1. Let \((X, c)\) and \((Y, d)\) be metric spaces and \(Z = X \times Y\). Define \(e : Z \times Z \to \mathbb{R}\) by \(e((x_1, y_1), (x_2, y_2)) = c(x_1, x_2) + d(y_1, y_2)\). Then \((X \times Y, e)\) is also a metric space. Show that if \(U\) is open in \(X\) and \(V\) is open \(Y\) then \(U \times V\) is open in \(X \times Y\).

**Solution:** Let \((x, y) \in U \times V\). Since \(U\) and \(V\) are open, we can find \(r, s > 0\) such that \(B_r(x) \subseteq U\) and \(B_s(x) \subseteq V\). Then \(t = \min\{r, s\}\) is positive. If \((w, z) \in B_t((x, y))\) then \(t > e((x, y), (w, z)) = c(x, w) + d(y, z)\). Hence \(r > c(x, w)\) and \(s > d(y, z)\) and so \(w \in U\) and \(z \in V\). Thus \((w, z) \in U \times V\).

2. Let \((X, c)\) be a metric space and \(Y \subseteq X\). Let \(d = c|Y\) the restriction of \(c\) to \(Y\).
   a. Show that \((Y, d)\) is a metric space. It is called a subspace.
   b. Find an example that shows that a subset \(F\) of \(Y\) may be closed in \(Y\) but open in \(X\).
   c. Find an example that shows that a subset \(G\) of \(Y\) may be closed in \(Y\) but neither closed nor open in \(X\).

**Solution:** b. Let \(X = \mathbb{R}\), \(Y = (0, 1)\) and \(F = (0, 1)\).
   c. Let \(X = \mathbb{R}\), \(Y = (0, 1]\) and \(G = (0, 1]\).

3. Let \((X, c)\) and \((Y, d)\) be metric spaces. Show that \(f : X \to Y\) is continuous if and only if \(f^{-1}(U)\) is open in \(X\) for all open set \(U\) in \(Y\).

**Solution:** First suppose that \(f\) is continuous. Let \(U\) be open in \(Y\) and define \(V = f^{-1}(U)\). If \(x \in V\) then \(f(x) \in U\) and since \(U\) is open, we can find an \(r > 0\) such that \(B_r(f(x)) \subseteq U\). Since \(f\) is continuous, we can find an \(s > 0\) such that \(f(B_s(x)) \subseteq B_r(f(x))\). Hence \(B_s(x) \subseteq V\) which shows that \(V\) is open.

Next assume that \(f^{-1}(U)\) is open for all open \(U\). If \(x \in X\) and \(\epsilon > 0\) then \(W = f^{-1}(B_\epsilon(f(x)))\) is open and \(x \in W\). Hence there is a \(\delta > 0\) such that \(B_\delta(x) \subseteq W\). Then \(f(B_\delta(x)) \subseteq B_\epsilon(f(x))\) which shows that \(f\) is continuous at \(x\).