1. Show that a nonempty, countable, complete metric space \((X, d)\) has an isolated point, that is, a point \(x \in X\) such that \(\{x\}\) is open.

**Solution:** Assume \(X\) has no isolated points. Then \(\{x\}\) is not open and so \(A_x = \{x\}^C\) is not closed for any \(x \in X\). Hence \(A_x = X\) since the only subsets of \(X\) containing \(A_x\) are \(A_x\) and \(X\) and only \(X\) is closed. \(\{x\}\) is closed since finite sets in a metric space are closed. Hence \(A_x\) must be open. Thus \(\{A_x \mid x \in X\}\) is a countable collection of open, dense subsets of a complete metric space, and so by the Baire category theorem we have \(\emptyset = \cap_{x \in X} A_x\) is dense in \(X\) which is a contradiction.

2. Recall that a set \(N\) is nowhere dense in a topological space \((X, T)\) if the interior of the closure of \(N\) is the empty set, that is, \(N^0 = \emptyset\).

   a. Show that if \(D\) is open and dense in \(X\) then the complement \(D^C\) is closed nowhere dense in \(X\).
   b. Show that if \(N\) is nowhere dense in \(X\) then the complement of \(N\) is open and dense in \(X\).

   **Solution:** a. \(D^C\) is closed since \(D\) is open. So the closure of \(D^C\) is \(D^C\) itself. If \(U\) is a nonempty open set then \(U \cap D \neq \emptyset\) since \(D\) is dense. So there are no open subsets of \(D^C\) and so the interior of the closure \(D^C\) is the empty set.

   b. The closure of \(N\) is closed and so its complement is open. A nonempty open set \(U\) is not a subset of \(\overline{N}\) since the interior of \(\overline{N}\) is empty. Hence any nonempty open set intersects the complement of \(\overline{N}\).