A numerical method for finding sign-changing solutions of superlinear Dirichlet problems

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Abstract. In a recent result (See Castro-Cossio-Neuberger [2]), it was shown via a variational argument that a class of superlinear elliptic boundary value problems has at least three nontrivial solutions, a pair of one sign and one which sign changes exactly once. These three and all other nontrivial solutions are saddle points of an action functional, and are characterized as local minima of that functional restricted to a codimension one submanifold of the Hilbert space $H_0^{1,2}(\Omega)$ or appropriate higher codimension subsets of that manifold.

In this paper we present a numerical Sobolev steepest descent algorithm (see [10], [11], and [12]) for finding these three solutions. Of primary interest is the method of projecting iterates of elements in $H_0^{1,2}(\Omega)$ onto the submanifold and its subsets. When applied to the ordinary differential equation, the algorithm is extended to find additional solutions possessing a greater number of internal zeroes, and in that case the solutions are compared to independent numerical calculations obtained by Euler's method. We further test the algorithm on partial differential equations on the unit square. With or without a symmetric nonlinearity, numerical computations for PDEs on the square yield four exactly-once sign-changing solutions of *Morse Index* 2 and supply evidence that suggests that there may exist four more of MI 3.

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1 Introduction

In this section we define our elliptic PDE, define the action functional whose critical points are the solutions we seek, and state the main existence result from [2]. In Section 2 we briefly summarize the method of proof in that work. The algorithm presented in Section 3 is based on the proof of that result, where the primary innovation lies in a projection of sign-changing functions onto a special subset of a codimension 1 submanifold of $H_0^{1,2}(\Omega)$. In Sections 4 and 5 we present numerical data for several examples. Section 6 contains a summary of interesting phenomena observed in our experiments and outlines several conjectures and future research projects.

Let Ω be a smooth bounded region in $\mathbf{R}^N$, Δ the Laplacian operator, and $f \in C^1(\mathbf{R}, \mathbf{R})$ such that $f(0) = 0$. We seek solutions to the boundary value problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{1}$$

We assume that there exist constants $A > 0$ and $p \in (1, \frac{N+2}{N-2})$ such that $|f'(u)| \leq A(|u|^{p-1} + 1)$ for all $u \in \mathbf{R}$. It follows that f is <u>subcritical</u>, i.e., there exists $B > 0$ such that $|f(u)| \leq B(|u|^p + 1)$. Also, we assume that there exists $m \in (0, 1)$ such that

$$\frac{m}{2} f(u)u \geq F(u), \tag{2}$$

where $F(u) = \int_0^u f(s)\, ds$, for all $u \in \mathbf{R}$. A vital assumption that we make is that f is <u>superlinear</u>, i.e.,

$$\lim_{|u| \to \infty} \frac{f(u)}{u} = \infty. \tag{3}$$

Finally, we make the assumption that f satisfies

$$f'(u) > \frac{f(u)}{u} \quad \text{for } u \neq 0. \tag{4}$$

Let H be the Sobolev space $H_0^{1,2}(\Omega)$, in which case the zero Dirichlet conditions allow the inner product $\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v\, dx$ (see [1], [6], or [10]). We define the action functional $J : H \to \mathbf{R}$ by

$$J(u) = \int_\Omega \{\frac{1}{2}|\nabla u|^2 - F(u)\}\, dx.$$

One easily sees that critical points of J are weak solutions to (1). In fact, by regularity theory for elliptic boundary value problems (see [6]), u is a solution (classical) to (1) if and only if u is a critical point of the action functional J.

Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. The following result is proved in [2]:

Theorem 1.1 *If $f'(0) < \lambda_1$, then (1) has at least three nontrivial solutions: $\omega_1 > 0$ in Ω, $\omega_2 < 0$ in Ω, and ω_3. The function ω_3 changes sign exactly once in Ω, i.e., $(\omega_3)^{-1}(\mathbf{R} - \{0\})$ has exactly two connected components. If nondegenerate, the one-sign solutions are Morse index 1 critical points of J, and the sign-changing solution has Morse index 2. Furthermore,*

$$J(\omega_3) \ge J(\omega_1) + J(\omega_2).$$

To the best of our knowledge, the above theorem is the first to establish the existence of a *sign-changing* solution to (1). We note that if $f'(0) > \lambda_1$, then by multiplying (1) by an eigenfunction corresponding to λ_1 and integrating by parts, it is easily seen that (1) does not have one-signed solutions. We conjecture that a pair of sign-changing exactly-once solutions exist provided that $f'(0) < \lambda_2$. This is known to be true for the ODE and was numerically observed in one of our experiments for a specific PDE. We will use the terminology “1-Hump” function to refer to functions with no internal zeroes, “2-Hump” for exactly-once sign-changing functions with exactly one connected internal zero set, and so on.

2 Variational characterization

In this section we include some definitions and theorems from [2]. Our assumptions on f imply that $J \in C^2(H, \mathbf{R})$ (see [13]), and that

$$J'(u)(v) = \langle \nabla J(u), v \rangle = \int_\Omega \{(\nabla u \cdot \nabla v - f(u)v)\}\, dx, \quad \text{for all } v \in H. \tag{5}$$

Define $\gamma : H \to \mathbf{R}$ by $\gamma(u) = \langle \nabla J(u), u \rangle = \int_\Omega \{|\nabla u|^2 - uf(u)\}\, dx$, and compute

$$\begin{aligned} \gamma'(u)(v) &= \langle \nabla \gamma(u), v \rangle \\ &= 2\textstyle\int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega f(u)v\, dx - \int_\Omega f'(u)uv\, dx. \end{aligned} \tag{6}$$

Definition 2.1 *For $u \in L^1(\Omega)$, we define $u_+(x) = \max\{u(x), 0\} \in L^1(\Omega)$ and $u_-(x) = \min\{u(x), 0\} \in L^1(\Omega)$. If $u \in H$ then $u_+, u_- \in H$ (see [5]). We say that $u \in L^1(\Omega)$ changes sign if $u_+ \neq 0$ and $u_- \neq 0$. For $u \neq 0$ we say that u is positive (and write $u > 0$) if $u_- = 0$, and similarly, u is negative ($u < 0$) if $u_+ = 0$.*

Lemma 2.2 *The function $h : H \to H$ defined by $h(u) = u_+$ is continuous. Also, h defines a continuous function from $L^{p+1}(\Omega)$ into itself.*

The two most important subsets of H defined in [2] are

$$S = \{u \in H - \{0\} : \gamma(u) = 0\}$$

$$S_1 = \{u \in S : u_+ \neq 0, u_- \neq 0, \gamma(u_+) = 0\},$$

where we note that nontrivial solutions to (1) are in S (a closed subset of H) and sign-changing solutions are in S_1 (a closed subset of S).

Lemma 2.3 *Under the above assumptions we have:*

(a) 0 is a local minimum of J. If $u \in H - \{0\}$, then there exists a unique $\bar{\lambda} = \bar{\lambda}(u) \in (0, \infty)$ such that $\bar{\lambda}u \in S$. Moreover, $J(\bar{\lambda}u) = \max_{\lambda > 0} J(\lambda u) > 0$.

(b) The function $\bar{\lambda} \in C^1(S^\infty, (0, \infty))$. The set S is closed, unbounded, and a connected C^1- submanifold of H diffeomorphic to S^∞.

(c) $u \in S$ is a critical point of J iff u is a critical point of $J|_S$.

(d) $J|_S$ is coercive, i.e., $J(u) \to \infty$ as $||u|| \to \infty$ in S. Also, $0 \notin S$ and $\inf_S J > 0$.

Lemma 2.4 *Given $w \in S$ which changes sign, there exists a path $r \in C^1([0, 1], S)$ such that*

(a) $r(0) = aw_+ \in S$ for some $a > 0$, $r(1) = bw_- \in S$ for some $b > 0$, and $r(\frac{a}{a+b}) = w$.

(b) $r(\frac{1}{2}) = aw_+ + bw_- \in S_1$ and $r([0, 1]) \cap S_1 = \{r(\frac{1}{2})\}$.

(c) $J(r(0)) < J(r(t)) < J(r(\frac{1}{2}))$ for $t \in (0, \frac{1}{2})$ and $J(r(1)) < J(r(t)) < J(r(\frac{1}{2}))$ for $t \in (\frac{1}{2}, 1)$.

From (2.3) one sees that there exists $\alpha \in C^1(H - \{0\}, (0, \infty))$ so that

$$r(t) = \alpha(t)(a(1-t)w_+ + btw_-) \in S. \tag{7}$$

This path satisfies the above conditions and is essential to the proof of the following lemma.

Lemma 2.5 *(a) Positive and negative elements of S are separated by S_1.*

(b) If $w \in S$, $w > 0$ and $J(w) = \min_{\{u \in S : u > 0\}} J$, then w is a critical point of J.

The proof of the following sign-changing lemma relies on an application of the Deformation Lemma to the path in (7).

Lemma 2.6 *If $w \in S_1$ and $J(w) = \min_{S_1} J$, then w is a critical point of J.*

Sketch of proof of Theorem 1.1.

Using the Sobolev Imbedding Theorem, the coercivity of J, and properties of γ, one can show that the minimizers of J from Lemma 2.5 (b) and Lemma 2.6 exist, and hence the solutions $\omega_1 > 0$, $\omega_2 < 0$, and ω_3 (sign-changing) of Theorem 1.1 exist. Furthermore, since ω_3 is a classical solution and hence continuous, one can show that ω_3 changes sign exactly once. Loosely speaking, a *solution* with three or more humps must have an action functional value higher than that of ω_3, since each hump (or rather its zero extension to all of Ω) would itself be on S. We observe that

$$J(\omega_3) = J((\omega_3)_+) + J((\omega_3)_-) \geq J(\omega_1) + J(\omega_2),$$

where we have used the fact that ω_1 and ω_2 are minimizers of J over the positive and respectively the negative elements of S. In the case that our critical points are nondegenerate, the Morse index of our three solutions follows naturally from their variational characterization as local minima on S and S_1, respectively.

3 Numerical algorithm

The algorithms presented in this section depend on choice of grid, differentiation, and integration methods. For purposes of simplicity and since we consider only simple regions Ω, we use a regular grid, central differencing, and standard rectangular integration schemes. One expects that increased accuracy and efficiency would be gained by using more sophisticated methods; the descent method should similarly benefit from such refinements as the conjugate gradient method or optimal-step-size calculation (see [10], [11], [12]).

Given a nonzero element $u \in H$ and a piece-wise smooth region $\Omega \subset \mathbf{R}^N$, we will use the notation $\mathbf{u}$ to represent an array of real numbers agreeing with u on a grid $\mathbf{\Omega} \subset \bar{\Omega}$. In our experiment, we will take the grid to be regular, but it is clear that irregular grids with finer meshes near large gradient locations would increase accuracy and efficiency. Also, more sophisticated grid techniques will be necessary to investigate solutions when Ω is itself more complicated than an interval or a square.

At each step of our iterative process, we are required to project nonzero elements of H onto the submanifold S. By Lemma 2.3, we see that the projection of $\nabla J(u)$ onto the ray $\{\lambda u : \lambda \geq 0\}$ is given by

$$P_u(\nabla J(u)) = \frac{\langle \nabla J(u), u\rangle}{\langle u, u\rangle} u = \frac{\gamma(u)}{||u||^2} u.$$

Thus, the sign of $\gamma(u)$ determines the uphill direction, which suggests the following one-dimensional gradient *ascent* method:

Algorithm 3.1 *Let u be a nonzero element of H, represented by $\mathbf{u}$ over the grid $\mathbf{\Omega}$. Let $s_1 = .5$, or another perhaps optimally determined small positive constant. Define $\mathbf{u}_0 = \mathbf{u}$ and*

$$\mathbf{u}_{k+1} = \mathbf{u}_k + s_1 \frac{\gamma(\mathbf{u}_k)}{||\mathbf{u}_k||^2} \mathbf{u}_k \quad \text{for } k \geq 0.$$

We will use the notation $P_1(\mathbf{u}) = \lim \mathbf{u}_k$, provided that limit exists, to represent the unique positive multiple of u lying on S (See Lemma 2.3 a)).

In all of our numerical experiments, this process converged numerically, provided only that the hypothesis to Theorem 1.1 was satisfied and that u_0 was "reasonable" <u>or</u> s_1 was sufficiently small.

Remark 3.2 *The map $u \to u_+$ is effected on $\mathbf{u}$ by a simple grid loop and conditional test. A more sophisticated handling of internal zero* *points may lead to increased accuracy and efficiency, as differencing schemes tend to be sensitive to approximation. Using the well-ordered property of the real numbers, in the ODE case we can extend the notion of a positive and negative part to parsing sign-changing functions with more internal zeroes to three (or more) "hump" functions.*

Algorithm 3.3 *Let u be a sign-changing element of H, represented by $\mathbf{u}$ over the grid $\mathbf{\Omega}$. By applying Algorithm 3.1 to the arrays $\mathbf{u}_+$ and $\mathbf{u}_-$ representing u_+ and u_-, we define the resulting sum $P_2(\mathbf{u}) = P_1(\mathbf{u}_+) + P_1(\mathbf{u}_-)$. This array represents the element of S_1 corresponding to $r(\frac{1}{2})$ in (7).*

The standard L^2 gradient is definitely not the gradient we are considering. It is of paramount importance that we use the Sobolev gradient (see [10], [11], and [12]), which in our case can be written down in closed form after integrating by parts:

$$\begin{aligned}\langle \nabla J(u), v\rangle &= J'(u)(v) = \int_\Omega \{(\nabla u \cdot \nabla v - f(u)v)\}\, dx \\ &= \int_\Omega \{(\nabla u \cdot \nabla v - (-\Delta)(-\Delta)^{-1}(f(u))v)\}\, dx \\ &= \int_\Omega \{(\nabla u \cdot \nabla v + \nabla(-\Delta)^{-1}(f(u)) \cdot \nabla v)\}\, dx \\ &= \langle u - (-\Delta)^{-1}(f(u)), v\rangle.\end{aligned} \tag{8}$$

Algorithm 3.4 *Let u be an element of H, represented by $\mathbf{u}$ over the grid $\mathbf{\Omega}$. Then solving the linear system $-\Delta g = f(\mathbf{u})$ for g allows one to explicitly construct the array $\nabla J(\mathbf{u}) \equiv \mathbf{u} - g$, representing $\nabla J(u)$.*

In order to solve this system, we used Gaussian-Elimination for the ODE (N=1), and Gauss-Sidel with successive overrelaxation (SOR, $\omega = 1.73$) for the PDE (see [9]).

Remark 3.5 *At one time this author projected Sobolev gradients $\nabla J(u)$ onto tangent spaces T_uS, thinking this necessary in order to keep iterates "near" the surface S. This turns out not to be necessary (although not particularly harmful), since projections of iterates $\mathbf{u}_k$ onto S by P_1 or P_2 after each gradient step effectively performs this operation.*

Algorithm 3.6 *Let u be an element of H, represented by $\mathbf{u}$ over the grid $\mathbf{\Omega}$. Let $s_2 = 1$ or another perhaps optimally determined small positive constant. Define $\mathbf{u}_0 = \mathbf{u}$ and*

$$\mathbf{u}_{k+1} = P_1(\mathbf{u}_k - s_2 \nabla J(\mathbf{u}_k)) \quad \text{for } k \geq 0,$$

where $\nabla J(\mathbf{u}_k)$ is computed from Algorithm 3.4. The limit of this iterative process, provided it exists, represents a positive solution to (1).

Algorithm 3.7 *Let u be a sign-changing element of H, represented by $\mathbf{u}$ over the grid $\mathbf{\Omega}$. Let $s_2 = 1$ or another perhaps optimally determined small positive constant. Define $\mathbf{u}_0 = \mathbf{u}$ and*

$$\mathbf{u}_{k+1} = P_2(\mathbf{u}_k - s_2 \nabla J(\mathbf{u}_k)) \quad \text{for } k \geq 0.$$

The limit of this iterative process, provided it exists, represents a sign-changing solution to (1).

In all of our numerical experiments the iterative processes of Algorithms 3.6 and 3.7 converged numerically, provided only that the hypothesis to Theorem 1.1 was satisfied, u_0 was "reasonable" or s_1 was sufficiently small, and s_2 was sufficiently small. If there exists more than one sign-changing solution of Morse Index 2, the Algorithm 3.7 should find them provided the initial estimate $\mathbf{u}_0$ is "reasonably close", i.e., of approximately the correct nodal structure.

4 Numerical results: the ODE

Let $\lambda = 1$, $f(u) = u^3 + \lambda u$, and $\Omega = [0, 1]$. We numerically computed 1-hump, 2-hump, and 3-hump (i.e., solutions with 0, 1, and respectively 3 internal zeroes) solutions via our algorithm and compared the results with independent Euler method calculations.

The data below used the first three eigenfunctions of $-u'' = \lambda u$ as initial guesses. In Section 6 we note that it in fact suffices to use any initial function with the correct nodal structure. As the well known bifurcation diagram in this ODE case indicates, additional numerical experiments verified that one does in fact get sign-changing solutions for $\lambda > \pi^2 = \lambda_1$, and solutions which change sign exactly once for $4\pi^2 = \lambda_2 > \lambda > \pi^2 = \lambda_1$. Additionally, using the eigenfunctions

Table **I**: Convergence data for the ODE.

j	grid	its	$\|\|\nabla J(u)\|\|_{L^2}$	J(u)	seconds
0	$n = 5000$	7	$3.67 \cdot 10^{-6}$	12.66	21
1	$n = 5000$	40	$2.53 \cdot 10^{-5}$	239.68	220
2	$n = 40000$	151	$2.73 \cdot 10^{-4}$	1248.11	10117

Table **II**: Approximations of solutions to the ODE.

# places gives agreement with Euler's Method				
.1	.3	.5	.7	.9
0.93583	2.67632	3.50717	2.67631	0.93581
3.82151	6.83475	0.0000	-6.83475	-3.82152
8.3832	2.9041	-11.0588	2.9042	8.3830

$-sin(j\pi x)$, $j = 1, 2, 3$ for initial estimates lead to numerical solutions which were the negative of those in Table **II**. Care was needed when switching the 3-hump code from expecting 2 positive humps to 2 negative ones. Satisfactory runs were also made which in the obvious way lead to solutions with still more internal zeroes. As one would expect, however, due to increasing values of $||\nabla J(u)||$ accuracy and efficiency were increasingly hard to maintain. Finally, all experiments were repeated with asymmetric nonlinearities which still satisfied the hypothesis in Section 1. In all cases, the well known bifurcation diagram was reproduced. As expected, increasing the number of divisions of the unit interval lead to behavior which more accurately approximated the continuous case. Stability was confirmed by allowing the program to execute for many hours past apparent convergence.

5 Numerical results: the PDE

Let $\lambda = 9$, $f(u) = u^3 + \lambda u$, and $\Omega = [0,1] \times [0,1]$. Using initial functions $u_0 = \pm v = \pm \sin(\pi x) \sin(\pi y)$, we numerically computed a positive and a negative solution. Using initial functions of the form $u_0 =$

$\pm y^1 = \pm\sin(2\pi x)\sin(\pi y)$, $u_0 = \pm y^2 = \pm\sin(\pi x)\sin(2\pi y)$, and linear combinations $u_0 = \alpha y^1 + \beta y^2$, our sign-changing algorithm converged numerically to one of four exactly-once sign-changing solutions.

In Tables **III** - **V** we present a subset of the grid values for the positive ($j = 0$) and one of the four sign-changing solutions ($j = 1$), together with some convergence data. The initial functions used for these two experiments were $u_0 = \sin(\pi x)\sin(\pi y)$ and $u_0 = \sin(2\pi x)\sin(\pi y) + \sin(\pi x)\sin(2\pi y)$, respectively.

As in the ODE case, the exact choice of initial estimate u_0 was not crucial as long as the nodal structure was roughly close to that of the desired sign-changing solution. There was a surprise in the nodal structure of our sign-changing solutions.
Namely, regardless of which of the above initial estimates (sign-changing) were used, our solutions were diagonally antisymmetric, i.e., of the same nodal structure as $\pm(y^1 \pm y^2)$.

Table **III**: Convergence data for the PDE.

j	grid	its	$\|\|\nabla J(u)\|\|_{L^2}$	$\|\|\Delta u + f(u)\|\|_{L^2}$	J(u)	seconds
0	$n = 50$	7	$5.01 \cdot 10^{-3}$	$1.09 \cdot 10^{-1}$	12.00	119
	$n = 100$	7	$7.18 \cdot 10^{-4}$	$3.21 \cdot 10^{-2}$	11.97	1173
1	$n = 50$	13	$1.49 \cdot 10^{-2}$	$8.72 \cdot 10^{-1}$	149.49	743
	$n = 100$	13	$1.51 \cdot 10^{-3}$	$9.59 \cdot 10^{-2}$	148.78	3653

Table **IV**: Approximation of the positive solution to the PDE, $n = 100$.

	.1	.3	.5	.7	.9
.1	0.3229	0.8958	1.1490	0.8959	0.3229
.3	0.8958	2.5915	3.4243	2.5917	0.8960
.5	1.1490	3.4243	4.6284	3.4246	1.1491
.7	0.8959	2.5917	3.4246	2.5919	0.8961
.9	0.3229	0.8960	1.1491	0.8961	0.3230

Initial guesses of the form $\pm y^1$ and $\pm y^2$ appeared to lead to convergence to similar horizontally and vertically antisymmetric functions, only to "fall off" to the diagonally antisymmetric solutions. The four solutions with diagonal antisymmetry are of Morse index 2 and true minimizers of J over S_1. We conjecture that the four unstable "almost convergent" horizontally and vertically antisymmetric functions are in fact Morse index 3 solutions (See Section 6). It is important to note that *oddness plays no essential role* in these experiments. Similar results followed for a variety of asymmetric nonlinearities, although in those cases the interior zero set was slightly curved so that the sign-changing solution was no longer diagonally antisymmetric. As in the ODE experiments, stability was confirmed by allowing the program to execute for many hours past apparent convergence, particularly in light of the appearance of the four almost convergent functions mentioned above.

6 Conclusions

A number of conjectures concerning improvements or special cases of Theorem 1.1 can be investigated with code operating on the principals of Section 3. In this section we describe various experiments in which we seek to better understand the variational characterization of the solutions to (1).

As long as the initial guess function $\underline{u}_0$ is "nodally correct", i.e. has roughly the same zero set structure as the desired sign changing solution, the process converges to said expected solution even when the initial estimate is quite far off under the H norm. For example, when $\Omega = [0, 1]$, the initial estimate functions $\sin(2\pi x) + \sin(20\pi x)/100$ and $x(x-.2)(x-1)$ both result in satisfactory convergence. Experimentally, one observes the "small support equals large sup-norm" phenomenon in the latter example. In fact, in this case it is initially necessary to dampen the process by reducing the step size, since for the first few iterations the gradient is large for $x \in (0, .2)$.

It is important to note that we do not know whether S_1 is a true submanifold of S. Were $\gamma \circ h$ to be continuously differentiable, this would be so. In fact, in that case $T_u S_1$ could be defined as $\{v \in T_u S : v \perp \nabla\gamma(u_+)\}$. Using *Mathematica* to render 3 dimensional graphics, we are gaining insight into the structure of S and S_1 by using Algorithm 3.1 to intersect linear subspaces spanned by trios of eigenfunctions or solutions with S. These images lend credibility to the conjecture that there should be a second exactly-once sign changing solution, roughly corresponding to $-\omega_3$, whether or not f is odd. As far as the general shape of such "peanut" shaped objects, oddness does not appear to be a special property. Properties such as asymmetry of the nonlinearity or multiplicity of eigenvalues correspond to distinctive features when thus visualized.

This author believes that the idea from Remark 3.2 should generalize to higher dimensions. We attempted to apply this notion to find a 3-Hump solution resembling $\sin(3\pi x)\sin(\pi y)$ on the unit square, but the process did not converge to a solution. This seems to be due to the multiplicity 2 second eigenvalue in that case, whereupon the algorithm does not have enough constraints and falls away from a local minimum of a higher codimension subset of S_1. Solving these numerical difficulties may be synonymous with finding an existence proof of more solutions of higher Morse index and a greater number of internal zero sets: one seeks higher codimension analog to S and S_1 and a precise definition of such "hump" functions (corresponding to the part functions of Definition 2.1), together with continuity theorems similar to Lemma 2.2.

The multiplicity 2 issue mentioned above was also investigated in the course of finding the four sign-changing solutions in Section 5. When horizontally or vertically antisymmetric initial guess functions such as $\pm\sin(2\pi x)\sin(\pi y)$ or $\pm\sin(\pi x)\sin(2\pi y)$ were used, the process almost stalled on nodally similar points before falling off into a basin containing one of the four diagonally antisymmetric solutions. The gradient remained extremely small throughout the iterations, but it seems likely to this author that the four stall points represented additional exactly-once sign-changing solutions of Morse index 3. At this time this author is trying to prove analytically the existence of the aforementioned eight exactly-once sign-changing solutions on the square. Note that if one restricts to the case where f is odd, eight Morse index 2 solutions can easily be constructed by piecing together w_1 and w_2 solutions on upper/lower triangles/rectangles. Our numerical experiments with non-odd nonlinearities lead us to believe in the existence of these eight solutions in the general case. Lastly, in [3] we are in the process of proving that when Ω is a disk in $\mathbf{R}^2$, the sign-changing solution w_3 is nonradial. One would then see that in fact there are infinitely many such solutions, obtained by rotation by any angle. This author visualizes the eight "solutions" in the case $\Omega = [0,1] \times [0,1]$ as "optimally rotated", i.e., there is a path in S_1 of sign-changing elements, roughly corresponding to projections of linear combinations of the first two sign-changing eigenfunctions, containing four local minima and four local maxima. This is one of the approaches being considered our efforts to construct such proofs.

The work [3] mentioned above contains several proofs building on the variational results of [2]. One important technique used in [3] is to show that in a given situation an element of S_1 with a certain symmetry or antisymmetry cannot be a minimizer of the action functional J. We hope that the numerical results of this paper together with the analytical results from [2] and [3] will lead to a more complete understanding of the variational characterization of all solutions to (1). In particular, we plan to investigate the bifurcation diagram numerically as we seek to prove sign-changing multiplicity results.

It has been worthwhile to run code modified for more general regions. Experiments on disks and annuli in $\mathbf{R}^2$ are providing information which complements our current research (see for example [3]). Additionally, since symmetry of the region is in no more relevant than symmetry of the nonlinearity to our scheme, nodal properties of solutions in regions such as the dumbbell can also be investigated. In particular, it will be interesting to see how the structure changes when the symmetry of the region Ω is perturbed slightly, turning a multiplicity of eigenvalues into a close pair. We are currently trying to prove existence theorems along these lines.

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Table **V**: Approximation of a Sign-Changing Solution to the PDE, $n = 100$.

	.1	.3	.5	.7	.9
.1	1.4804	3.5414	2.1449	0.4804	0.0000
.3	3.5414	9.3817	4.1729	0.0003	-0.4802
.5	2.1449	4.1729	0.0006	-4.1722	-2.1448
.7	0.4804	0.0003	-4.1722	-9.3823	-3.5418
.9	0.0000	-0.4802	-2.1448	-3.5418	-1.4806