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A sign-changing solution for a superlinear Dirichlet problem with a reaction term nonzero at zero

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1. Introduction

Let Ω be a smooth bounded region in $\mathbf{R}^N$, Δ the Laplacian operator, and $f \in C^1(\mathbf{R},\mathbf{R})$ such that $f(0)=0$. We will take f to satisfy the assumptions below and consider the nonlinearity $f(u)-\varepsilon$, where $|\varepsilon|$ will be taken to be sufficiently small. We assume that there exist constants $A>0$ and $p \in (1, \frac{N+2}{N-2})$ such that $|f'(u)| \le A(|u|^{p-1}+1)$ for all $u \in \mathbf{R}$. Hence f is subcritical, i.e. there exists $B>0$ such that $|f(u)| \le B(|u|^p+1)$. Also, we assume that there exists $m \in (0,1)$ such that

$$f(u)u - 2F(u) \ge muf(u), \tag{1}$$

where $F(u) = \int_0^u f(s)\,\mathrm{d}s$, for all $u \in \mathbf{R}$. Finally, we make the assumptions that f satisfies

$$f'(u) > \frac{f(u)}{u} \quad \text{for } u \neq 0 \quad \text{and} \quad \lim_{|u|\to\infty} \frac{f(u)}{u} = \infty, \tag{2}$$

i.e. that f is superlinear. In this paper we study the boundary value problem

$$\begin{cases} \Delta u + f(u) - \varepsilon = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{3}$$

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Let H be the Sobolev space $H_0^{1,2}(\Omega)$ with inner product $\langle u,v\rangle = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$ (see [1] or [2]). We define $J: H\to\mathbf{R}$ by

$$J(u)=\int_\Omega \{\tfrac{1}{2}|\nabla u|^2 - F(u)+\varepsilon u\}\,\mathrm{d}x = J_0(u)+\varepsilon\int_\Omega u\,\mathrm{d}x,$$

where J_0 is the functional for the case $\varepsilon=0$ (see [3]). By regularity theory for elliptic boundary value problems (see [2]), u is a solution to (3) if and only if u is a critical point of the action functional J. Let $0<\lambda_1<\lambda_2\leq\lambda_3\leq\cdots$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. We prove the following Main Theorem:

Theorem 1.1. *Let ε be a sufficiently small positive real number (semipositone case). Then* (3) *has at least four nontrivial solutions: $\omega_0<0$ in Ω, $\omega_1^-<0$ in Ω, ω_1^+, and ω_2. The function ω_1^+ has a nontrivial positive part and the function ω_2 changes sign, i.e. has nontrivial positive and negative parts. If nondegenerate, the solutions have Morse index corresponding to the their subscripts. Furthermore,*

$$J(\omega_2)\geq J(\omega_1^+)+J(\omega_1^-)>J(\omega_1^+)+J(\omega_1^-)+J(\omega_0).$$

Remark 1.2. If we assume that condition (1) holds only for $|u|>\rho$, some $\rho>0$, we can extend all of this paper's proofs (and those of [3]) to hold for a wider class of nonlinearities. We need the assumption that $f'(0)<\lambda_1$ for our proofs. Since (1) implies that $f'(0)\leq 0$ and we already have $\lambda_1>0$, we need only add this assumption if we assume this weaker form of (1). The expanded class includes, for example, nonlinearities of the form $f(u)=\lambda u+u^3$, where $f'(0)=\lambda<\lambda_1$.

While the proof of Theorem 1.1 does follow the method of proof given in [3], each new step requires modification since $f(0)\neq 0$ implies that there is no longer a trivial solution; the local minimum of the action functional now corresponds to the nontrivial solution ω_0. As this new result demonstrates, the general technique is useful for obtaining sign-changing existence theorems and paves the way for obtaining further generalizations to an even wider class of nonlinear elliptic PDEs, including asymptotically linear, sublinear, and p-Laplacian type problems. In fact, we have recently determined that the sign-changing existence proof for the p-Laplacian case requires no modification from our original proof in [3]; only the assumptions of f need change. In [4] we are taking the different approach of applying the original $\varepsilon=0$ result as a tool rather than following its method.

Additionally, this paper notes a possible loosening of the coercivity condition (1) (see Remark 1.2), contains a more detailed analysis of the behavior of the action functional along certain key paths (see for example (11)), and reports current progress in numerical investigations of solutions. We refer the reader to [5] for a detailed explanation of the numerical algorithm used and [6] for a general development of constructive variational methods.

Remark 1.3. If we assume ε is negative (Positone case) and that $|\varepsilon|$ is sufficiently small, we can obtain a similar existence result where the roles of positive and negative parts are reversed. That is to say, (3) has at least four nontrivial solutions: $\omega_0 > 0$ in Ω, $\omega_1^+ > 0$ in Ω, ω_1^-, and ω_2. The function ω_1^- has a nontrivial negative part and the function ω_2 changes sign, i.e. has nontrivial positive and negative parts. Since the argument is nearly identical to that of Theorem 1.1, we omit the proof and consider only the semipositone case $\varepsilon > 0$ in the sequel.

To the best of our knowledge, [3] was the first to establish the existence of a *sign-changing* solution to (3) for a general region in the superlinear case where $\varepsilon = 0$, and this result in turn is the first to establish it where $\varepsilon \neq 0$. We wish to acknowledge the complementary works of [7] (which preceded [3]) and [8] (which is currently in submission), where completely different topological techniques are leading towards closely related results. Their and our methods are revealing different information and should both prove useful in future investigations. In the semipositone and radial symmetry cases, much is known about the existence of positive solutions and, respectively, infinitely many radial solutions (see for example [9–15]). In this paper, our focus is on the existence of the sign-changing solution ω_2; we emphasize that neither f nor Ω need any special symmetry.

For completeness, and due to the fact that these proofs occur naturally in our analysis, we also establish the existence of the negative solutions ω_0 and ω_1^-, as well as the mostly-positive solution ω_1^+. We are unable to establish the existence of a purely positive or a sign-changing solution which changes sign exactly once. As observed in our numerical experiments and work such as [11] and [10], there are problems in our class where such solutions do not exist. By treating $f'(0)$ as a bifurcation parameter or by choosing ε sufficiently small, one can sometimes obtain the existence of these "nodally pure" solutions.

In the final section we include a brief outline of a numerical algorithm based on our variational proofs (see [5]). Application of the algorithm requires an understanding of the variational structure, and conversely, provides insight in to it. This algorithm is useful not only for calculating approximations to solutions, but also as a tool for investigating the topological and geometrical structure of the submanifolds and subsets containing the critical points and verifying the nature of bifurcation. It is our belief that such investigations will aid us in understanding the variational structure of related problems, hopefully to the end of obtaining more solutions of higher Morse index and a more complex nodal structure to this and related problems.

2. Preliminary lemmas

Our assumptions on f imply that J_0, $J \in C^2(H, \mathbf{R})$ (see [16]) and that

$$J'(u)(v) = \langle \nabla J(u), v \rangle = \int_\Omega \{\nabla u \cdot \nabla v - f(u)v + \varepsilon v\}\,\mathrm{d}x, \quad \text{for all } v \in H. \tag{4}$$

Define $\gamma : H \to \mathbf{R}$ by $\gamma(u) = \langle \nabla J(u), u \rangle = \|u\|^2 + \int_\Omega \{\varepsilon u - uf(u)\}\,\mathrm{d}x = \gamma_0(u) + \varepsilon \int_\Omega u\,\mathrm{d}x$ and compute

$$\gamma'(u)(v) = \langle \nabla\gamma(u), v \rangle = 2\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x - \int_\Omega f(u)v\,\mathrm{d}x - \int_\Omega f'(u)uv\,\mathrm{d}x. \tag{5}$$

Definition 2.1. For $u \in L^1(\Omega)$, we define $u_+(x) = \max\{u(x), 0\} \in L^1(\Omega)$ and $u_-(x) = \min\{u(x), 0\} \in L^1(\Omega)$. If $u \in H$ then $u_+, u_- \in H$ (see [17]). We say that $u \in L^1(\Omega)$ changes sign if $u_+ \neq 0$ and $u_- \neq 0$. For $u \neq 0$ we say that u is positive (and write $u > 0$) if $u_- = 0$, and similarly, u is *negative* ($u < 0$) if $u_+ = 0$.

In Lemma 2 of [3] we showed that the map $h : H \to H$ defined by $u \to u_+$ is continuous. We observe that this is also true as a map from $L^{p+1}(\Omega)$ into itself. An important consequence of Lemmas A.3 and A.4 of [17] is the fact $J(u) = J(u_+) + J(u_-)$ and $\gamma(u) = \gamma(u_+) + \gamma(u_-)$ for all $u \in H$.

We define $S \subset H$ and various subsets of S:

$$\begin{aligned}
&S = \{u \in H - \{0\} : \gamma(u) = 0,\ \gamma'(u)(u) < 0\} = \{u \in H - \{0\} : J(u) \geq J(\lambda u),\ \lambda > 0\},\\
&S_* = \{u \in H : \gamma(u) = 0,\ \gamma'(u)(u) \geq 0\},\\
&S_1 = \{u \in H : u_+ \in S,\ u_- \in S\}, \qquad S_2 = \{u \in H : u_+ \in S,\ u_- \in S_*\},\\
&G^+ = \{u \in S : u_- = 0\}, \qquad G^- = \{u \in S : u_+ = 0\}.
\end{aligned} \tag{6}$$

We define W^+ and W^- to be the connected components of $S - S_1$ containing G^+ and, respectively, G^-. We will see that the disjoint union $W^+ \cup S_1 \cup W^- = S$. We note that nontrivial solutions to (3) are in $S \cup S_*$, one-sign solutions are in $G_v^+ \cup G_v^- \cup S_*$, sign-changing solutions are in $S_1 \cup S_2$, and $S_2 \subset W^+$. We will see that S_* is "inside" S, i.e. S separates S_* from infinity, and thus define u to be *mostly positive* if $u_+ \in G^+$ while $u_- \in S_*$. Note that $0 \in S_*$, so that positive functions are also mostly positive. We restate Theorem 1.1 in terms of the sets defined in (6):

Theorem 2.2. *There exists solutions $\omega_0 \in S_*$, $\omega_1^+ \in S_2$, $\omega_1^- \in G^-$, and $\omega_2 \in S_1$ to* (3) *with the variational characterizations $J(\omega_0) = \min_{S_*} J$, $J(\omega_1^+) = \min_{S_2} J = \min_{W^+} J$, $J(\omega_1^-) = \min_{G^-} J = \min_{W^-} J$, and $J(\omega_2) = \min_{S_1} J$.*

We summarize important properties of J, S, and S_* which we prove in a series of lemmas:

(a) $J|_{S \cup S_*}$ is coercive, i.e. $J(u) \to \infty$ as $\|u\| \to \infty$ in $S \cup S_*$. Also, there exists $\delta > 0$ with $\|u\| > \delta$ for all $u \in S$.
(b) If $u \in H - \{0\}$, then there exists a unique $\bar{\lambda} = \bar{\lambda}(u) \in (0, \infty)$ such that $\bar{\lambda} u \in S$. Moreover, $J(\bar{\lambda} u) = \max_{\lambda > 0} J(\lambda u) > 0$ and there exists $c > 0$ such that for $u \in S$ we have $c \leq J(u)$.
(c) If $u \in H$ with $\int_\Omega u\,\mathrm{d}x < 0$, then there exists a unique $\lambda_* = \lambda_*(u) \in (0, \bar{\lambda}(u))$ such that $\lambda_* u \in S_*$. Moreover, $J(\lambda u)$ has a local minimum at $\lambda = \lambda_*$ and there exists $K > 0$ such that for $u \in S_*$ we have $-K \leq J(u) \leq 0$.

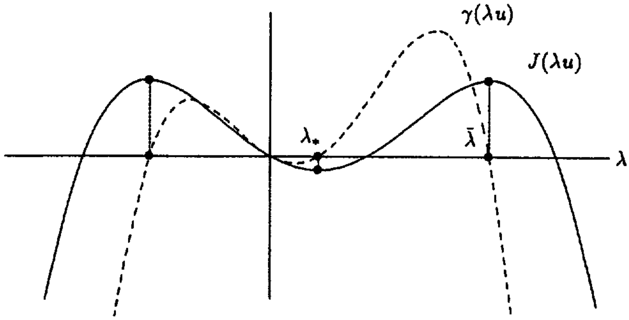

Fig. 1. Graphs of $J(\lambda u)$ and $\gamma(\lambda u)$.

(d) The set S is a closed C^1-submanifold of H. The set S_* is closed and locally (at least) a C^1 submanifold away from $0\in H$.

(e) $u\in H$ is a solution to (3) $\Leftrightarrow$ u is a critical point of $J|_S$ or $J|_{S_*}$.

Let $u\in H$ so that $\int_\Omega u\,\mathrm{d}x<0$. Let us see that Fig. 1 represents the graph of $\phi:\mathbf{R}\to\mathbf{R}$ given by $\phi(\lambda)=J(\lambda u)$; the graph of ϕ for $u\in H$ with $\int_\Omega u\,\mathrm{d}x=0$ is the the same except that the local minimum is at $\lambda_*=0$. For convenience, we overlay the graph of $\gamma(\lambda u)=\lambda\phi'(\lambda)$. Note that

$$\phi(\lambda)=\frac{\lambda}{2}\|u\|^2-\int_\Omega\{F(\lambda u)-\lambda\varepsilon u\}\,\mathrm{d}x$$

so that

$$\phi'(\lambda)=\lambda\|u\|^2-\int_\Omega\{uf(\lambda u)-\varepsilon u\}\,\mathrm{d}x \quad\text{and}\quad \phi''(\lambda)=\|u\|^2-\int_\Omega u^2 f'(\lambda u)\,\mathrm{d}x.$$

Clearly $\phi(0)=0$. Since f is superlinear, we also see that $\lim_{\lambda\to\infty}\phi(\lambda)=-\infty$. From $\phi'(0)=\varepsilon\int_\Omega u\,\mathrm{d}x$, it follows that ϕ is decreasing at $\lambda=0$. Also, $\phi''(0)=\|u\|^2>0$ and $\phi''(\lambda)\downarrow-\infty$ as $|\to\infty$. Thus, there exists a unique local minimum of ϕ at $\bar{\lambda}_1>0$ and $\phi(\bar{\lambda}_1)<0$. Similarly, there exists a unique global maximum $\bar{\lambda}_3<0$ of $\phi|_{\lambda\le 0}$ and $\phi(\bar{\lambda}_3)>0$. There can be only one other possible critical point, a global maximum of $\phi|_{\lambda\ge 0}$. To see that this critical point $\bar{\lambda}_3$ exists and that $J(\bar{\lambda}_3)>0$, we first need to prove several lemmas.

Lemma 2.3. *There exists $\delta\equiv\delta_\varepsilon>0$ such that $\|u\|\ge\delta$ for all $u\in S$.*

Proof. We first obtain an estimate for the term $|s^2f'(s)|$. Since $f'(0)<\lambda_1$, there exists $\rho>0$ and $\alpha>0$ such that $f'(s)<\alpha<\lambda_1$ for all $|s|<\rho$. For $|s|\ge\rho$, since f is subcritical we see that there exists $\beta>0$ such that $|f'(s)|\le A((\frac{|s|}{\rho})^{p-1}+|s|^{p-1})=\beta|s|^{p-1}$. Then for all $s\in\mathbf{R}$ we have $|f'(s)|\le\alpha+\beta|s|^{p-1}$ so that $|s^2f'(s)|\le\alpha|s|^2+\beta|s|^{p+1}$. Now consider $u\in S$. We know that $\gamma(u)=0$ and that $\gamma'(u)(u)<0$, hence

$$2\|u\|^2-\int_\Omega\{u^2f'(u)+uf(u)-\varepsilon u\}\,\mathrm{d}x=\gamma(u)+\|u\|^2-\int_\Omega u^2f'(u)\,\mathrm{d}x$$

$$=\|u\|^2-\int_\Omega u^2f'(u)\,\mathrm{d}x<0.$$

Thus,

$$\|u\|^2<\int_\Omega u^2f'(u)\,\mathrm{d}x\le\alpha\|u\|_2^2+\beta\|u\|_{p+1}^{p+1}\le\frac{\alpha}{\lambda_1}\|u\|^2+c\beta\|u\|^{p+1},$$

where we have used Poincare's inequality and the Sobolev Imbedding Theorem (S.I.T.) to obtain the imbedding constants λ_1 and c. The above inequality implies that

$$\|u\|\ge\left(\frac{1-\alpha/\lambda_1}{c\beta}\right)^{1/(p-1)}>0.$$

We take δ to be the largest such lower bound. □

The next lemma shows that $J|_{S\cup S_*}$ is coercive and that J is bounded away from 0 on S.

Lemma 2.4. *Let $u_k\in S\cup S_*=\{u\in H:\gamma(u)=0\}$ with $\|u_k\|\to\infty$. Then $J(u_k)\to\infty$. Furthermore, given ε sufficiently small there exists $c>0$ such that $\inf_S J\ge c>0$.*

Proof. Let $\varepsilon_1=(1-m/2-m)\delta\sqrt{\frac{\lambda_1}{|\Omega|}}$, where $|\Omega|$ denotes the measure of the region Ω and $m\in(0,1)$ is taken from equation (1). We assume that $\varepsilon\le\varepsilon_1/2$ so that the following inequality holds:

$$\frac{1-m}{2}\delta-\varepsilon\frac{2-m}{2}\sqrt{\frac{|\Omega|}{\lambda_1}}\ge\frac{1-m}{4}\delta>0. \qquad (7)$$

Note that given $u\in H$, we have $\int_\Omega|u|\,\mathrm{d}x\le(|\Omega|\int_\Omega u^2\,\mathrm{d}x)^{\frac12}\le\sqrt{\frac{|\Omega|}{\lambda_1}}\|u\|$. For $u\in S$, recall that $\gamma(u)=0$ so that $\|u\|^2=\int_\Omega\{uf(u)-\varepsilon u\}\,\mathrm{d}x$. Thus, by making use of equation (1) we see that for $u\in S\cup S_*$ we have

$$J(u)=\frac12\|u\|^2-\int_\Omega F(u)\,\mathrm{d}x+\varepsilon\int_\Omega u\,\mathrm{d}x=\int_\Omega\left\{\frac12uf(u)-F(u)+\frac{\varepsilon}{2}u\right\}\,\mathrm{d}x$$

$$\ge\frac{1-m}{2}\int_\Omega uf(u)\,\mathrm{d}x+\frac{\varepsilon}{2}\int_\Omega u\,\mathrm{d}x=\frac{1-m}{2}\left[\|u\|^2+\varepsilon\int_\Omega u\,\mathrm{d}x\right]+\frac{\varepsilon}{2}\int_\Omega u\,\mathrm{d}x$$

$$=\frac{1-m}{2}\|u\|^2+\frac{\varepsilon}{2}(2-m)\int_\Omega u\,\mathrm{d}x\ge\frac{1-m}{2}\|u\|^2-\varepsilon\frac{2-m}{2}\sqrt{\frac{|\Omega|}{\lambda_1}}\|u\|. \qquad (8)$$

From the quadratic lower bound in (8), we see that given $u = u_k \in S \cup S_*$ with $\|u_k\| \to \infty$, we have $J(u_k) \to \infty$. Furthermore, since given $u \in S$ we have $\|u\| \geq \delta$, it follows by (7) that

$$J(u) \geq \|u\| \left(\frac{1-m}{2}\|u\| - \varepsilon\frac{2-m}{2}\sqrt{\frac{|\Omega|}{\lambda_1}} \right) \geq \|u\|\frac{1-m}{4}\delta \geq \frac{1-m}{4}\delta^2 = c > 0. \qquad \square$$

In fact, we now see that given $u \in H$, there exists a unique $\lambda > 0$ so that $\lambda u \in S$ and that $J(\lambda u) \geq c > 0$. That is to say, we now have the third critical point $\bar{\lambda}_2$ of ϕ, which completes the proof of parts (a) and (b) in our list of important properties of J and S. The proof of part (c) is completed in the next lemma, where we provide a lower bound for J "inside" S, or more relevantly, for $J|_{S_*}$.

Lemma 2.5. *There exists $K > 0$ such that $J(u) \geq -K$ for all $u \in S_*$.*

Proof. For $u \in S_*$ we have $\gamma(u) \geq 0$, whence we see that

$$\|u\|^2 \geq \int_\Omega \{uf(u) - \varepsilon u\}\,\mathrm{d}x \geq \int_\Omega uf(u)\,\mathrm{d}x.$$

Since f is superlinear, we see that there exists A_1 and A_2 so that $|f(u)| \geq A_1|u| - A_2$. It then follows that

$$\begin{aligned} J(u) &\geq \frac{1-m}{2}\int_\Omega uf(u)\,\mathrm{d}x - \varepsilon\int_\Omega |u|\,\mathrm{d}x \\ &\geq \frac{1-m}{2}\left[\int_\Omega \{A_1|u|^2 - A_2|u|\}\,\mathrm{d}x\right] - \varepsilon\int_\Omega |u|\,\mathrm{d}x \\ &= \frac{A_1(1-m)}{2}\int_\Omega |u|^2\,\mathrm{d}x - (A_2+\varepsilon)\int_\Omega |u|\,\mathrm{d}x \\ &\geq \frac{A_1(1-m)}{2|\Omega|}\left(\int_\Omega |u|\,\mathrm{d}x\right)^2 - (A_2+\varepsilon)\int_\Omega |u|\,\mathrm{d}x \\ &\geq -K = -K(A_1, A_2, m, |\Omega|, \varepsilon). \end{aligned} \tag{9}$$

We take $K > 0$ so that $\inf_{\{u \in H:\, \gamma_0(u) \geq 0\}} J(u) = \inf_{S_*} J = -K$. $\square$

Note that since γ is continuous, $S \cup S_*$ is closed. Since the open set $J^{-1}(-\infty, \frac{c}{2})$ separates S and S_*, we see that S and S_* are closed as well. Since $\gamma(u) = 0$ and $\gamma'(u)(u) < 0$ for all $u \in S$, we appeal to the Implicit Function Theorem to conclude that S is a codimension 1 C^1-submanifold of H. Since $\nabla\gamma(u)$ is nonvanishing on $S_* - \{0\}$, we see that $S_* - \{0\}$ is locally a submanifold. This concludes the proof of part (d) of our list of properties. The proof of the final part (e) is made by the following lemma.

Lemma 2.6. *Functions $u \in H$ are solutions to* (3) *if and only if they are critical points of $J|_S$ or $J|_{S_*}$.*

Proof. We first recall that, by regularity theory for elliptic PDEs, u is a weak solution if and only if it is a classical one and that by definition critical points of J are weak solutions (see [2]).

(a) If $u \in S$ is a critical point of $J|_S$, then by the method of LaGrange multipliers, there exists $\lambda \in \mathbf{R}$ so that $\nabla J(u) = \lambda \nabla \gamma(u)$, since $\nabla \gamma(u)$ is a normal vector to S at u. Observe that

$$0 = \gamma(u) = \langle \nabla J(u), u \rangle = \lambda \langle \nabla \gamma(u), u \rangle. \tag{10}$$

Since for $u \in S$ the last inner product is negative, we see that $\lambda = 0$ and hence $\nabla J(u) = 0$.

(b) If $u \in S_*$ is a critical point of $J|_{S_*}$, first observe that $u \neq 0$ since $J'(0)(w) = \varepsilon \int_\Omega w \, dx \neq 0$ for some $w \in H$. For $u \neq 0$, $u \in S_*$, we have $\langle \nabla \gamma(u), u \rangle > 0$, whence again equation (10) implies that $\nabla J(u) = 0$. □

3. Existence of the small negative solution ω_0

Let K be as in Lemma 2.5 so that $\inf_{S_*} J = -K$ and take $v_n \in S_*$ with $J(v_n) \downarrow -K$. It follows that $\gamma(v_n) = 0$ and $\gamma'(v_n)(v_n) > 0$. Since J is coercive on S_* (see Lemma 2.4), we can invoke the S.I.T. and without loss of generality find $v \in H$ so that

$$v_n \rightharpoonup v \quad \text{in } H, \qquad v_n \to v \quad \text{in } L^{p+1}.$$

If we suppose that the convergence in H is not strong, then as in [3] we may assume without loss of generality that $\|v\| < \liminf \|v_n\|$, whereby $\gamma(v) < \liminf \gamma(v_n) = 0$. Note that there are only two possible regions of H where $\gamma(v) < 0$ may hold.

1. If v is "outside" S, then there exists $\alpha < 1$ so that $\alpha v \in S$. Since $v_n \in S_*$ and $\alpha < 1$, it follows that $J(\alpha v_n) < 0$ so that $J(\alpha v) < \liminf J(\alpha v_n) \leq 0$. This contradicts $J|_S > 0$.
2. If v is "inside" S, then in fact v is "inside" S_*. Thus there exists $\alpha > 1$ such that $\alpha v \in S_*$, whence $J(\alpha v) < J(v) < \liminf J(v_n) = -K$. This contradicts the definition of K.

The above two contradictions imply that we have $v_n \to v$ in H and $J(v) = -K$. We can easily show that v is a solution to (3). One of several proofs is that the "inside" of S is an open subset of H, so that v minimizes J on an open set.

We now show that v is a negative solution. Suppose to the contrary that $v_+ \neq 0$. Since v is a solution it follows that $\gamma(v_+) = 0$ and $v_+ \in S$, whereby $J(v_+) > 0$. This leads to the inequality $J(v_-) = J(v) - J(v_+) < J(v) = -K$, which contradicts $J(v) = -K = \min_{S_*} J$ since $J(v_-) < 0$ and $\gamma(v_-) = 0$ imply that $v_- \in S_*$. We set $\omega_0 = v$, which completes the proof of the existence of our small negative solution of (3). Note that since ω_0 is a local minimum of J, if it is a nondegenerate critical point then it is of Morse index 0.

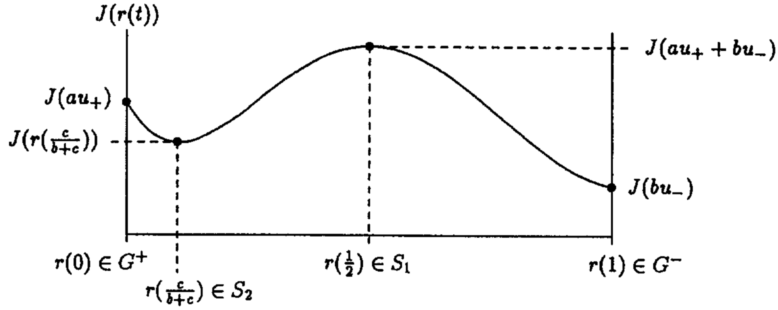

Fig. 2. Graph of $J(r(t))$.

4. Construction of paths on S

In this section we construct explicit paths on S and provide properties of the functional J restricted to these paths. From this analysis we will obtain the remaining three solutions, ω_1^+, ω_1^-, and ω_2.

Let $u\in S$ be such that $u_+,u_-\neq 0$, i.e. u changes sign. Then there exist positive constants a and b such that $au_+,bu_-\in S$. We define the convex linear combination

$$z(t)=(1-t)au_+ + tbu_-$$

and as in [3], we consider $\alpha\in C^1([0,1],(0,\infty))$ so that we can construct the smooth path $r:[0,1]\to S$ by $r(t)=\alpha(t)z(t)\in S$. In Fig. 2 we have displayed the graph of $J(r(t))$ with several important features labeled. Let us see that Fig. 2 is correct. Easily we see that $r(0)=au_+\in G^+$, $r(a/(a+b))=u$, $r(\frac{1}{2})=au_+ + bu_-\in S_1$, and $r(1)=bu_-\in G_-$. Since r is continuous, for t near 0 we see that $r(t)_-$ is near 0 and hence $\gamma(r(t)_-)<0$, $\gamma(r(t)_+)=\gamma(r(t))-\gamma(r(t)_-)=-\gamma(r(t)_-)>0$. Similarly, for t near 1 we have $\gamma(r(t)_+)>0$ and $\gamma(r(t)_-)<0$. Also, for some $t\in(0,\frac{1}{2})$ we have $r(t)_-\in S_*$ which implies that $\gamma(r(t)_-)=0$, $\gamma(r(t)_+)=0$, $r(t)_+\in G^+$, and hence $r(t)\in S_2$. Since $J|_S>0$, we see that

$$J(r(0))=J(au_+)<J\left(r\left(\frac{1}{2}\right)\right)=J(au_+ + bu_-),$$

$$J(r(1))=J(bu_-)<J\left(r\left(\frac{1}{2}\right)\right)=J(au_+ + bu_-).$$

For $t\neq\frac{1}{2}$,

$$J(r(t))=J(r(t)_+)+J(r(t)_-)<J(au_+)+J(bu_-)=J(\tfrac{1}{2}).$$

Thus, we see that the minimum $J(r(\bar{t}))=\min_{t\in[0,1]}$ occurs when $r(\bar{t})\in S_2$. Since there also exists $c>0$ (with $c<b$) so that $cu_-\in S_*$, we can explicitly determine the above

$\bar{t}$ as $c/(b+c)$. From the definition of J and r we obtain

$$\begin{aligned}\frac{\partial}{\partial t}J(r(t)) &= J'(r(t))(r'(t)) = \int_\Omega \{\nabla(r(t))\cdot\nabla(r'(t)) - r'(t)f(r(t))\}\,\mathrm{d}x \\ &= \frac{\alpha'(t)}{\alpha(t)}\gamma(r(t)) + \frac{1}{t}\gamma(r(t)_-) - \frac{1}{1-t}\gamma(r(t)+) = \frac{1}{t(1-t)}\gamma(r(t)_-), \qquad (11)\end{aligned}$$

for all $t\in(0,1)$. Since $\gamma((r(c/(b+c)))_-)=\gamma(cu_-)=0$ and $\gamma(r(\frac{1}{2})_-)=\gamma(bu_-)=0$, this confirms that indeed $J\circ r$ has a unique minimum at $t=c/(b+c)$ and maximum at $t=\frac{1}{2}$. As a new piece of information not specifically used in our proof but of intrinsic interest in understanding the behavior of the functional J on S, we note that

$$\begin{aligned}\frac{1}{t(1-t)}\gamma(r(t)_-) &= \frac{1}{t(1-t)}\left[\alpha^2(t)b^2t^2\|u_-\|^2 - \int_\Omega \alpha btu_- f(\alpha btu_-)\,\mathrm{d}x\right] \\ &= \left[\alpha^2(t)b^2\frac{t}{1-t}\|u_-\|^2 - \frac{\alpha b}{1-t}\int_\Omega u_- f(\alpha btu_-)\,\mathrm{d}x\right] \qquad (12) \\ &\to \varepsilon\alpha b\int_\Omega u_-\,\mathrm{d}x \quad \text{as } t\to 0.\end{aligned}$$

Similarly, the limit as $t\to 1$ is given by $-\varepsilon\alpha a\int_\Omega u_+\,\mathrm{d}x$. Thus in this case $(\partial/\partial t)J(r(t)) <0$ for $t\in\{0,1\}$, as opposed to $(\partial/\partial t)J(r(t))=0$ for $t\in\{0,1\}$ in the original $\varepsilon=0$ case found in [3].

We conclude this section by further analyzing the topological properties of important subsets of H given in (6), as well as defining three additional subsets of S. As in [3], we can easily show that G^+ and G^- are connected. Indeed, given any two one-sign elements of the same sign, we can project the convex linear combination joining the two in a line segment onto S. This path lies entirely in the appropriate set G^+ or G^-. We define subsets of sign-changing elements of S by first recalling that if $u\in H$ with $u_+,u_- \neq 0$ then there exist $a,b>0$ such that $au_+\in G^+$ and $bu_-\in G^-$. With a and b so chosen for each sign-changing $u\in H$,

$$\hat{S}=\{u\in S: u_+,u_-\neq 0\},$$

$$\hat{S}^+ = \left\{u\in\hat{S}: \frac{a}{a+b}<\frac{1}{2}\right\} \quad\text{and}\quad \hat{S}^- = \left\{u\in\hat{S}: \frac{a}{a+b}>\frac{1}{2}\right\}.$$

We observe that we have the disjoint union $\hat{S}=\hat{S}^+\cup S_1\cup\hat{S}^-$ and that we can obtain equivalent definitions (see the paragraph following (6)) $W^+=G^+\cup\hat{S}^+$ and $W^-=G^-\cup\hat{S}^-$. It is clear that W^+ and W^- are the (only) two connected components of $S-S_1$ and that S_1 separates G^+ from G^- (see also [3].) From the continuity of γ and the map $u\to u_+$, we see that S_1, S_2, G^+, and G^- are closed, as well as that $\hat{S}^+$, $\hat{S}^-$, $\hat{S}=\hat{S}^-\cup\hat{S}^+$, W^+, and W^- are open. Direct proofs that the above subsets are connected are easy, with the exception of S_1 and S_2; since we do not need these two sets to be connected, we will not pursue the matter further.

5. Existence of the remaining three solutions

We have seen that $\omega_0 \in S_*$ is a negative solution to (3). We claim that there exist solutions $\omega_1^+ \in S_2 \subset W^+$, $\omega_1^- \in G^- \subset W^-$, and $\omega_2 \in S_1$ so that

$$J(\omega_1^+) = \min_{W^-} J = \min_{S_2} J, \qquad J(\omega_1^-) = \min_{W^-} J = \min_{G^-} J, \quad \text{and} \quad J(\omega_2) = \min_{S_1} J.$$

Existence of the large negative solution: ω_1^-. Let $\{u_n\} \subset G^-$ be such that $\lim J(u_n) \downarrow \inf_{G^-} J$. By the coercivity of J and by again appealing to the S.I.T., as in Section 3 it follows that without loss of generality there exists $u \in H$ such that $u_n \rightharpoonup u$ in H and $u_n \to u$ in L^{p+1}. Since the map $u \to u_+$ is continuous in L^{p+1}, it follows that $u_+ = 0$. Since

$$\int_\Omega \{uf(u) - \varepsilon u\}\,\mathrm{d}x = \lim \int_\Omega \{u_n f(u_n) - \varepsilon u_n\}\,\mathrm{d}x = \lim \|u_n\|^2 \ge \delta > 0,$$

we see that $u = u_- \neq 0$. Suppose that $u_n \not\to u$ in H. Then without loss of generality $\|u\| < \liminf \|u_n\|$, so that as before, $\gamma(u) < \liminf \gamma(u_n) = 0$. Then there exists $\alpha \neq 1$ so that $\alpha u \in G^-$. This leads to a contradiction, since

$$J(\alpha u) < \liminf J(\alpha u_n) \le \liminf J(u_n) = \inf_{G^-} J.$$

Thus $u_n \to u$ in H and $J(u) = \min_{G^-} J$. Since for all $u \in \hat{S}^- = W^- - G^-$ the path $r \equiv r_u$ as constructed in the previous section provides $r(1) \in G^-$ and $J(r(1)) < J(r(\frac{a}{a+b})) = J(u)$, we see that $\min_{G^-} J = \min_{W^-} J$. Since W^- is an open subset of the C^1-submanifold S, we conclude that $\omega_1^- \equiv u$ is a critical point of J and hence a negative solution of (3). Since ω_1^- is a local minimizer of $J|_S$, if it is a nondegenerate critical point it has Morse index 1.

Existence of the sign-changing solution: ω_2. We take $\{u_n\} \subset S_1$ so that $J(u_n) \downarrow \inf_{S_1} J$. Again appealing to the coercivity of J and the S.I.T., there exists $u \in H$ so that $u_n \rightharpoonup u$ in H and $u_n \to u$ in L^{p+1}. We can find additional elements as weak limits in H and strong limits in L^{p+1} of $(u_n)_+$ and $(u_n)_-$, but easily we see these in fact correspond to u_+ and u_-. Similarly to the above argument and that of [3], we see that $u_+, u_- \neq 0$. We proceed by supposing that $(u_n)_+ \not\to u$ in H, whereby (as before without loss of generality) $\gamma(u_+) < 0$ and there exists $\alpha \neq 1$ such that $\alpha u_+ \in G^+$. Also, there exists $\beta > 0$ so that $\beta u_- \in G^-$, whence we can construct $z = \alpha u_+ + \beta u_- \in S_1$. Then

$$\begin{aligned} J(z) &< \liminf\{J(\alpha(u_n)_+) + J(\beta(u_n)_-)\} \\ &\le \liminf\{J((u_n)_+) + J((u_n)_-)\} = \lim J(u_n) = \inf_{S_1} J. \end{aligned}$$

This contradiction implies that $(u_n)_+ \to u_+$ in H. A similar argument shows that $(u_n)_- \to u_-$ in H, whence we see that $u_n \to u$ in H, $J(u) = \min_{S_1} J$, and $u \in S_1$. Exactly as in [3], we use the separation property of S_1 and a form of the deformation lemma to conclude that $\omega_2 \equiv u \in S_1$ is a sign-changing solution to (3). If nondegenerate, this critical point is of Morse index 2. This final step of the sign-changing existence proof can be found in full detail in [3], but for convenience, we outline it below.

1. Assume that ω_2 is not a solution.
2. Construct the path r_{ω_2} connecting the positive and negative parts $(\omega_2)_+$ and $(\omega_2)_-$.
3. Deform this path along the negative gradient flow.
4. Observe that the resulting deformed path still connects the positive and negative parts of ω_2, and hence intersects S_1 at some element w'.
5. By construction, this new element $w' \in S_1$ satisfies $J(w') < J(\omega_2)$, contradicting the fact that ω_2 minimizes $J|_{S_1}$.

Unlike the sign-changing solution in [3], we cannot conclude that ω_2 changes sign exactly once. An identical argument as in that document shows that $(\omega_2)^{-1}(0,\infty)$ is connected, there is only one positive "hump". Since $J(v)<0$ for all $v \in S_*$, it may be that $(\omega_2)_-$ can be decomposed in to a single negative "hump" on S (in G^-) and possibly multiple elements of S_*. The key to the above analysis is the observation that as a solution to (3), the zero extension of $u=(\omega_2)|_A$ to all of Ω for any connected component $A \subset (\omega_2)^{-1}(\mathbf{R} - \{0\})$ satisfies $\gamma(u)=0$. Thus, $u \in S \cup S_*$. If the one-sign function u is positive, then $u \in S$ and $J(u)>0$ so that there may be only one such positive portion of ω_2. As observed above, however, $J|_{S_*}<0$, so that we may not conclude the same for negative portions u.

Existence of the mostly-positive solution: ω_1^+. Let $\{u_n\} \subset S_2$ be such that $J(u_n) \downarrow \inf_{S_2} J$. As a final application of the coercivity of J and the S.I.T., we see that there exists $u \in H$ such that without loss of generality $u_n \rightharpoonup u$ in H and $u_n \to u$ in L^{p+1}. Easily we see that $u_+ \neq 0$. We proceed in two cases, reflective of our eventual uncertainty as to whether ω_1^+ is positive on Ω or has a small negative component as well.

Case I. Suppose that $u_- = 0$. Then by the continuity of the map $u \to u_-$, we see that $(u_n)_- \to 0$ in H. Suppose that $(u_n)_+ = u_n \not\to u_+ = u$ in H. Then without loss of generality as before we have $\gamma(u)<0$, whence there exists $\alpha \neq 1$ such that $\alpha u \in G^+ \subset S^2 \subset S$. It then follows that

$$J(\alpha u) < \liminf J(\alpha u_n) \leq \lim J(u_n) = \inf_{S_2} J.$$

This contradiction implies that $u_n \to u = u_+ \in G^+$, whereby $\omega_1^+ \equiv u$ satisfies $J(\omega_1^+) = \min_{G^+} J = \min_{S_2} J = \min_{W^+} J$. Since W^+ is an open subset of S, we conclude that ω_1^+ is a critical point of $J|_S$ and hence a positive solution of (3). If nondegenerate, this solution is of Morse index 1.

Case II. Suppose that $u_- \neq 0$. Again if we suppose that $(u_n)_+ \not\to u_+$ in H, we can find $\alpha < 1$ so that $\alpha(u_n)_+ \in G^+$.

(a) Suppose that $(u_n)_- \to u_-$ in H. Then since S_* is closed, we see that $u_- \in S_*$, whence

$$\begin{aligned} J(\alpha u_+ + u_-) &< \liminf \{J(\alpha(u_n)_+) + J((u_n)_-)\} \\ &\leq \lim \{J((u_n)_+) + J((u_n)_-)\} = \lim J(u_n) = \inf_{S_2} J. \end{aligned}$$

This is a contradiction since $\alpha u_+ + u_- \in S_2$.

(b) Suppose instead that $(u_n)_- \not\to u_-$ in H. Then without loss of generality $\gamma(u_-)<0$, where as in Section 3 there are only two subregions of H to which u_- may belong.

1. If u_- is "outside" S, then there exists $\beta<1$ so that $\beta u_- \in G^- \subset S$. From this we would have $J(\beta u_-) < \liminf J(\beta(u_n)_-) \leq 0$, where the last inequality holds since $(u_n)_- \in S_*$ and $\beta<1$. This contradicts $J|_S>0$.
2. If u_- is "inside" S, then in fact v is "inside" S_*. Thus there exists $\beta>1$ such that $\beta u_- \in S_*$, whereby we obtain the contradiction

$$J(\alpha u_+ + \beta u_-) < J(\alpha u_+) + J(u_-) < \liminf\{J(\alpha(u_n)_+) + J((u_n)_-)\}$$
$$\leq \liminf\{J((u_n)_+) + J((u_n)_-)\} = \lim J(u_n) = \inf_{S_2} J.$$

The above contradictions imply that $(u_n)_+ \to u_+$ in H. Assuming that $(u_n)_- \not\to u_-$ in H leads to identical inequalities and contradictions. We conclude that $u_n \to u$ in H, $u \in S_2$, and $J(u) = \min_{S_2} J$. Define $\omega_1^+ \equiv u$. Refer to Section 4 and recall that $G^+ \subset S_2 \subset W^+$. For $w \in W^+ - S_2$, we see that $r(c/(b+c)) \in S_2$ and $J(r(c/(b+c))) < J(w)$. Thus $\min_{W^+} J = \min_{S_2} J = J(\omega_1^+)$, hence ω_1^+ minimizes J over the open set $W^+ \subset S$. It follows that w_1^+ is a critical point of $J|_S$ and hence a solution to (3). If nondegenerate, this critical point is of Morse index 1. We see that $(\omega_1^+)^{-1}(0,\infty)$ is connected, but we cannot determine if $u_- = 0$ or if $(\omega_1^+)^{-1}(-\infty,0)$ consists of possibly multiple supports of elements of S_*.

6. Numerical algorithm and related semipositone results

In this section we outline a numerical algorithm for computing the four solutions detailed in our Main Theorem and leave fuller detail of implementation to [5] (see also [6] for more on gradient descent in general). We also report on a numerical experiment on the disk in $\mathbf{R}^2$ and compare with a theorem from the paper [11].

Numerical Algorithm

1. Initialize u_0 with appropriate nodal structure *projected* on to S to find w_1^+ and w_1^-, or on to S_1 to find w_2. Finding the local minimum w_0 requires no projection.
2. Begin Loop with $k=0$.
   (a) Solve linear system to obtain $\nabla J(u_k)$.
   (b) Take gradient descent step and reproject (as needed) on to S or S_1.
   (c) Increment k and repeat steps (a) and (b) until convergence criteria are met: $\|\nabla J(u_k)\|_2^2 \approx 0$, $\|\Delta u + f(u) - \varepsilon\|_2^2 \approx 0$, etc.

The projections can be implemented by iteratively following gradient *ascent* in the ray direction of u on to S or u_+, u_- on to S and hence u on to S_1. These projections are the new features of the algorithm making it differ from pre-existing steepest descent algorithms. The grid of approximation is composed of intervals, squares, or (potentially) cubes, on which standard differencing and integration schemes can compute the values of u, J, γ, $\|u\|^2$, etc. . . The linear system can be solved by Gaussian elimination as in our ODE runs or, more efficiently, by iterative methods such as Gauss–Sidel or SOR in the PDE case.

As a final note, we observe that in [11] radially symmetric solutions to problems such as ours are treated on the ball in $\mathbf{R}^N$ and the bifurcation diagram almost completely understood. That work considers the equation $\Delta u+\lambda f(u)=0$ and (among other conclusions) shows that there exists a value of $\hat{\lambda}$ so that for $\lambda<\hat{\lambda}$ there exists a positive solution, whereas for $\lambda>\hat{\lambda}$ only sign-changing solutions exist. Unlike the $f(0)=0$ case, the positive branch does not bifurcate from the trivial solution, instead makes a continuous unbounded loop passing to sign-changing solutions at $\hat{\lambda}$, where the solution satisfies the zero Neumann boundary condition as well as the zero Dirichlet condition. We have observed this phenomenon while running our numerical experiments. Specifically, when we choose $f(u)=u^3-\varepsilon$ and varied λ from $a=\lambda_1/f'(\beta)$ to $b=\lambda_2/f'(\beta)$ where $\beta=\varepsilon^{1/3}$ is the zero of f, we observed the behavior specified in [11]. For example, this verified that the value of $\hat{\lambda}$ did indeed fall between a and b and that the positive solution at $\lambda=\hat{\lambda}$ was also a solution to the zero Neumann problem with minimal sup-norm. That is to say, $w_1^+(\hat{\lambda})$ is the bottom most point of the bifurcation loop as shown in [11].

An important question is: where do the sign-changing solutions $w_2(\lambda)$ fit in to the bifurcation diagram? A conjecture supported by our numerical experiments is that w_2 is *nonradial* when Ω is a ball or annulus, and hence does not appear in the literature studying radial solutions via the corresponding singular ODE. Our full conjecture is somewhat more general, applying to general regions and for wider classes of nonlinearites. Specifically, we believe that the internal zero set of our minimal action value sign-changing solution intersects the boundary of Ω. This would imply, for instance, that one gets infinitely many solutions w_2 on the disk and four on the square. We are currently trying to prove this conjecture, at least in special cases for specific regions or narrower classes of nonlinearites. As we make progress on this matter we should be able to add nonradial branches to the existing diagram.

The method is also proving useful in investigating qualitative properties of solutions on the annulus and for a wide variety of nonlinearites. We are able to generate good approximations to not only superlinear problems, but also asymptotically linear and sublinear problems. The general concept is applicable (as is our method of proof) to many other related problems, including the p-Laplacian.

Of great interest to the author are the many graphics (not included) of *actual data* detailing finite dimensional slices of the surface of S and the behavior of J when restricted to them. It is particularly thought provoking to intersect a two- or three-dimensional eigenspace with S or S_1 in order to visualize the mountain pass hierarchy of projected eigenfunctions increasing in norm and action value, or the role of symmetry (of both f and Ω) in the geometrical relationship between solutions viewed as points on our manifold. It is the author's hope and belief that these numerical investigations will be an aid in the further analytic development of existence, multiplicity, and bifurcation theories of semilinear elliptic boundary value problems. To that end, experiments have already been performed which have yielded solutions to superlinear, asymptotically linear, and sublinear equations with symmetry and asymmetry on the interval, square, disk, annuli, and dumbell. There is nothing to stop the interested programmer from easily adapting the code (FORTRAN code available upon request) to investigate more unusual regions or different boundary conditions on a wide range of nonlinearites.

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