1. Show that in a Hausdorff topological space every convergent sequence has a unique limit.

Solution: Suppose that \( x_n \to x \), \( x_n \to y \) and \( x \neq y \). Since the space is Hausdorff, we can find disjoint, open neighborhoods \( U \) and \( V \) of \( x \) and \( y \) respectively. Because of the convergence, the tail of \((x_n)\) needs to be in both \( U \) and \( V \). This is impossible.

2. Show that in a Hausdorff topological space every singleton set is closed.

Solution: Let \( x \in X \). For all \( y \in X \) with \( x \neq y \) there are disjoint, open sets \( U_y \) and \( V_y \) such that \( x \in U_y \), \( y \in V_y \). Then \( \{x\} = \bigcap\{V_y \mid y \in X \setminus \{x\}\} \) is closed since it’s an intersection of closed sets.

3. Show that every metric space is Hausdorff.

Solution: If \( x \neq y \) then let \( D = B_{d(x,y)/2}(x) \cap B_{d(x,y)/2}(y) \). If \( z \in D \) then \( d(x,y) \leq d(x,z) + d(z,y) < d(x,y)/2 + d(x,y)/2 = d(x,y) \) gives a contradiction. This \( D \) is empty.

4. Let \( X \) be a topological space and \( A, B \subseteq X \).
   a. Show that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \).
   b. Show that \( (A^C)^c = A^C \).

Solution: a. \( A \subseteq A \cup B \) and so \( \overline{A} \subseteq \overline{A \cup B} \). Similarly \( B \subseteq A \cup B \). Hence \( \overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \). \( \overline{A \cup B} \) is closed and \( A \cup B \subseteq \overline{A \cup B} \). Hence \( \overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \).
   b.
   \[
   A^C = (\cap \{ F \mid F \text{ is closed, } A \subseteq F \})^C
   = \cup \{ F^C \mid F \text{ is closed, } F^C \subseteq A^C \}
   = \cup \{ U \mid U \text{ is open, } U \subseteq A^C \}
   = (A^C)^C.
   \]