1. Show that if \( f : \mathbb{R} \to \mathbb{R} \) is uniformly continuous and \( f_n(x) := f \left( x + \frac{1}{n} \right) \) then \( f_n \to f \) uniformly.

**Solution:** Let \( \epsilon > 0 \). Since \( f \) is uniformly continuous there is a \( \delta > 0 \) such that for all \( x, y \in \mathbb{R} \), \(|x - y| < \delta\) implies \(|f(x) - f(y)| < \epsilon\). Choose a \( k \in \mathbb{N} \) such that \( 1/k < \delta \). Then for all \( n \geq k \) and \( x \in \mathbb{R} \) we have

\[
\left| \left( x + \frac{1}{n} \right) - x \right| = \frac{1}{n} < \frac{1}{k} < \delta.
\]

and so

\[
|f_n(x) - f(x)| = \left| f \left( x + \frac{1}{n} \right) - f(x) \right| < \epsilon.
\]

2. Is the statement of the previous problem true if \( f \) continuous but not uniformly continuous?

**Solution:** The statement is not true if the continuity of \( f \) is not uniform. For example let \( f(x) = x^2 \). Then \( f_n(x) = (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \). Let \( \epsilon = 1 \) and \( N \in \mathbb{N} \). If \( x > \frac{N}{2} \) then

\[
|f_N(x) - f(x)| = \left| \frac{2x}{N} + \frac{1}{N^2} \right| > \frac{2x}{N} > 1 = \epsilon.
\]

This means no threshold index \( N \) could work for all \( x \in \mathbb{R} \) and so \( f_n \) does not converge to \( f \) uniformly.

3. Find the largest domain on which the function series \( \sum_{n=1}^{\infty} \frac{n}{n+1} \left( \frac{x}{2x+1} \right)^n \) is convergent.

**Solution:** We apply the ratio test.

\[
\frac{n+1}{n+2} \left| \frac{x}{2x+1} \right| \frac{n+1}{n} \left| \frac{x}{x+1} \right|^n = \left( \frac{n+1}{n} \right) \frac{x}{2x+1} \to \left| \frac{x}{2x+1} \right|
\]

If \( x \in (-\infty, -1) \cup (-\frac{1}{3}, \infty) \) then \( \left| \frac{x}{2x+1} \right| < 1 \) and so the series is absolutely convergent. If \( x \in (-1, -\frac{1}{3}) \) then \( \left| \frac{x}{2x+1} \right| > 1 \) and so the series is divergent. It remains to check the case \( x = -1 \) and \( x = -\frac{1}{3} \). If \( x = -1 \) then the series fails the n-th term test since \( \frac{n}{n+1} \left( -\frac{1}{2} \right)^n \to 1 \neq 0 \). If \( x = -\frac{1}{3} \) then again the series fails the n-th term test since \( \frac{n}{n+1} \left( -\frac{1}{2/3+1} \right)^n = \frac{n}{n+1}(-1)^n \) diverges. So the series is absolutely convergence on \( (-\infty, -1) \cup (-\frac{1}{3}, \infty) \).