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A numerical investigation of sign-changing solutions to superlinear elliptic equations on symmetric domains

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Abstract

In this paper, we investigate numerically sign-changing solutions of superlinear elliptic equations on symmetric domains. Based upon the symmetric criticality principle of Palais, the existence of sign-changing solutions which reflect the symmetry of Ω is studied first. A simple numerical algorithm, the modified mountain pass algorithm, is then proposed to compute the sign-changing solutions. This algorithm is discussed and compared with the high-linking algorithm for sign-changing solutions developed by Ding et al. [Nonlinear Anal. 37 (1999) 151–172]. By implementing both algorithms on several numerical examples, the sign-changing solutions and their nodal curves are displayed and discussed. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\Omega \subset \Re^N$ be a bounded connected domain with regular boundary $\partial\Omega$. Consider the following semilinear elliptic equation subject to Dirichlet boundary condition:

$$\begin{cases} -\Delta w = f(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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The differential operator $-\Delta$ can be replaced by a more general second-order uniformly elliptic operator in divergence form. Nonlinear elliptic equations of type (1.1) arise naturally in physics, engineering, mathematical biology, ecology, geometry, etc. It is known that many such equations have multiple solutions, and most of them are sign-changing solutions. The study of sign-changing solutions of (1.1) has attracted the attention of many pure and applied mathematicians during the past two decades.

Let us assume that f satisfies the following regularity and growth conditions:

(A1) $f \in C^1(\mathfrak{R}, \mathfrak{R})$, $f(0) = 0$, and $f'(0) < \lambda_1$ (the first eigenvalue of $-\Delta$);

(A2) there are constants C_1 and C_2 such that

$$|f(t)| \leqslant C_1 + C_2|t|^p,$$

where $0 \leqslant p < (N+2)/(N-2)$ for $N \geqslant 3$. If $N = 1$, (A2) can be dropped. For $N = 2$, it suffices that

$$|f(t)| \leqslant C_3 \exp(\psi(t)),$$

where $\psi(t)/t^2 \to 0$ as $t \to \infty$ and C_3 is a constant.

(A3) there are constants $\mu > 2$ and $M > 0$ such that for $|t| \geqslant M$,

$$0 < \mu F(t) \leqslant t f(t),$$

where $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

Condition (A1) implies that (1.1) has the trivial solution $w(x) = 0$, and condition (A2) says that f is *subcritical*. Under conditions (A1) and (A2), it is well-known [18] that the functional $J : H_0^1(\Omega) \to \mathfrak{R}$ defined by

$$J(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - F(w) \right) \mathrm{d}x \tag{1.2}$$

is continuously Fréchet differentiable, i.e., $J \in C^1(H_0^1(\Omega), \mathfrak{R})$, and its critical points correspond to weak solutions of (1.1). Moreover, any weak solution is also in $C^\infty(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ by standard elliptic regularity estimates. Condition (A3) implies that $F(t)$ grows at a "superquadratic" rate and $f(t)$ grows at a "superlinear" rate as $|t| \to \infty$. Thus, problem (1.1) is a "superlinear" Dirichlet problem. With assumptions (A1)–(A3), it can be verified [18] that J satisfies the Palais–Smale condition. Then, by applying the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1] after suitable truncations on $f(t)$, it can be shown that problem (1.1) has at least two nontrivial solutions, one of which is a positive mountain pass solution and the other is a negative mountain pass solution.

The question whether (1.1) has, in general, one or more sign-changing solution is much more delicate. When $f(t)$ is an odd nonlinearity (i.e., $f(-t) = -f(t)$), the existence of infinitely many sign-changing solutions can be established [1,2]. Also, when Ω has certain symmetry properties, multiple sign-changing solutions can be obtained by applying the Mountain Pass Theorem to $J(w)$ in suitable symmetric subspaces of $H_0^1(\Omega)$ and using the principle of symmetric criticality due to Palais [17]. However, if $f(t)$ is not an odd nonlinearity and Ω has no symmetric properties, the question of existence of multiple sign-changing solutions is more challenging. Under assumptions (A1)–(A3), Wang [20] first proved that (1.1) admits at least three nontrivial solutions. Under conditions (A1)–(A4), where

$$\text{(A4)}\ f'(t) > \frac{f(t)}{t} \quad \text{for any } t \neq 0,$$

Castro, Cossio and Neuberger [4] proved that (1.1) has at least two mountain pass solutions and one sign-changing solution. Under conditions (A1)–(A3) with the weaker assumption $f'(0) < \lambda_2$, where λ_2 is the second eigenvalue of $-\Delta$, Bartsch and Wang [3] proved that (1.1) has at least one sign-changing solution.

In the past few years, numerical algorithms for computing unstable solutions of (1.1) have attracted the attention from both pure and applied mathematicians [6–9,11,14,16]. In general, those unstable solutions of (1.1) correspond to "saddle-type" critical points of $J(w)$. The first ingenious numerical algorithm for computing mountain pass solutions of semilinear elliptic equations, the Mountain Pass Algorithm, was proposed by Choi and McKenna [8]. In general, this algorithm will find only solutions of mountain pass type of Morse index 1 or 0. When the domain Ω is symmetric about some hyperplanes in $\mathfrak{R}^N$ and $f(t)$ is an odd nonlinearity, the Mountain Pass Algorithm may also give some sign-changing solutions of (1.1) through the symmetry properties of Ω. Mountain-pass-type algorithms have been successfully applied to semilinear wave equations [9], the suspension bridge equation [14], the nonlinearly suspended beam [6], etc. Many interesting mountain-pass-type solutions of semilinear elliptic equations on several typical symmetric and nonsymmetric domains have been displayed in [7]. However, when Ω is not symmetric or $f(t)$ is not symmetric, the mountain pass algorithm cannot be used to find any sign-changing solutions.

When Ω is not symmetric and $f(t)$ is not an odd nonlinearity, the task of computing numerically sign-changing solutions of (1.1) is much more challenging. This difficulty was overcome by Ding et al. [11] and Neuberger [16] based upon different ideas. The algorithm proposed in [11], the High-Linking Algorithm, was inspired by the elegant work in [20], and provides at least two sign-changing solutions by starting at two mountain pass solutions obtained by the Mountain Pass Algorithm and by constructing a local linking. This algorithm was tested on problem (1.1) with Ω being several typical symmetric and nonsymmetric domains. The sign-changing solutions were also displayed in [11]. The algorithm proposed in [16], the Projection Algorithm, was inspired by the elegant work in [4]. However, the algorithm was only tested on problem (1.1) with $\Omega = [0,1]$ and $\Omega = [0,1] \times [0,1]$, and with $f(t)$ being an odd nonlinearity.

The main objective of this paper is to investigate numerically sign-changing solutions of superlinear elliptic equation (1.1) on symmetric domains. Based upon the symmetric criticality principle due to Palais [17], the existence of sign-changing solutions which reflect the symmetry of Ω is proved first. By implementing numerically that principle for group actions which reflect the symmetry of Ω, we propose a simple numerical algorithm, the modified mountain pass algorithm, to compute the sign-changing solutions. This algorithm is inspired by the Mountain Pass Algorithm [8] and the elegant work in [4]. The High-Linking Algorithm [11] is also adopted to compute the sign-changing solutions. Both algorithms are tested and compared on several numerical examples. The sign-changing solutions and their nodal curves are displayed and discussed.

The organization of this paper is as follows. In Section 2, we discuss the existence of sign-changing solutions to (1.1), which reflect the symmetry of Ω. In Section 3, numerical algorithms for computing sign-changing solutions are introduced and discussed. Numerical examples on several typical symmetric domains are presented in Section 4, while some further discussions on numerical algorithms and sign-changing solutions are given in Section 5.

2. Existence of symmetric sign-changing solutions

Let $H = H_0^1(\Omega)$. Assume that there is a unitary representation $\{T(k)\}_{k\in G}$ of a compact topological group G on the Hilbert space H [15]. In other words, $T(k): H \to H$ is a linear isometric operator for any $k \in G$, and satisfies the following properties:
(a) $T(0)u = u$, $\forall u \in H$;
(b) $T(k_1)T(k_2) = T(k_1 + k_2)$ for any $k_1,\ k_2 \in G$;
(c) $(k, u) \mapsto T(k)u$ is continuous on $G \times H$.
The fixed set of the representation $\{T(k)\}_{k\in G}$, denoted by Fix(G), is defined as the closed subspace of H given by

$$\mathrm{Fix}(G) = \{u \in H \mid T(k)u = u,\ \forall k \in G\}.$$

In the following discussion, we always assume that Fix(G) is a nontrivial subspace of H. Assume that the functional $J: H \mapsto \Re$ defined by (1.2) is invariant under $\{T(k)\}_{k\in G}$, i.e.,
(d) $J(T(k)u) = J(u)$, for all $u \in H$, $k \in G$.
In this paper, we will consider representations which satisfy also an additional condition,
(e) $u \in \mathrm{Fix}(G)$ implies $u_+,\ u_- \in \mathrm{Fix}(G)$,
where $u_+(x) = \max\{u(x),\ 0\}$, $u_-(x) = \min\{u(x),\ 0\}$ and $u = u_+ + u_-$.

There are many examples of such group representations [15]. One example is the following. Let $\Omega \subset \Re^2$ be an equilateral polygonal domain centered at the origin with n corners, $G = Z_n$, and $x \in \Omega$ be represented by the polar coordinates (r, θ). Define the representation $\{T(k)\}_{k\in Z_n}$ on $H = H_0^1(\Omega)$ by rotations:

$$T(k)u(r,\theta) = u\left(r, \theta + \frac{2k\pi}{n}\right), \quad k = 0, \ldots, n-1.$$

The fixed set of $\{T(k)\}_{k\in Z_n}$ is given by

$$\mathrm{Fix}(Z_n) = \{u \in H \mid T(k)u = u,\ \forall k \in Z_n\}.$$

It consists of those functions which are "generated" by the n rotations $T(1), \ldots, T(n)$ of the functions $v \in H^1(\Omega_0)$ defined on a "fundamental subdomain" $\Omega_0 = \{(r,\theta) \mid 0 \leqslant \theta \leqslant 2\pi/n\}$. It is easy to verify that $\{T(k)\}_{k\in Z_n}$ satisfy the above properties (a)–(e). Another example is the isometric representation $\{T(k)\}_{k\in S^1}$ of S^1 on $H = H_0^1(\Omega)$, where $\Omega \subset \Re^2$ is the unit disk [10].

Under the above assumptions, we are interested in sign-changing solutions of (1.1) in Fix(G). The following theorem can be obtained.

Theorem 2.1. *Under assumptions* (A1)–(A4) *and* (a)–(e), *Eq.* (1.1) *admits at least one sign-changing solution in Fix*(G).

The proof of this theorem follows from an application of the principle of symmetric criticality [17] together with the argument in [4]. Indeed, from the *equivariance* of ∇J, we have $\nabla J(T(k)u) =$

$T(k)\nabla J(u)$. Thus $\nabla J(u) \in \text{Fix}(G)$ if $u \in \text{Fix}(G)$. It then follows that a critical point of J restricted to $\text{Fix}(G)$ is also a critical point of J on H. Next, for the sake of completeness and for understanding the numerical algorithms to be introduced later, we include a sketch of the proof of Theorem 2.1.

Let $\gamma(u) = \langle \nabla J(u), u \rangle = \int_\Omega (|\nabla u|^2 - uf(u))\,\mathrm{d}x$ for any $u \in H$, and introduce the sets

$$\mathscr{S} = \{u \in \text{Fix}(G) \,|\, u \neq 0,\ \gamma(u) = 0\}, \qquad \mathscr{S}_1 = \{u \in \text{Fix}(G) \,|\, u_+ \in \mathscr{S},\ u_- \in \mathscr{S}\}.$$

Under assumptions (A1)–(A4), it is well-known that $\mathscr{S}$ is a closed and unbounded C^1-submanifold of $\text{Fix}(G)$ with codimension 1. Furthermore, $J(u)$ is coercive on $\mathscr{S}$ and $\inf_{u\in\mathscr{S}} J(u) > 0$. By the continuity of γ and of the mappings $u \mapsto u_\pm$ on H, $\mathscr{S}_1$ is a closed subset of $\mathscr{S}$. From the definitions of $\mathscr{S}$ and $\mathscr{S}_1$, all nontrivial solutions of (1.1) in $\text{Fix}(G)$ belong to $\mathscr{S}$, and all sign-changing solutions of (1.1) in $\text{Fix}(G)$ belong to $\mathscr{S}_1$. So, the proof of Theorem 2.1 can be obtained in two steps:

Step 1: Prove that there exists an $w \in \mathscr{S}_1$ such that $J(w) = \inf_{u\in\mathscr{S}_1} J(u)$;

Step 2: Prove that if $w \in \mathscr{S}_1$ and $J(w) = \inf_{u\in\mathscr{S}_1} J(u)$, then w is a critical point of J restricted on $\text{Fix}(G)$.

To accomplish Step 1, let $c_3 = \inf_{u\in\mathscr{S}_1} J(u) > 0$ and $\{u_n\} \subset \mathscr{S}_1$ be such that $\lim_{n\to\infty} J(u_n) = c_3$. By using the coercivity of $J(u)$ on $\mathscr{S}$ and the local weak compactness of H, there exists a subsequence of $\{u_n\}$ (which we continue to denote by $\{u_n\}$) and $w \in H$ such that

$$\lim_{n\to\infty} u_n = w \quad \text{weakly in } H.$$

Since $\text{Fix}(G)$ is a closed subspace of H, we have $w \in \text{Fix}(G)$ and, consequently, $w_+ \in \text{Fix}(G)$ and $w_- \in \text{Fix}(G)$. Recalling $J(u) = J(u_-) + J(u_+)$, we have

$$\lim_{n\to\infty} (u_n)_+ = w_+ \quad \text{weakly in } H.$$

By using the Sobolev Embedding Theorem and the subcriticality of f, we also have

$$\lim_{n\to\infty} \int_\Omega F((u_n)_+)\,\mathrm{d}x = \int_\Omega F(w_+)\,\mathrm{d}x \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega f((u_n)_+)(u_n)_+\,\mathrm{d}x = \int_\Omega f(w_+)w_+\,\mathrm{d}x.$$

Noting that $(u_n)_+ \subset \mathscr{S}$, we have $\gamma((u_n)_+) = 0$ and

$$\int_\Omega f(w_+)w_+\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega f((u_n)_+)(u_n)_+\,\mathrm{d}x = \lim_{n\to\infty} ||(u_n)_+||^2 > 0.$$

Thus $w_+ \neq 0$. Similarly $w_- \neq 0$. By using the weak lower semicontinuity of $||\cdot||$, we obtain $J(w_+) \leqslant \liminf_{n\to\infty} J((u_n)_+)$. Similarly $J(w_-) \leqslant \liminf_{n\to\infty} J((u_n)_-)$. Thus,

$$J(w) = J(w_+) + J(w_-) \leqslant \liminf_{n\to\infty}\, [J((u_n)_+) + J((u_n)_-)] = \liminf_{n\to\infty} J(u_n) = c_3.$$

Step 1 will be complete once we prove that $w \in \mathscr{S}_1$. Suppose $w \notin \mathscr{S}_1$. Then, without loss of generality, we may assume that $w_+ \notin \mathscr{S}$. Since

$$\gamma(w_+) = ||w_+||^2 - \int_\Omega f(w_+)w_+\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \left[||(u_n)_+||^2 - \int_\Omega f((u_n)_+)(u_n)_+\,\mathrm{d}x \right] = 0,$$

we have $||w_+||^2 < \liminf_{n\to\infty} ||(u_n)_+||^2$. Under conditions (A1) and (A4), it is straightforward to show that there is an $\alpha > 0$ such that $\alpha w_+ \in \mathscr{S}$. Similarly, there exists an β satisfying $\beta > 0$ such that $\beta w_- \in \mathscr{S}$. Thus $\alpha w_+ + \beta w_- \in \mathscr{S}_1$ and

$$\begin{aligned} J(\alpha u_+ + \beta w_-) &< \liminf_{n\to\infty} J(\alpha(u_n)_+ + \beta(u_n)_-) \\ &= \liminf_{n\to\infty} [J(\alpha(u_n)_+) + J(\beta(u_n)_-)] \\ &\leqslant \liminf_{n\to\infty} [J((u_n)_+) + J((u_n)_-)] \\ &= \liminf_{n\to\infty} J(u_n) = c_3, \end{aligned}$$

which is a contradiction. Therefore $w \in \mathscr{S}_1$.

To accomplish Step 2, we need to prove that $\nabla J(w) = 0$. Let $T_w\mathscr{S}$ be the tangent space to $\mathscr{S}$ at w, where $T_w\mathscr{S} = \{u \in \mathrm{Fix}(G) \mid \langle \nabla\gamma(w), u\rangle = 0\}$ and

$$\langle \nabla\gamma(w), u\rangle = 2\int_\Omega \nabla w \nabla u \,\mathrm{d}x - \int_\Omega f(w)u \,\mathrm{d}x - \int_\Omega f'(w)wu \,\mathrm{d}x.$$

By using condition (A4), we have $\langle \nabla\gamma(w), w\rangle < 0$. Thus $w \notin T_w\mathscr{S}$. Since $\nabla J(w) = 0$ is equivalent to $P_w\nabla J(w) = 0$, where P_w is the orthogonal projection from $\mathrm{Fix}(G)$ to $T_w\mathscr{S}$, we only need to prove that $P_w\nabla J(w) = 0$. Assume $P_w\nabla J(w) \neq 0$. Let $0 < \epsilon < \frac{1}{2}\min\{\|w_+\|, \|w_-\|\}$ and $\mathscr{B} = \{u \in \mathscr{S} \mid \|u - w\| \geqslant \epsilon\}$. By a version of the Deformation Lemma on Banach manifolds [12, p. 55], there exists a flow $\eta \in C^1([0,1] \times \mathscr{S}, \mathscr{S})$ and $t_0 > 0$ such that for all $0 \leqslant t < t_0$,

(i) $\eta(0,u) = u$ for all $u \in \mathscr{S}$,
(ii) $\eta(t,u) = u$ for all $u \in \mathscr{B}$,
(iii) $\eta(t,\cdot)$ is a homeomorphism from $\mathscr{S}$ onto $\mathscr{S}$,
(iv) $J(\eta(t,u)) \leqslant J(u)$ for all $u \in \mathscr{S}$,
(v) $J(\eta(t,w)) \leqslant J(w) - \frac{t}{4}\|P_w\nabla J(w)\|$.

Consider the linear combination $(1-t)w_+ + tw_-$ for $0 \leqslant t \leqslant 1$. Under conditions (A1)–(A4), it is straightforward to show that there exists $\alpha \in C^1([0,1], \mathfrak{R}_+)$ such that

$$r(t) = \alpha(t)[(1-t)w_+ + tw_-] \in \mathscr{S}.$$

It is obvious that $\alpha(0)=1$, $\alpha(\frac{1}{2})=2$, $\alpha(1)=1$, and $J(r(t)) < J(w)$ for $t \neq \frac{1}{2}$. Define $r_1(t)=\eta(t_0/2, r(t))$. By properties (iv) and (v), we have

$$J(r_1(t)) \leqslant J(r(t)) < J(w) \quad \text{for } t \neq \tfrac{1}{2}$$

and

$$J\left(r_1\left(\frac{1}{2}\right)\right) = J\left(\eta\left(\frac{t_0}{2}, w\right)\right) \leqslant J(w) - \frac{t_0}{8}\|P_w\nabla J(w)\| < J(w).$$

Hence

$$\max_{t\in[0,1]} J(r_1(t)) < J(w) = \inf_{u\in\mathscr{S}_1} J(u).$$

On the other hand, it is known [4] that, under conditions (A1)–(A4), $\mathscr{S}$ is separated by $\mathscr{S}_1$ into two connected components,

$$\mathscr{S}_+ = \{u \in \mathscr{S} \mid u \geqslant 0 \text{ or } \gamma(u_+) < 0\} \quad \text{and} \quad \mathscr{S}_- = \{u \in \mathscr{S} \mid u \leqslant 0 \text{ or } \gamma(u_-) < 0\}.$$

Since $r_1(0) \in \mathscr{S}_+$ and $r_1(1) \in \mathscr{S}_-$, $r_1(t) \cap \mathscr{S}_1$ is not empty, which provides a contradiction to the last inequality. Therefore $P_w\nabla J(w) = 0$. This concludes the sketch of the proof of Theorem 2.1. □

Finally, we note that an application of Theorem 2.1 to the equilateral n-polygonal domain (with the representation $\{T(k)\}_{k\in Z_n}$) considered in the beginning of this section yields the existence of a sign-changing solution w "generated" by the n rotations $T(1),\ldots,T(n)$ of some sign-changing function $v\in H^1(\Omega_0)$ defined on the "fundamental subdomain" $\Omega_0=\{(r,\theta)\,|\,0\leqslant\theta\leqslant 2\pi/n\}$. Similarly, in the case of the unit disc (and the representation $\{T(k)\}_{k\in S^1}$), we obtain a radially symmetric sign-changing solution.

3. Numerical algorithms for sign-changing solutions in Fix(G)

In this section, we focus on developing numerical algorithms for computing sign-changing solutions of the superlinear Dirichlet problem (1.1) in Fix(G), whose existence have been discussed in the previous section. Two numerical algorithms are proposed in this section. One algorithm is inspired by the proof outlined in Section 2, while the other is adopted from the High-Linking Algorithm [11].

Before presenting these two algorithms, let us first introduce the Mountain Pass Algorithm (MPA) proposed by Choi and McKenna [8] for computing mountain pass solutions of (1.1). We have adopted their algorithm, whose flowchart is given below.

Mountain Pass Algorithm

Step 1: Take an initial guess $w_0\in \mathrm{Fix}(G)$ such that $w_0\neq 0$ and $J(w_0)\leqslant 0$;
Step 2: Find $t^*>0$ such that $J(t^*w_0)=\max_{t\in[0,\infty)}J(tw_0)$, and set $w_1=t^*w_0\in\mathscr{S}$;
Step 3: Compute $\nabla J(w_1)$ and set $v=\nabla J(w_1)\in \mathrm{Fix}(G)$;
Step 4: If $||v||\leqslant\varepsilon$, then output w_1 and stop;
else, goto the next step;
Step 5: Let $w=-v+w_1$ and find $t^*>0$ such that $J(t^*w)=\max_{t\in[0,\infty)}J(tw)$;
Step 6: If $J(t^*w)<J(w_1)$, set $w_1=t^*w\in\mathscr{S}$ and goto Step 3;
else, set $v=\frac{1}{2}v$ and goto Step 5.

Remark 3.1. The element $v=\nabla J(w_1)$ in Step 3 is understood as follows: v is the unique element in E such that $\langle J'(w_1),u\rangle_{E',E}=\langle v,u\rangle$ for any $u\in E$, where E' is the dual of E, $J'(w_1)\in E'$ and $\langle\cdot,\cdot\rangle_{E',E}$ is the duality pairing between E' and E. By the *equivariance* of ∇J, $v\in\mathrm{Fix}(G)$ because $w_1\in\mathrm{Fix}(G)$. Thus $-v$ is, in fact, the steepest descent direction of $J|_{\mathrm{Fix}(G)}$ at w_1. To calculate v in Step 3, we first need to compute the gradient of J at w_1. For any $u\in E$,

$$\begin{aligned}\langle J'(w_1),u\rangle_{E',E}&=\lim_{\varepsilon\to 0}\frac{J(w_1+\varepsilon u)-J(w_1)}{\varepsilon}\\&=\int_\Omega(\nabla w_1\cdot\nabla u-f(w_1)u)\,\mathrm{d}x\\&=\int_\Omega(-\Delta w_1-f(w_1))u\,\mathrm{d}x.\end{aligned}$$

Note that

$$\begin{aligned}\langle J'(w_1), u\rangle_{E',E} &= \langle v, u\rangle \\ &= \int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x \\ &= \int_\Omega (-\Delta v) u \,\mathrm{d}x.\end{aligned}$$

Therefore, v can be determined by the following linear elliptic equation:

$$\begin{aligned}\Delta v &= \Delta w_1 + f(w_1) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{3.1}$$

Eq. (3.1) can be solved by any numerical elliptic solver. Even though it is theoretically true that $v \in \mathrm{Fix}(G)$, the numerical solution v of (3.1) may however not be in $\mathrm{Fix}(G)$ due to round off errors. Therefore, v should be replaced by the average of $\{T(k)v \mid k \in G\}$ to guarantee $v \in \mathrm{Fix}(G)$. □

Inspired by the proof of Theorem 2.1 and the MPA described above, we propose a numerical algorithm, called the Modified Mountain Pass Algorithm (MMPA), to compute sign-changing solutions of (1.1) in $\mathrm{Fix}(G)$. The algorithm is described below.

Modified Mountain Pass Algorithm

Step 1: Take an initial guess $w_0 \in \mathrm{Fix}(G)$ such that $(w_0)_+ \neq 0$, $(w_0)_- \neq 0$, $J((w_0)_+) \leqslant 0$, and $J((w_0)_-) \leqslant 0$;

Step 2: Find $t_1^* > 0$ such that $J(t_1^*(w_0)_+) = \max_{t\in[0,\infty)} J(t(w_0)_+)$, and find $t_2^* > 0$ such that $J(t_2^*(w_0)_-) = \max_{t\in[0,\infty)} J(t(w_0)_-)$, and set $w_1 = t_1^*(w_0)_+ + t_2^*(w_0)_- \in \mathscr{S}_1$;

Step 3: Compute $\nabla J(w_1)$ and set $v = \nabla J(w_1) \in \mathrm{Fix}(G)$;

Step 4: If $\|v\| \leqslant \varepsilon$, then output w_1 and stop;
else, goto the next step;

Step 5: Let $w = -v + w_1$ and find $t^* > 0$ such that $J(t^* w) = \max_{t\in[0,\infty)} J(tw)$;

Step 6: If $J(t^* w) < J(w_1)$, set $w_0 = t^* w \in \mathscr{S}$ and goto Step 2;
else, set $v = \frac{1}{2} v$ and goto Step 5.

Remark 3.2. The idea of splitting a sign-changing function into its positive and negative parts and implementing the steepest descent technique was also used in [16].

Remark 3.3. Instead of projecting w_0 to $\mathscr{S}$ as in the MPA, Step 2 of the MMPA projects w_0 to $\mathscr{S}_1$, which is the only difference between the MPA and MMPA. It should be pointed out that even though the MMPA works on some numerical examples to be presented in the next section, this algorithm, however, has two disadvantages in implementation. One is that the choice of w_0 in Step 1 is sometimes difficult if the geometry of Ω is complicated. The other is that the convergence is not stable. Even though Step 6 forces the functional value of J to decrease on $\mathscr{S}$ along the steepest descent direction, the value of J, however, may increase or oscillate on $\mathscr{S}_1$. Enforcing that the value J decreases on $\mathscr{S}_1$ along the steepest descent direction in Step 6 will put the algorithm to death due to the nature of line searching of Step 2 and the fact that the codimension of a tangent space to $\mathscr{S}_1$ is 2 in general. These disadvantages have been overcome in the next algorithm.

Inspired by an elegant work by Wang [20], who proved that (1.1) has at least three nontrivial solutions by using linking and Morse-type arguments, a High-Linking Algorithm (HLA) for computing sign-changing solutions of (1.1) on a general domain was proposed in [11]. The idea of the HLA is to construct a local linking from a known critical point, which is a local variant of the global linking in [20], leading to a new critical point. To construct such a linking, one needs to have some knowledge of the local behavior of the functional $J(w)$ at the known critical points. Since the local behavior of the functional $J(w)$ at a positive and a negative mountain pass solution is known [20], one is able to find sign-changing solutions from them by using the HLA. By adopting the HLA in $\mathrm{Fix}(G)$, we introduce the following algorithm called again the High-Linking Algorithm since there is no significant difference between the HLA in [11] and the algorithm below.

The HLA preassumes that a mountain pass solution $w_1 \in \mathrm{Fix}(G)$ of (1.1) is given or computed by the MPA. It is known [13] that the Morse index of w_1 is either 1 or 0, where the Morse index of w_1 is defined as the maximal dimension of a subspace where $J''(w_1)$ is negative definite. By applying the Morse Lemma [5], one shows [20] that there exist $u_1, u_2 \in \mathrm{Fix}(G)$, and a $\delta > 0$ such that

$$J(w_1 + tu_1) < J(w_1), \quad J(w_1 + tu_2) > J(w_1), \quad \text{for } 0 < |t| \leqslant \delta. \tag{3.2}$$

Thus a local linking can be constructed by using $\{w_1, u_1, u_2\}$. The HLA is described below.

High-Linking Algorithm

Step 1: Take an initial guess $w_0 \in \mathrm{Fix}(G)$ such that $w_0 \neq 0$ and $J(w_0) \leqslant 0$;
Step 2: Apply the MPA to find a mountain pass solution $w_1 \in \mathrm{Fix}(G)$, and $u_1, u_2 \in \mathrm{Fix}(G)$ satisfying (3.2);
Step 3: Find $t_1 > 0$ and $t_2 < 0$ such that $J(w_1 + t_1 u_1) \leqslant 0$ and $J(w_1 + t_2 u_1) \leqslant 0$, and set $g_1 = w_1 + t_1 u_1$ and $g_2 = w_1 + t_2 u_1$;
Step 4: Find $t_3 > 0$ such that $J(w_1 + t_3 u_2) \leqslant J(w_1)$, and set $g_3 = w_1 + t_3 u_2$;
Step 5: Construct the triangle $\triangle$ in $\mathrm{Fix}(G)$ by
$$\triangle = \{\lambda_1 g_1 + \lambda_2 g_2 + (1 - \lambda_1 - \lambda_2) g_3, \mid \lambda_1 \geqslant 0,\ \lambda_2 \geqslant 0,\ \lambda_1 + \lambda_2 \leqslant 1\},$$
and find $w^* \in \triangle$ such that $J(w^*) = \max_{g \in \triangle} J(g)$;
Step 6: If w^* is an interior point of $\triangle$, then goto the next step; else, set $u_2 = w^* - w_1$ and goto Step 4;
Step 7: Set $w_2 = w^*$, compute $\nabla J(w_2)$ and set $v = \nabla J(w_2) \in \mathrm{Fix}(G)$;
Step 8: If $||v|| \leqslant \varepsilon$, then output w_2 and stop; else, set $u_2 = (-v + w_2) - w_1$ and goto the next step;
Step 9: Repeat the same procedures as Steps $4 - 6$ to construct a new triangle $\triangle$ and to find an interior point $w^* \in \triangle$ such that $J(w^*) = \max_{g \in \triangle} J(g)$;
Step 10: If $J(w^*) < J(w_2)$, goto Step 7; else, set $v = \frac{1}{2} v$ and $u_2 = (-v + w_2) - w_1$, and goto Step 9.

Remark 3.4. A more detailed discussion and justification of the HLA can be found in [11]. Our numerical examples (to be introduced in the next section) have indicated that the HLA is convergent and stable, and is more reliable than the MMPA due to the fact pointed out in Remark 3.3.

Remark 3.5. In Step 2, the direction u_1 can be obtained by $u_1 = w_1/||w_1||$, and the direction u_2 can be found while computing w_1 by the MPA. When implementing Step 5 of the MPA, we set $u_2 = w_1 - t^* w$ and normalize u_2 by taking $u_2/||u_2||$. If a mountain pass solution w_1 is obtained, then u_2 is obtained also. Both u_1 and u_2 found in this way will satisfy (3.2), which was justified in [11].

Remark 3.6. The implementations of Steps 5 and 6 are explicit. The maximum point w^* will not be at any of the three corners, but it may be on the line segment joining g_1 and g_3 or on the line segment joining g_2 and g_3. If this is the case, we need to adjust the triangle $\triangle$ along the direction $w^* - w_1$. Note that all of these adjusted triangles are on the same plane, and the functional J can always be expressed as $J(w_1 + tu_1 + su_2)$. Under assumption (A3), for any given nonzero $u \in \mathrm{Fix}(G)$, there is always a $t_0 > 0$ such that $J(w_1 + t_0 u) \leqslant 0$. Thus, after a finite number of adjustments of $\triangle$, we can find w^* as an interior point of $\triangle$ such that $J(w^*) = \max_{g \in \triangle} J(g)$. It must be pointed out that the adjustment of $\triangle$ in such a way that w^* is an interior point is crucial for the linking argument. Without such adjustments of $\triangle$, the HLA may converge to the mountain pass solution w_1, and thus no additional solution is generated.

4. Numerical examples

In this section, we will apply both MMPA and HLA to the superlinear Dirichlet problem (1.1) on Ω with three typical geometries: equilateral triangle, square and disc. For all examples discussed in this section, conditions (A1)–(A4) are satisfied. We use the finite element method with linear splines to approximate solutions of the Dirichlet problems, and employ the uniform finite element triangulation on Ω that is Z_n–rotational symmetric, and fix the partition number of Ω to be 100. A detailed discussion of the finite element method can be found, for example, in [19]. We set $\varepsilon = 1.0 \times 10^{-2}$ as the control of convergence in all numerical computations of this paper, and use the Maple software to graph numerical solutions.

Example 4.1. Let $\Omega = \{(x_1, x_2) \in \Re^2 | -1 < x_1 < 1, 0 < x_2 < \sqrt{3}(1 - |x_1|)\}$. Let $p > 1$ and $q > 1$. Consider the Dirichlet problem

$$\begin{cases} -\Delta w = w_+^p + w_- |w_-|^{q-1} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Define

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - \frac{1}{p+1}\int_\Omega w_+^{p+1} \, dx - \frac{1}{q+1}\int_\Omega |w_-|^{q+1} \, dx.$$

It is easy to verify that $w = 0$ is a (trivial) solution of (4.1), and a local minimum of $J(w)$. Let $G = Z_3$, and $x \in \Omega$ be represented by the polar coordinates (r, θ). Define the representation $\{T(k)\}_{k \in Z_3}$ on $H = H_0^1(\Omega)$ by rotations

$$T(k)u(r, \theta) = u\left(r, \theta + \frac{2k\pi}{3}\right), \quad k \in Z_3.$$

We are interested in sign-changing solutions in $\mathrm{Fix}(Z_3)$.

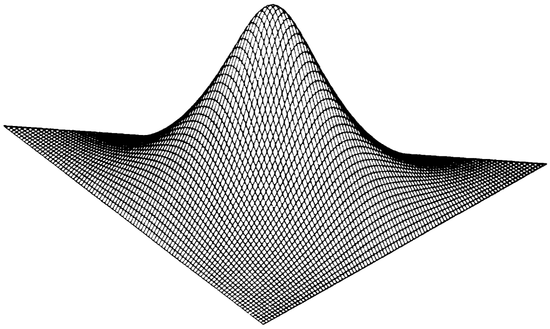

Fig. 1. Positive mountain pass solution of (4.1) with $p = q = 5$.

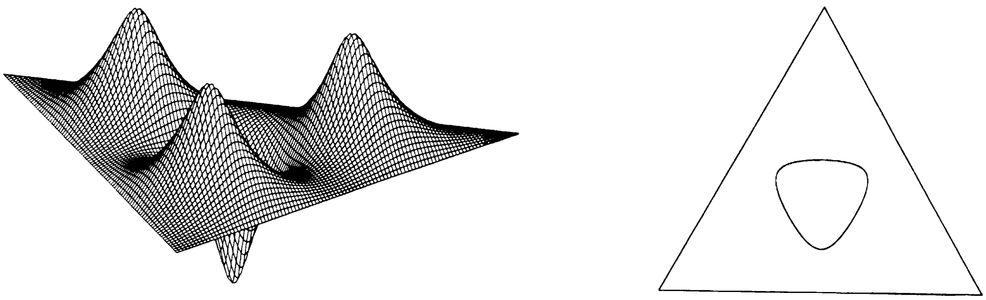

Fig. 2. A sign-changing solution of (4.1) with $p = q = 5$.

We first consider the odd nonlinearity case: $p = q$. By letting $w_0(x) = 10x_2(x_2 - \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 - \sqrt{3})$ and $p = q = 5$, the positive mountain pass solution of (4.1) obtained by applying the MPA is shown in Fig. 1, where $J(w) = 7.3859$ and $\max_{x\in\Omega} w(x) = 2.8376$. The negative mountain pass solution of (4.1) with $p = q = 3$, which is the negative of the solution shown in Fig. 1, can be obtained by letting $w_0(x) = -10x_2(x_2 - \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 - \sqrt{3})$ and by applying the MPA. By applying the HLA at the positive mountain solution shown in Fig. 1, we obtain a sign-changing solution of (4.1) with $p = q = 5$. The solution and its nodal curve are shown in Fig. 2, where $J(w) = 60.4952$, $\max_{x\in\Omega} w(x) = 3.8994$ and $\min_{x\in\Omega} w(x) = -4.5047$.

By letting $w_0(x_1, x_2) = 10x_2(x_2 - \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1)(x_2 - \sqrt{3}x_1)(x_2 - 0.5\sqrt{3})$ and by applying the MMPA to (4.1) with $p = q = 5$, we obtain a different sign-changing solution. The solution and its nodal curve are shown in Fig. 3, where $J(w) = 59.6959$, $\max_{x\in\Omega} w(x) = 4.1560$ and $\min_{x\in\Omega} w(x) = -3.7277$. This solution can be also reproduced by applying the HLA with $u_2 = x_2(x_2 - \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 + 0.2\sqrt{3})(x_2 - \sqrt{3}x_1 + 0.2\sqrt{3})(x_2 - 0.6\sqrt{3})$. The comparison between the HLA and the MMPA is given in Table 1.

Table 1 shows that both algorithms are effective on computing the similar type of solutions as shown in Fig. 3. We wish to point out that, due to its nature of looking for the infimum of $J(w)$ on $\mathscr{S}_1$, the MMPA cannot reproduce the solution shown in Fig. 2 because its functional value $J(w)$ is greater than that of the solution shown in Fig. 3.

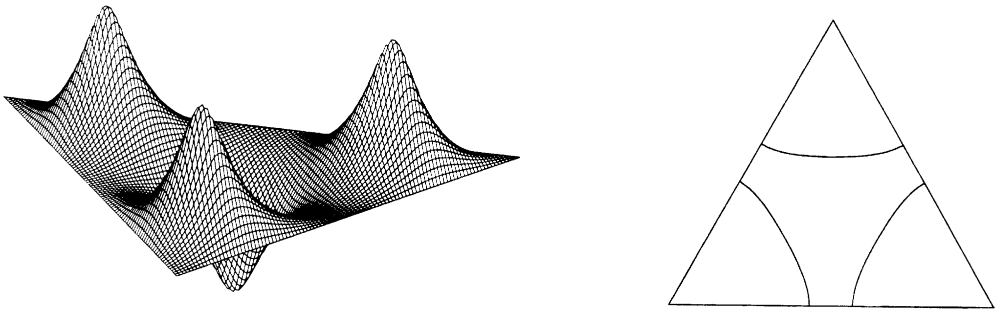

Fig. 3. A sign-changing solution of (4.1) with $p=q=5$.

Table 1
Comparison between the HLA and the MMPA

	$p=q=3$			$p=q=5$		
	$J(w)$	$\max w$	$\min w$	$J(w)$	$\max w$	$\min w$
MMPA	408.9845	10.8102	−10.7336	59.6959	4.1560	−3.7277
HLA	409.2215	10.9053	−10.6563	59.7447	4.1458	−3.7131

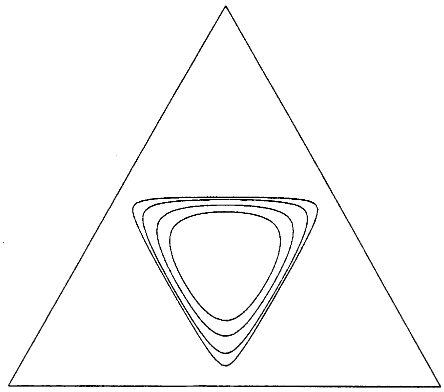

Fig. 4. Comparison of nodal curves.

Even though both sign-changing solutions shown in Figs 2 and 3 have similar profiles, they are indeed very different. The solution shown in Fig. 3 has a lower value of $J(w)$ than that of the solution shown in Fig. 2. Furthermore, the nodal curves of both solutions have completely different geometries. The comparison of nodal curves of sign-changing solutions similar to Fig. 2 is shown in Fig. 4, where the nodal curves from the outside to the inside correspond to $p=q=3$, $p=q=4$, $p=q=5$ and $p=q=6$. As the value of $p=q$ increases, the nodal curve shrinks toward the center of Ω. The comparison of nodal curves of sign-changing solutions similar to Fig. 3 is shown in Fig. 5, where

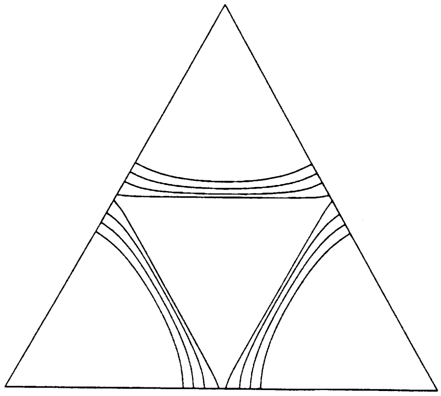

Fig. 5. Comparison of nodal curves.

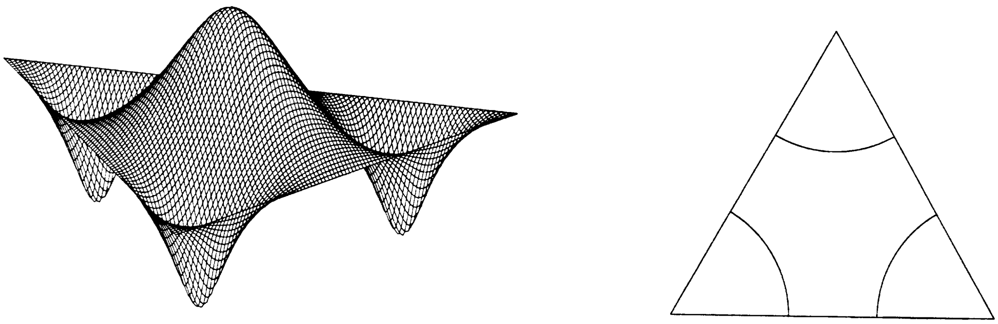

Fig. 6. A sign-changing solution of (4.1) with $p=3$ and $q=4$.

the nodal curves from the inside to the outside correspond to $p=q=3$, $p=q=4$, $p=q=5$ and $p=q=6$. As the value of $p=q$ increases, the nodal curve moves to the corners of Ω.

Next we consider the non odd nonlinearity case $p \neq q$. By letting $w_0(x) = 10x_2(x_2 - \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 - \sqrt{3})$, and by applying the HLA at the positive mountain pass solution, we obtain a sign-changing solution of (4.1) with $p = 3$ and $q = 4$. The solution and its nodal curve are shown in Fig. 6, where $J(w) = 155.1658$, $\max_{x\in\Omega} w(x) = 7.1790$ and $\min_{x\in\Omega} w(x) = -6.3139$. By applying the HLA at the negative mountain pass solution, we obtain another sign-changing solution of (4.1) with $p = 3$ and $q = 4$. The solution and its nodal curve are shown in Fig. 7, where $J(w)=278.4613$, $\max_{x\in\Omega} w(x)=8.6114$ and $\min_{x\in\Omega} w(x)=-9.2286$. By letting $w_0(x_1,x_2)=10x_2(x_2-\sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1 - \sqrt{3})(x_2 + \sqrt{3}x_1)(x_2 - \sqrt{3}x_1)(x_2 - 0.5\sqrt{3})$, the MMPA can reproduce the solution shown in Fig. 6. However, the MMPA fails to reproduce the solution shown in Fig. 7 due to the instability of the algorithm discussed in Remark 3.3 and due to the fact that the support of the negative part of the solution shown in Fig. 7 is relatively small.

An important observation from our numerical computation of sign-changing solutions of (4.1) in $\mathrm{Fix}(Z_3)$ is that we obtained at least four sign-changing solutions for the odd nonlinearity case $p=q$, and at least two sign-changing solutions for the non odd nonlinearity case $p \neq q$. Further numerical

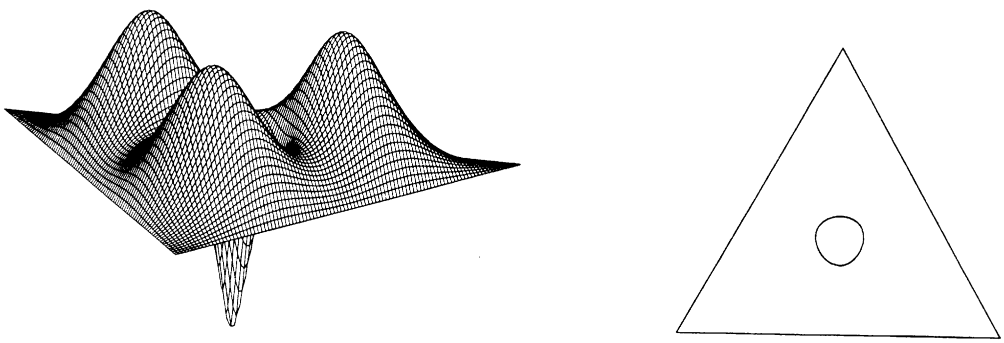

Fig. 7. A sign-changing solution of (4.1) with $p=3$ and $q=4$.

investigations have revealed that if $p \neq q$ and $|p-q|$ is very small, then we can obtain at least four sign-changing solutions. Due to the length of this paper, we omit the detail here.

Example 4.2. Let $\Omega = \{(x_1, x_2) \in \Re^2 | 0 < x_1 < 1, 0 < x_2 < 1\}$. Let $p > 1$ and $q > 1$. Consider the Dirichlet problem

$$\begin{cases} -\Delta w = w_+^p + w_-|w_-|^{q-1} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Define

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_\Omega w_+^{p+1} \,\mathrm{d}x - \frac{1}{q+1}\int_\Omega |w_-|^{q+1} \,\mathrm{d}x.$$

One can check that $w=0$ is a (trivial) solution of (4.2), and a local minimum of $J(w)$. Let $G = Z_4$, and $x \in \Omega$ be represented by the polar coordinates (r, θ). Define the representation $\{T(k)\}_{k \in Z_4}$ on $H = H_0^1(\Omega)$ by rotations:

$$T(k)u(r,\theta) = u\left(r, \theta + \frac{k\pi}{2}\right), \quad k \in Z_4$$

We are interested in sign-changing solutions in $\mathrm{Fix}(Z_4)$.

We first consider the odd nonlinearity case: $p=q$. By letting $p=q=5$ and $w_0(x) = 20\sin(\pi x_1)\sin(\pi x_2)$, the positive mountain pass solution of (4.2) obtained by applying the MPA is shown in Fig. 8, where $J(w) = 8.9591$ and $\max_{x\in\Omega} w(x) = 3.1610$. The negative mountain pass solution of (4.2) with $p=q=5$, which is the negative of the solution shown in Fig. 8, can be obtained by letting $w_0(x) = -20\sin(\pi x_1)\sin(\pi x_2)$ and by applying the MPA. By applying the HLA at the positive mountain pass solution shown in Fig. 8, we obtain a sign-changing solution of (4.2) with $p=q=5$. The solution and its nodal curve are shown in Fig. 9, where $J(w) = 97.4241$, $\max_{x\in\Omega} w(x) = 4.2496$ and $\min_{x\in\Omega} w(x) = -5.9402$. By letting $w_0(x) = 20\sin(\pi x_1)\sin(\pi x_2)(x_1+x_2-0.4)(x_1+x_2-1.6)(x_1-x_2-0.6)(x_1-x_2+0.6)$, the MMPA produces the same solution as shown in Fig. 9. The comparison of nodal curves of sign-changing solutions similar to Fig. 9 is shown in Fig. 10, where the nodal curves from the outside to the inside correspond to $p=q=3$, $p=q=4$, $p=q=5$, $p=q=6$ and $p=q=7$. As the value of $p=q$ increases, the nodal curve shrinks toward the center of Ω.

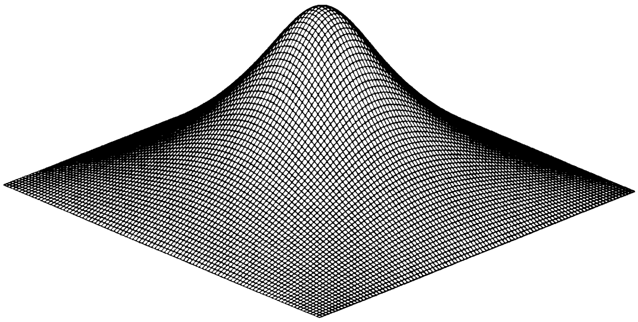

Fig. 8. Positive mountain pass solution of (4.2) with $p = q = 5$.

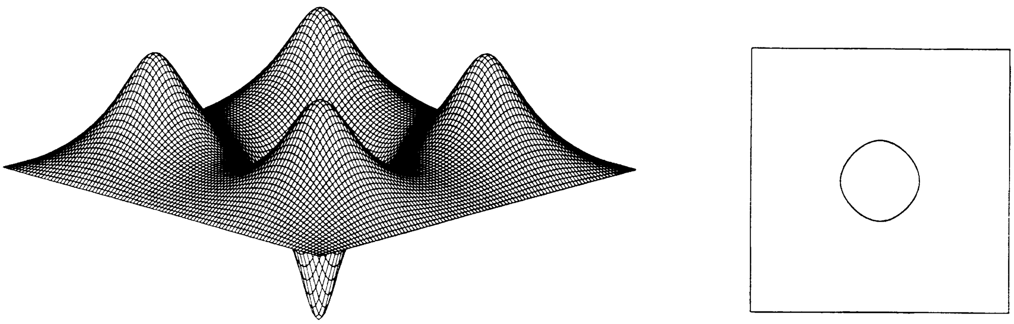

Fig. 9. A sign-changing solution of (4.2) with $p = q = 5$.

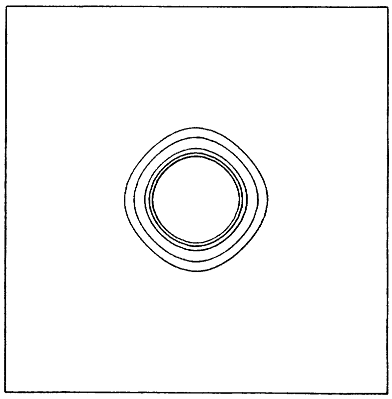

Fig. 10. Comparison of nodal curves.

Unlike the equilateral triangle case in Example 4.1, we obtained at least two sign-changing solutions in $\mathrm{Fix}(Z_4)$.

Next we consider the non odd nonlinearity case $p \neq q$. By letting $w_0(x) = 20\sin(\pi x_1)\ \sin(\pi x_2)$, and by applying the HLA at the positive mountain pass solution, we obtain a sign-changing solution

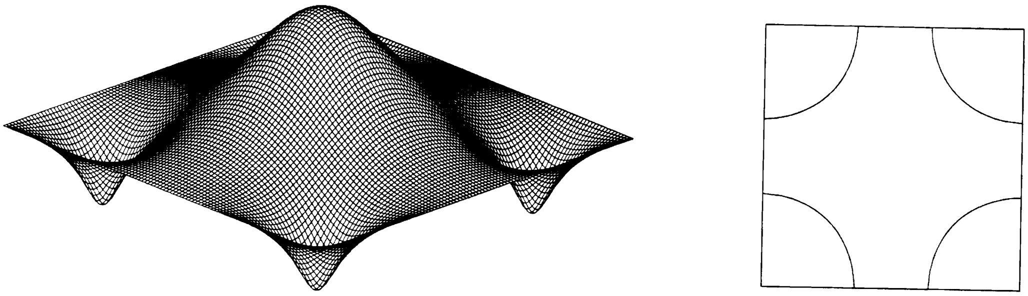

Fig. 11. A sign-changing solution of (4.2) with $p = 3$ and $q = 5$.

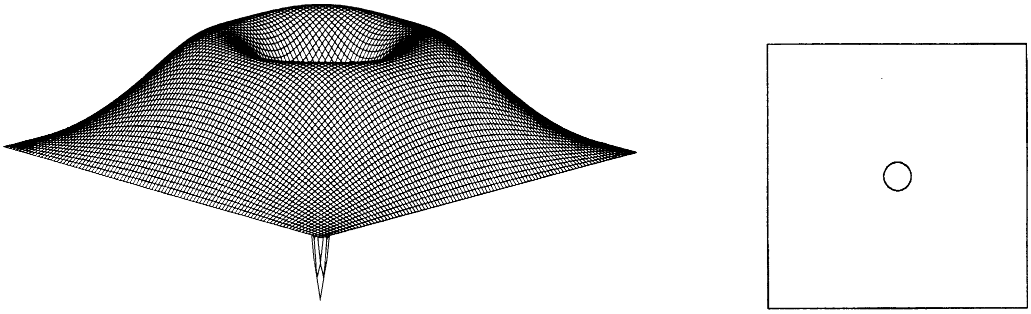

Fig. 12. A sign-changing solution of (4.2) with $p = 3$ and $q = 5$.

of (4.2) with $p=3$ and $q=5$. The solution and its nodal curve are shown in Fig. 11, where $J(w) = 188.0203$, $\max_{x\in\Omega} w(x) = 9.1191$ and $\min_{x\in\Omega} w(x) = -5.6627$. By applying the HLA at the negative mountain pass solution, we obtain another sign-changing solution of (4.2) with $p=3$ and $q=5$. The solution and its nodal curve are shown in Fig. 12, where $J(w)=274.2947$, $\max_{x\in\Omega} w(x)=7.4345$ and $\min_{x\in\Omega} w(x)=-9.6297$. By letting $w_0(x_1,x_2)=20\sin(\pi x_1)\sin(\pi x_2)(x_1+x_2-0.4)(x_1+x_2-1.6)(x_1-x_2-0.6)(x_1-x_2+0.6)$, the MMPA can reproduce the solution shown in Fig. 11. However, the MMPA fails to reproduce the solution shown in Fig. 12 due to the instability of the algorithm discussed in Remark 3.3 and due to the fact that the support of the negative part of the solution shown in Fig. 12 is relatively small.

Example 4.3. Let $\Omega = \{(x_1,x_2) \in \Re^2 \,|\, x_1^2 + x_2^2 < 1\}$. Let $p > 1$ and $q > 1$. Consider the Dirichlet problem

$$\begin{cases} -\Delta w = w_+^p + w_-|w_-|^{q-1} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Define

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x - \frac{1}{p+1}\int_\Omega w_+^{p+1}\,\mathrm{d}x - \frac{1}{q+1}\int_\Omega |w_-|^{q+1}\,\mathrm{d}x.$$

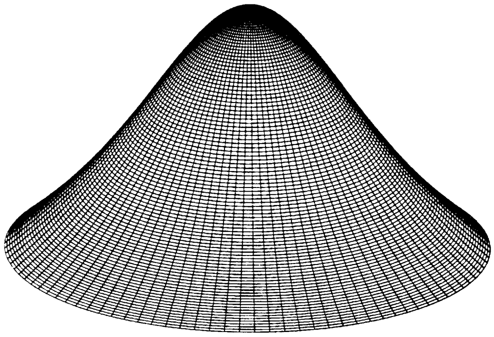

Fig. 13. Positive mountain pass solution of (4.3) with $p=q=3$.

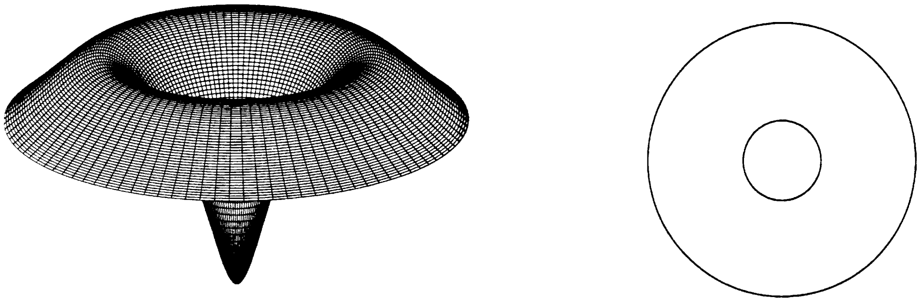

Fig. 14. A sign-changing solution of (4.3) with $p=q=3$.

One can verify that $w=0$ is a (trivial) solution of (4.3), and a local minimum of $J(w)$. Let $G=S^1=\mathfrak{R}/2\pi\mathscr{Z}$, and $x\in\Omega$ be represented by the polar coordinates (r,θ). Define the representation $\{T(s)\}_{k\in S^1}$ on $H=H_0^1(\Omega)$ by

$$T(s)u(r,\theta)=u(r,\theta+s),\quad s\in S^1.$$

We are interested in sign-changing solutions in $\mathrm{Fix}(S^1)$.

We first consider the odd nonlinearity case: $p=q$. By letting $w_0(x)=20\cos((\pi/2)\sqrt{x_1^2+x_2^2})$ and $p=q=3$, the positive mountain pass solution of (4.3) obtained by applying the MPA is shown in Fig. 13, where $J(w)=11.3232$ and $\max_{x\in\Omega}w(x)=3.5937$. The negative mountain pass solution of (4.3) with $p=q=3$, which is the negative of the solution shown in Fig. 13, can be obtained by letting $w_0(x)=-20\cos((\pi/2)\sqrt{x_1^2+x_2^2})$ and by applying the MPA. By applying the HLA at the positive mountain solution shown in Fig. 8, we obtain a sign-changing solution of (4.2) with $p=q=3$. The solution and its nodal curve are shown in Fig. 14, where $J(w)=306.5837$, $\max_{x\in\Omega}w(x)=5.2555$ and $\min_{x\in\Omega}w(x)=-12.9917$. By letting $w_0(x)=-20\cos((\pi/2)\sqrt{x_1^2+x_2^2})$, the MMPA produces the same solution as shown in Fig. 14, but is unable to reproduce those sign-changing solutions for $p=q$ relatively large. The comparison of nodal curves of sign-changing solutions similar to Fig. 14 is shown in Fig. 15, where the nodal curves from the outside to the inside correspond to $p=q=3$, $p=q=4$, $p=q=5$, $p=q=6$ and $p=q=7$. As the value of $p=q$ increases, the nodal curve shrinks toward the center of Ω.

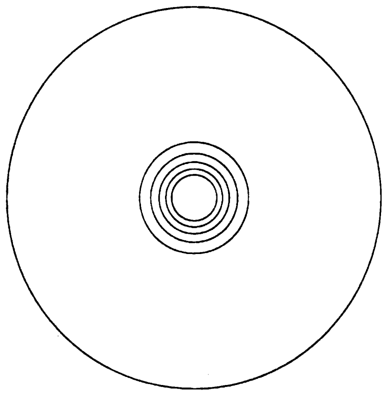

Fig. 15. Comparison of nodal curves.

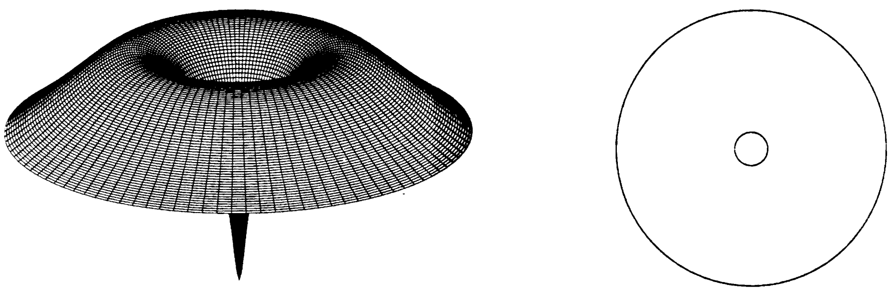

Fig. 16. A sign-changing solution of (4.3) with $p=3$ and $q=5$.

Next, we consider the non-odd nonlinearity case $p \neq q$. By applying the HLA at the positive mountain pass solution and by letting $w_0(x) = 20\cos((\pi/2)\sqrt{x_1^2+x_2^2})$, we obtain a sign-changing solution of (4.3) with $p=3$ and $q=5$. The solution and its nodal curve are shown in Fig. 16, where $J(w) = 106.8036$, $\max_{x\in\Omega} w(x) = 4.1372$ and $\min_{x\in\Omega} w(x) = -6.8617$. By applying the HLA at the negative mountain pass solution, we obtain another sign-changing solution of (4.3) with $p=3$ and $q=5$. The solution and its nodal curve are shown in Fig. 17, where $J(w)=146.7561$, $\max_{x\in\Omega} w(x)= 9.4084$ and $\min_{x\in\Omega} w(x) = -2.6805$. However, the MMPA cannot reproduce the solutions shown in Figs. 16 and 17. In other words, no matter how to choose the initial guess, the MMPA diverges. The failure of the MMPA to generate sign-changing solutions in this case may be due to the nature of the MMPA and the instability of the algorithm discussed in Remark 3.3. Among two sign-changing solutions shown in Figs. 16 and 17, the one shown in Fig. 16 has the lower value of $J(w)$. One can also note that the support of the negative part of this solution is relatively small. Since the MMPA can produce in general sign-changing solutions with the smallest energy $J(w)$ on $\mathscr{S}_1$, it would generate the solution shown in Fig. 16 instead of that shown in Fig. 17 if it worked out. Due to the relatively small support of the negative part of the solution shown Fig. 16, the instability problem pointed out in Remark 3.3 becomes more significant in this example than in the previous two examples.

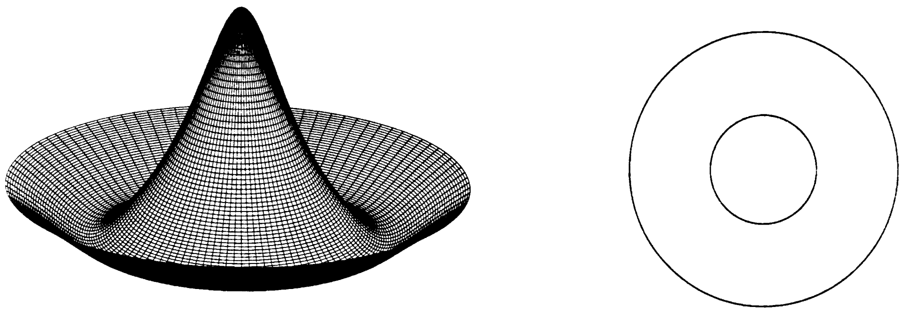

Fig. 17. A sign-changing solution of (4.3) with $p=3$ and $q=5$.

5. Some further discussions

For the two numerical algorithms given in this paper for sign-changing solutions of superlinear Dirichlet problems, our numerical examples indicate that each of them has certain advantages and disadvantages. Before giving further discussions on these, we wish to point out first that the group G in both algorithms could not only be groups reflecting the domain symmetries as discussed in the examples, but also be some other group actions. As a typical example, by letting G be the trivial group, then both algorithms can be applied to a superlinear Dirichlet problem on any domain. The numerical investigation of sign-changing solutions of a superlinear Dirichlet problem on symmetric and nonsymmetric domains is given in [11].

The MMPA is a modified version of the MPA, and is then algorithmically simple. First, this algorithm can produce positive and negative mountain pass solutions of a superlinear Dirichlet problem if the initial guesses are positive or negative functions in $\mathrm{Fix}(G)$. In this case, the MMPA is the same as the MPA, which is a very stable algorithm for mountain pass solutions (see the detailed discussion in [8]). For the MPA, one can also use sign-changing functions as initial guesses. Second, the MMPA can also produce those sign-changing solutions of a superlinear Dirichlet problem with the smallest energy in $\mathscr{S}_1$ provided it is convergent. However, as pointed out in Remark 3.3, this algorithm is not stable as shown in the Examples in Section 3. Furthermore, if the sign-changing solution with the smallest energy $J(w)$ on $\mathscr{S}_1$ has a relatively small support for its positive or negative parts, then the MMPA may be dead in Step 2. It seems that the local grid adaptive technique may help to overcome this problem. However, due to the uncertainty of the location and geometry of the support, such a technique is extremely difficult to apply.

In contrast to the MMPA, the HLA is a very stable algorithm, and can provide more reliable sign-changing solutions of a superlinear Dirichlet problem on any domain. For any superlinear Dirichlet problem, this algorithm can always produce at least two sign-changing solutions (including those sign-changing solutions found by the MMPA). However, this algorithm is not algorithmically simple as the MMPA. Since implementing the MMPA or the MPA for mountain pass solutions is a necessary step for implementing the HLA, one should be recommended to test the MMPA first for sign-changing solutions, if one is only interested in the sign-changing solutions with the smallest energy $J(w)$ on $\mathscr{S}_1$ and if the MMPA is convergent. However, at present, the HLA is the only reliable numerical algorithm for sign-changing solutions of a superlinear Dirichlet problem on any domain.

By using the HLA and the MMPA, we have investigated sign-changing solutions in Fix(G) for superlinear elliptic equations on symmetric domains, and have obtained at least two numerical sign-changing solutions in Fix(G). For a superlinear Dirichlet problem with an odd nonlinearity, the two sign-changing solutions with the smallest energy $J(w)$ on $\mathscr{S}_1$ obtained by either the HLA or the MMPA are the negative of each other, and have the same nodal curves. Except for the equilateral triangle case, Figs 10 and 15 indicate that the nodal curves shrink toward the center of the domain when $p = q$ increases.

Among three domain geometries, it was observed that the equilateral triangle discussed in Example 4.1 displays an interesting bifurcation phenomenon. We found that a superlinear Dirichlet problem on an equilateral triangle may have at least four sign-changing solutions in Fix(G) for the odd nonlinearity case, and may have at least two sign-changing solutions in Fix(G) for the the non-odd nonlinearity case. As remarked at the end of Example 4.1, if $p \neq q$ and $|p - q|$ is very small, then we can obtain also at least four sign-changing solutions. This observation suggests that an interesting bifurcation phenomenon may exist for the number of nontrivial sign-changing solutions of (4.1). More precisely, it seems that there is a constant $\delta(\Omega) > 0$ such that when $|p - q| < \delta(\Omega)$, (4.1) has at least four sign-changing solutions, and when $|p - q| > \delta(\Omega)$, (4.1) has at least two sign-changing solutions. Our numerical examples have indicated that superlinear Dirichlet problems on other equilateral polygons or disc do not display that phenomenon.

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