p < 1$, $v - p < 1$, $v + q > m$, and $w + q > m$ then a journey $\{\langle a_i, b_i\rangle : 1 \leq i \leq k\}$ with the property that $a_i \neq -a_{i+1}$ for any i, cannot start at a vertex (v, y_1) and end at a vertex (w, y_2).*

Proof. Suppose we have a journey $\{\langle a_i, b_i\rangle : 1 \leq i \leq k\}$ from a vertex (v, y_1) to a vertex (w, y_2) must have $\langle a_1, b_1\rangle = \langle p, \pm q\rangle$ and $\langle a_k, b_k\rangle = \langle -p, \pm q\rangle$. A path which starts with a move $\langle p, \pm q\rangle$ seperated by single moves of types $\langle -q, \pm p\rangle$ satisfies the condition $a_i \neq -a_{i+1}$ for all i. Such a path cannot end at a vertex (v, y_2). □

Proposition 6.2. *Let $\{p, q\}$ be a knight allowed on an $[m, N]$ checker board. If $w - p < 1$, $v - p < 1$, $v + q > m$, and $w + q > m$ then there is no journey from cell (v, y_1) to (w, y_2) for any y_1 or y_2.*

Proof. Suppose that there is a path $\{\langle r_i, s_i\rangle : 1 \leq i \leq k\}$ from a vertex (v, y_1) to a vertex (w, y_2). Then, by Lemma 6.1 there exists j such that $r_j = -r_{j+1}$. A new path from the vertex (v, y_1) to a vertex (w, y_3) can be obtained by eliminating $\langle r_j, s_j\rangle$ and $\langle r_{j+1}, s_{j+1}\rangle$ from the original path. The new path has type $\{\langle u_i, v_i\rangle : 1 \leq i \leq k - 2\}$ where $\langle u_i, v_i\rangle = \langle a_i, b_i\rangle$ for $1 \leq i \leq j - 1$ and $\langle u_i, v_i\rangle = \langle a_{i+2}, b_{i+2}\rangle$ for $j \leq i \leq k - 2$. This process can be repeated and the journeys can be succesively reduced in length by the elimination of pairs of adjacent edges until we obtain a journey with no more than one edge from the vertex (v, y_1) to a vertex (w, y_4). It is impossible to get from (w, y_1) to (v, y_4) in one move since $w - v \neq p$ and $w - v \neq q$. Also, $(w, y_1) \neq (v, y_4)$. This is a contradiction. Therefore, we cannot reach any position (v, y_2) from a vertex (w, y_1). □

Proposition 6.3. *If $p \neq 1$ and $m \leq p + q - 3$ then $K^c(p, q, m, N)$ is disconnected for all $m \leq N$.*

Proof. Use Proposition 6.2 with $v = p$ and $w = p - 1$. □

Proposition 6.4. *Suppose the $\{p, q\}$ knight graph is connected on the boards with sizes $[\underline{x}, y]$ and $[x, \underline{y}]$ and is disconnected on the boards with sizes $[\underline{x} - 2, Y]$ for all $Y \geq \underline{x} - 2$, $[\underline{x}, y - 2]$, $[\underline{y} - 2, \underline{y} - 2]$ and $[x - 2, \underline{y}]$. If $\underline{x} = x - 2$ then we have exactly two minimal boards with sizes $[\underline{x}, y]$ and $[x, \underline{y}]$.*

Proof. Corollary 4.7 shows that $[\underline{x}, y]$ is a minimal board. Corollary 4.5 shows that $[x, \underline{y} - 2]$ is disconnected since it is smaller than $[\underline{y} - 2, \underline{y} - 2]$. Therefore, by Corollary 4.7 we know that $[x, \underline{y}]$ is a minimal board. Suppose that there is a third minimal board with size $[a, b]$. By Proposition 4.8, we cannot have $a = \underline{x}$, $a = x$, $b = \underline{y}$, or $b = y$ since all of these boards are either strictly smaller or strictly larger than a minimal board.

We cannot have $a < \underline{x}$ since this would mean that $[a, b]$ is smaller than $[\underline{x} - 2, b]$. Corollary 4.4 implies that $[a, b]$ is disconnected since $[\underline{x} - 2, Y]$ is disconnected for all $Y \geq \underline{x} - 2$. Therefore, $a > \underline{x}$.

We cannot have $b < \underline{y}$ since this would mean that $[a, b]$ is smaller than $[a, \underline{y} - 2]$. Corollary 4.4 than $[a, \underline{y} - 2]$ Corollary 4.4 implies that $[a, b]$ is disconnected since $[\underline{y} - 2, \underline{y} - 2]$ is disconnected. Therefore, $b > \underline{x}$.

We cannot have $b > \underline{y}$ and $a \geq x$ at the same time since then by Proposition 4.8, $[a, b]$ is not minimal since $[a, b]$ is strictly greater than $[x, \underline{y}]$. Therefore, $a < x$.

Now we have that $\underline{x} = x - 2 < a < x$. This is a contradiction. Therefore, we cannot have a third minimal board. □

Proposition 6.5. *Suppose the $\{p, q\}$ knight graph is connected on the boards with sizes $[\underline{x}, y]$ and $[x, \underline{y}]$ and is disconnected on the boards with sizes $[\underline{x} - 2, Y]$ for all $Y \geq \underline{x} - 2$, $[\underline{x}, y - 2]$, $[\underline{y} - 2, \underline{y} - 2]$. If $\underline{x} = x$ then we have exactly one minimal board.*

Proof. Since $\underline{x} = x$ we can replace x with $\underline{x}$. From Corollary 4.7 we have that $[\underline{x}, \underline{y}]$ is minimal since $[\underline{x} - 2, Y]$ for all $Y \geq \underline{x} - 2$ and $[\underline{y} - 2, \underline{y} - 2]$ are both disconnected. Therefore, by Proposition 4.8, the board with size $[\underline{x}, y]$ is not minimal since it is strictly larger than $[\underline{x}, \underline{y}]$. Therefore, $[\underline{x}, \underline{y}]$ is the only minimum board.

Now we look at the checker board results that correspond to the saw board results.

7. THEOREM 5.1

In this section we prove Proposition 7.7 that is the checker board version of Theorem 5.1. First we need some lemmas to show that $K^c(1, q, q + 2, 2q + 1)$ is connected.

Lemma 7.1. *The vertices* $(1,1),(1,3),...,(1,q+2)$ *and* $(q+1,2),(q+1,4),...,(q+1,q+1)$ *are in the same connected component of* $K^c(1,q,q+2,q+2)$.

Proof. A $\{1,q\}$ knight can travel from $(1,1)$ to $(1,q+2)$ in $q+1$ steps by alternating $\langle q,1\rangle$ and $\langle -q,1\rangle$ moves repeatedly. □

Example 7.2. In the case of $K^c(1,5,7,7)$ the route of the knight is shown in this picture:

6						
					5	
4						
					3	
2						
					1	
0						

Lemma 7.3. *All of the positions in a* 2 *square wide border of a* $[q+2,q+2]$ *checker board are in the same connected component of* $K^c(1,q,q+2,q+2)$.

Proof. We rotate the board that we obtain from Lemma 7.1 by 90° and place it on top of the original board. Since the two boards intersect at $(1,1)$, we now have a board with the first column of the checker positions, the first row of checker positions, the $q+1$ row of checker positions, and the $q+1$ column of checker positions in the same connected component of $K^c(1,q,q+2,q+2)$.

In the case of $K^c(1,5,7,7)$ the first picture shows the original board and the second picture shows the board rotated by 90° with $\times$'s and $\bigcirc$'s respectively marking the places the knight can travel. The third picture is the two boards placed on top of each other with the $\otimes$'s showing the intersections of the two boards.

$\times$						
					$\times$	
$\times$						
					$\times$	
$\times$						
					$\times$	
$\times$						

	$\bigcirc$		$\bigcirc$		$\bigcirc$	
$\bigcirc$		$\bigcirc$		$\bigcirc$		$\bigcirc$

$\times$						
	$\bigcirc$		$\bigcirc$		$\otimes$	
$\times$						
					$\times$	
$\times$						
					$\times$	
$\otimes$		$\bigcirc$		$\bigcirc$		$\bigcirc$

Next continue to rotate the original board by 90° two more times and place it on top of the resulting board. Finally we obtain a board whose connected component includes a checker border 2 blocks wide. The picture below shows the final border of $K^c(1,q,q+2,q+2)$ with $q=5$.

$\bullet$		$\bullet$		$\bullet$		$\bullet$
	$\bullet$		$\bullet$		$\bullet$	
$\bullet$						$\bullet$
	$\bullet$				$\bullet$	
$\bullet$						$\bullet$
	$\bullet$		$\bullet$		$\bullet$	
$\bullet$		$\bullet$		$\bullet$		$\bullet$

□

Lemma 7.4. *$K^c(1, q, q+2, 2q+1)$ is connected.*

Proof. From Lemma 7.3 we have a checker board with a two square wide border that is connected. From here we move the board two spaces up and place it on top of itself. Each time, the first two columns and the last two columns contain intersection points connecting the two boards. We do this $\frac{q-1}{2}$ times and end up with a $[q+2, 2q+1]$ board on which every checker position is connected. Therefore, $K^c(1, q, q+2, 2q+1)$ is connected. ☐

Lemma 7.5. *$K^c(1, q, q, Y)$ is disconnected for all $Y \geq q$.*

Proof. On a $[q, Y]$ board a $\{1, q\}$ knight cannot have a move of type $\langle \pm q, \pm 1\rangle$ because such a move would place the knight outside of the board. Hence any cell that can be reached from cell $(1, 1)$ is of the form $(a, 1+kq)$ for some $k \in \mathbf{Z}$ and $1 \leq a \leq q$. Since q is odd, $q \neq 2$. Also, $q \neq 1$ which implies that q divides kq but q does not divide 2. Therefore, $1 + kq \neq 3$ for any k. Hence the vertex $(3, 1)$ is not reachable by a $\{1, q\}$ knight from vertex $(1, 1)$. ☐

Lemma 7.6. *$K^c(1, q, 2q-1, 2q-1)$ is disconnected.*

Proof. On a $[2q-1, 2q-1]$ board, the point (q, q) will always be isolated. On our board, $1 \leq x \leq q+2$ and $1 \leq y \leq 2q-1$. If we travel $\langle \pm 1, q\rangle$ from (q, q) we arrive at $(q \pm 1, 2q)$. Since $2q > 2q-1$, we are outside of the board. If we travel $\langle \pm 1, -q\rangle$ from (q, q) we arrive at $(q \pm 1, 0)$. Since $0 < 1$, we are outside of the board. If we travel $\langle q, \pm 1\rangle$ from (q, q) we arrive at $(2q, q \pm 1)$. Since $q \geq 3$, $2q > q+2$ and we are outside of the board. If we travel $\langle -q, \pm 1\rangle$ from (q, q) we arrive at $\langle 0, q \pm 1\rangle$. Since $0 < 1$, we are outside of the board. It is impossible to travel from the point (q, q) to any other point on the board. Since (q, q) is an isolated point, the knight graph is not connected. ☐

Lemma 7.7. *$[q+2, 2q+1]$ is the only minimal board for a $\{1, q\}$ knight.*

Proof. Lemma 7.5 shows that q is the smallest that the height of the board can be. Similarly, Lemma 7.6 shows that $2q-1$ is the smallest that the length of the board can be. Therefore, $[q+2, 2q+1]$ is the only minimum board for a $\{1, q\}$ knight. ☐

8. CONJECTURE 5.2

In this section we will prove Conjecture 8.6 that is the checker board version of Conjecture 5.2.

Conjecture 8.1. *If $p \neq 1$ and $q = p+2$ then $K^c(p, q, p+q-1, 2q+3)$ is connected.*

Conjecture 8.2. *If $p \neq 1$ and $q = p+2$ then $K^c(p, q, p+q+1, 2q+1)$ is connected.*

Lemma 8.3. *If $p \neq 1$, $q = p+2$, and $Y \geq p+q-3$ then $K^c(p, q, p+q-3, Y)$ is disconnected for all $Y \geq p+q-3$.*

Proof. The proof follows from Proposition 6.3. ☐

Lemma 8.4. *If $p \neq 1$ and $q = p + 2$ then $K^c(p, q, 2q - 1, 2q - 1)$ is disconnected.*

Proof. On a $[2q - 1, 2q - 1]$ board, the point (q, q) will always be isolated. On our board, $1 \leq x \leq q + 2$ and $1 \leq y \leq 2q - 1$. If we travel $\langle \pm 1, q \rangle$ from (q, q) we arrive at $(q \pm 1, 2q)$. Since $2q > 2q - 1$, we are outside of the board. If we travel $\langle \pm 1, -q \rangle$ from (q, q) we arrive at $(q \pm 1, 0)$. Since $0 < 1$, we are outside of the board. If we travel $\langle q, \pm 1 \rangle$ from (q, q) we arrive at $(2q, q \pm 1)$. Since $q \geq 3$, $2q > q + 2$ and we are outside of the board. If we travel $\langle -q, \pm 1 \rangle$ from (q, q) we arrive at $\langle 0, q \pm 1 \rangle$. Since $0 < 1$, we are outside of the board. It is impossible to travel from the point (q, q) to any other point on the board. Since (q, q) is an isolated point, the knight graph is not connected. □

Conjecture 8.5. *If $p \neq 1$ and $q = p + 2$ then $K^c(p, q, p + q - 1, 2q + 1)$ is disconnected.*

Conjecture 8.6. *If $p \neq 1$ and $q = p + 2$ then there are two minimal boards with sizes $[p + q - 1, 2q + 3]$ and $[p + q + 1, 2q + 1]$.*

9. Conjecture 5.3

In this section we will prove Conjecture 9.7 that is the checker board version of Conjecture 5.3.

Conjecture 9.1. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$ then $K^c(p, q, p + q - 1, 3q - p - 1)$ is connected.*

Conjecture 9.2. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$ then $K^c(p, q, p + q + 1, 2q + 1)$ is connected.*

Lemma 9.3. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$, and $Y \geq p + q - 3$ then $K^c(p, q, p + q - 3, Y)$ is disconnected.*

Proof. First, we need to show that $p = s - r \neq 1$. Suppose $s - r = 1$ which implies that $s = r + 1$. Therefore, $r + 1 \bmod 2r = r - 1$. Since $r \neq 1$, we have $r \geq 2$ which implies that $r + 1 \bmod 2r = r + 1$. Then, $r + 1 = r - 1$ which is a contradiction.

The completion of this proof follows from Proposition 6.3. □

Conjecture 9.4. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$, and $Y \geq p + q - 3$ then $K^c(p, q, p + q - 1, 3q - p - 3)$ is disconnected.*

Lemma 9.5. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$ then $K^c(p, q, 2q - 1, 2q - 1)$ is disconnected.*

Proof. This proof is identical to the proof of Lemma 8.4. □

Conjecture 9.6. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$ then $K^c(p, q, p + q - 1, 2q + 1)$ is disconnected.*

Conjecture 9.7. *If $q \neq p + 2$ and $\frac{q+p}{2} \bmod (q - p) = \frac{q-p}{2} - 1$ then a $\{p, q\}$ knight has two minimal boards with sizes $[p + q - 1, 3q - p - 1]$ and $[p + q + 1, 2q + 1]$.*

10. CONJECTURE 5.4

In this section we will prove Conjecture 10.7 that is the checker board version of Conjecture 5.4.

Conjecture 10.1. *If* $p \neq 1$, $q \neq p+2$, *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$ *then* $K^c(p,q,p+q-1,3q-p+1)$ *is connected.*

Conjecture 10.2. *If* $p \neq 1$, $q \neq p+2$, *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$ *then* $K^c(p,q,p+q+1,2q+1)$ *is connected.*

Lemma 10.3. *If* $p \neq 1$, $q \neq p+2$, *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$, *and* $Y \geq p+q-1$ *then* $K^c(p,q,p+q-3,Y)$ *is disconnected.*

Proof. The result follows from Proposition 6.3. □

Conjecture 10.4. *If* $p \neq 1$, $q \neq p+2$, *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$, *and* $Y \geq p+q-1$ *then* $K^c(p,q,p+q-1,3q-p-1)$ *is disconnected.*

Lemma 10.5. *If* $p \neq 1$, $q-p \neq 2$ *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$ *then* $K^c(p,q,2q-1,2q-1)$ *is disconnected.*

Proof. This proof is identical to the proof of Lemma 8.4. □

Conjecture 10.6. *If* $p \neq 1$, $q-p \neq 2$ *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$ *then* $K^c(p,q,p+q-1,2q+1)$ *is disconnected.*

Conjecture 10.7. *If* $p \neq 1$, $q \neq p+2$, *and* $\frac{q+p}{2} \bmod (q-p) = \frac{q-p}{2}+1$ *then a* $\{p,q\}$ *knight has two minimal boards with sizes* $[p+q-1,3q-p+1]$ *and* $[p+q+1,2q+1]$.

11. CONJECTURE 5.5

In this section we will prove Conjecture 11.4 that is the checker board version of Conjecture 5.5.

Conjecture 11.1. *Let* $b = \frac{q+p}{2} \bmod (q-p)$. *If* $b < \frac{q-p}{2}$ *but* $b \neq \frac{q-p}{2}-1$ *then* $K^c(p,q,p+q-1,2q+1)$ *is connected.*

Lemma 11.2. *Let* $b = \frac{q+p}{2} \bmod (q-p)$. *If* $b < \frac{q-p}{2}$ *but* $b \neq \frac{q-p}{2}-1$, *and* $Y \geq p+q-3$ *then* $K^c(p,q,p+q-3,Y)$ *is disconnected.*

Proof. Suppose that $r = 1$. From Conjecture 5.5 we have that $a = s \bmod 2$. Therefore, a must be 0 or 1. Conjecture 5.5 states that $a < 1$ and $a \neq 0$ however. This is a contradiction. Therefore, $r \neq 1$.

Now we want to show that $p = s - r \neq 1$. Suppose $s - r = 1$ which implies that $s = r+1$. Then, from Conjecture 5.5, $r+1 \bmod 2r < r$. Since $r \neq 1$ we know that $r > 1$ which implies that $r+1 < 2r$. Therefore, $r+1 \bmod 2r = r+1$. This is a contradiction since $r+1 < r$. Thus, $p \neq 1$.

The completion of the proof follows from Proposition 6.3. □

Lemma 11.3. *Let* $b = \frac{q+p}{2} \bmod (q-p)$*. If* $b < \frac{q-p}{2}$ *but* $b \neq \frac{q-p}{2} - 1$ *then* $K^c(p,q,2q-1,2q-1)$ *is disconnected.*

Proof. This proof is similar to the proof of Lemma 8.4. □

Conjecture 11.4. *Let* $b = \frac{q+p}{2} \bmod (q-p)$*. If* $b < \frac{q-p}{2}$ *but* $b \neq \frac{q-p}{2} - 1$ *then a* $\{p,q\}$ *knight has one minimal board with size* $[p+q-1, 2q+1]$*.*

12. Conjecture 5.6

In this section we will prove Conjecture 12.7 that is the checker board version of Conjecture 5.6.

Conjecture 12.1. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$*. Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$*, and* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$ *then* $K^c(p,q,p+q-1,2q+b)$ *is connected.*

Conjecture 12.2. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$*. Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$*, and* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$ *then* $K^c(p,q,p+q+1,2q+1)$ *is connected.*

Lemma 12.3. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$*. Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$*,* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$*, and* $Y > p+q-3$ *then* $K^c(p,q,p+q-3,Y)$ *is disconnected.*

Proof. First we want to show that $p = s - r \neq 1$. Suppose $s - r = 1$ which implies that $s = r + 1$. Then, from Conjecture 5.5, $r + 1 \bmod 2r \neq r + 1$. Since $r > 3$ we know that $r + 1 < 2r$. Therefore, $r + 1 \bmod 2r = r + 1$. This is a contradiction. Thus, $p \neq 1$.

The completion of the proof follows from Proposition 6.3. □

Conjecture 12.4. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$ *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$ *and* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$ *then* $K^c(p,q,p+q-1,2q-b-2)$ *is disconnected.*

Lemma 12.5. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$ *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$ *and* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$ *then* $K^c(p,q,2q-1,2q-1)$ *is disconnected.*

Proof. This proof is similar to the proof of Lemma 8.4. □

Conjecture 12.6. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$ *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$ *and* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$ *then* $K^c(p,q,p+q-1,2q+1)$ *is disconnected.*

Conjecture 12.7. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2} + 1$*. Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$*. If* $b > 0$*, and* $\frac{q-p}{2} \bmod b = \frac{b \pm 1}{2}$ *then a* $\{p,q\}$ *knight has two minimal boards with sizes* $[p+q-1, 2q+b]$ *and* $[p+q+1, 2q+1]$*.*

13. Conjecture 5.7

In this section we will prove Conjecture 13.4 that is the checker board version of Conjecture 5.7.

Conjecture 13.1. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2}+1$. *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$. *If* $b > 0$ *and* $\frac{q-p}{2} \bmod b \neq \frac{b \pm 1}{2}$ *then* $K^c(p,q,p+q-1,2q+1)$ *is connected.*

Lemma 13.2. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2}+1$. *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$. *If* $b > 0$, $\frac{q-p}{2} \bmod b \neq \frac{b \pm 1}{2}$, *and* $Y > p+q-3$ *then* $K^c(p,q,p+q-3,Y)$ *is disconnected.*

Proof. This proof is the same as the proof for Lemma 12.3. □

Lemma 13.3. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2}+1$. *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$. *If* $b > 0$ *and* $\frac{q-p}{2} \bmod b \neq \frac{b \pm 1}{2}$ *then* $K^c(p,q,2q-1,2q-1)$ *is disconnected.*

Proof. This proof is similar to the proof of Lemma 8.4. □

Conjecture 13.4. *Suppose* $p > q+6$ *and* $\frac{q+p}{2} \bmod (q-p) \neq \frac{q-p}{2}+1$. *Let* $b = \frac{q+p}{2} \bmod (q-p) - \frac{q-p}{2}$. *If* $b > 0$ *and* $\frac{q-p}{2} \bmod b \neq \frac{b \pm 1}{2}$ *then a* $\{p,q\}$ *knight has one minimal board with size* $[p+q-1, 2q+1]$.

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