Graph Rubbling

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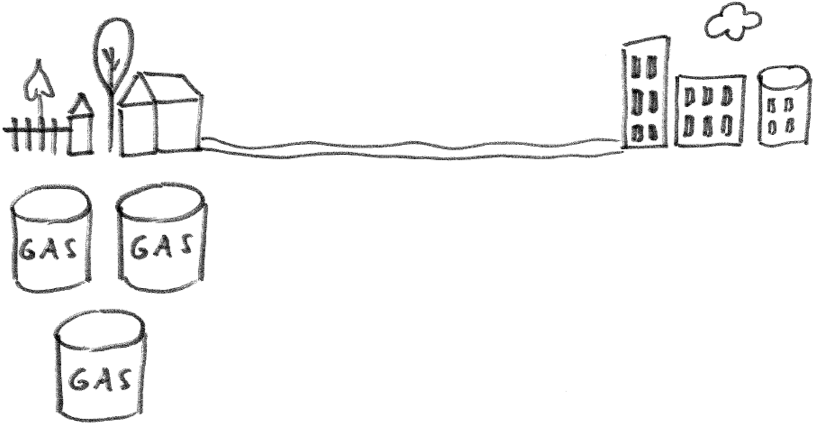
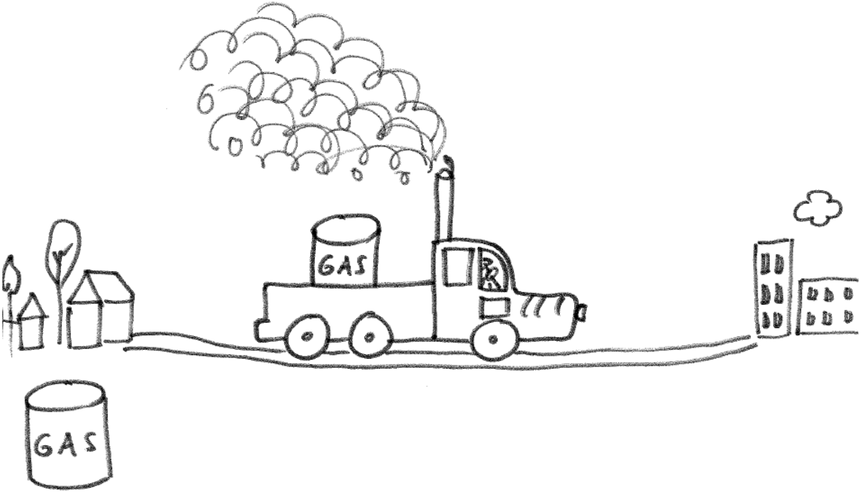
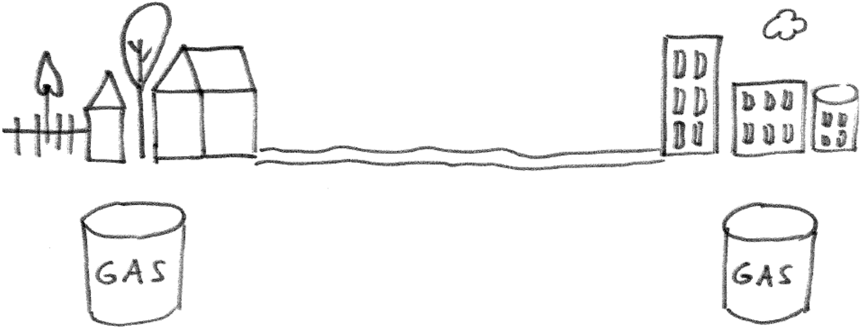
Pebbling



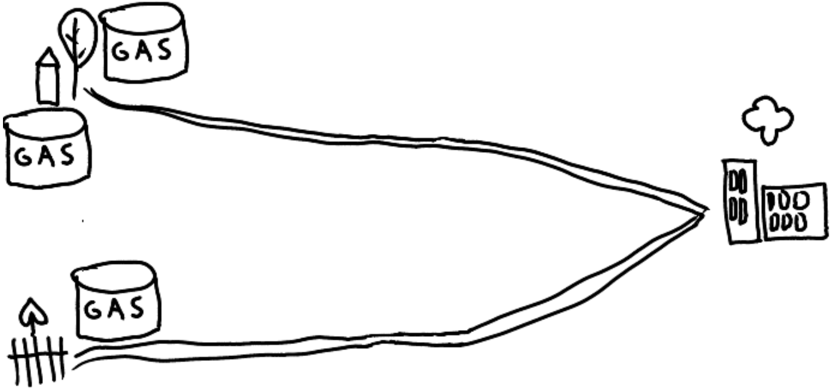
Pebbling



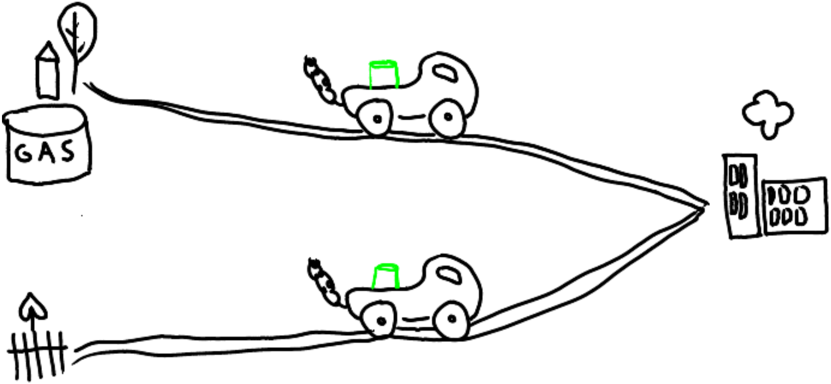
Pebbling



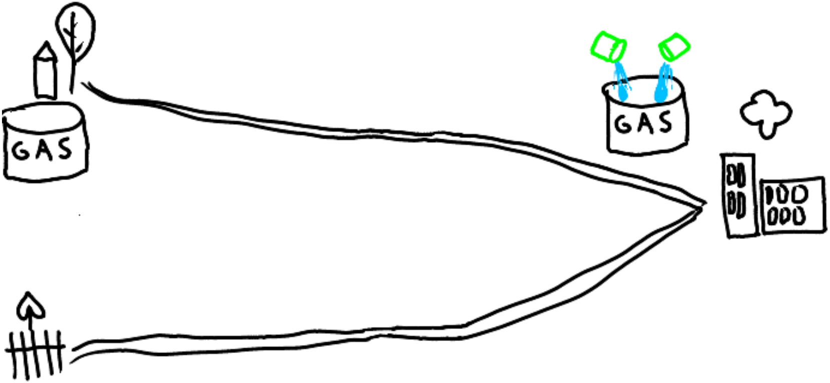
Rubbling



Rubbling

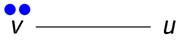


Rubbling



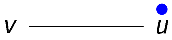
Rubbling moves

pebbling move $(v, v \to u)$:



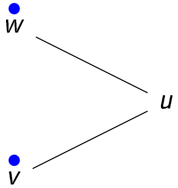
$p(v, u) = (2, 0)$

$\rightsquigarrow$



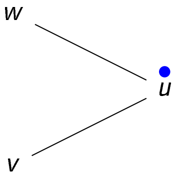
$p_{(v, v \to u)}(v, u) = (0, 1)$

strict rubbling move $(v, w \to u)$:



$p(v, w, u) = (1, 1, 0)$

$\rightsquigarrow$



$p_{(v, w \to u)}(v, w, u) = (0, 0, 1)$

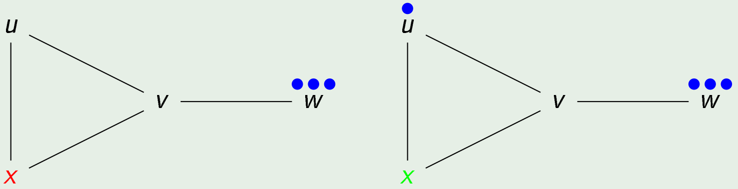
Rubbling number $\rho(G)$

Definition

Minimum number of pebbles to make any vertex reachable no matter how the pebbles are distributed on the vertices.

Example

$\rho(G) = 4$



Note: $\pi(G) = 5$
http://www.cefns.nau.edu/~ns46/pebbling/census.html

Basic properties

d =diameter, n =number of vertices

Fact

1. $\rho(G) \leq \pi(G)$
2. $2^d \leq \rho(G)$

$\rho(G) < n$ possible

Theorem

$\rho(G) \leq (n-d+1)(2^{d-1}-1)+2$

Recall [Chan, Godbole]

$\pi(G) \leq (n-d)(2^d-1)+1$

Simple graphs

Theorem

1. $\rho(P_n) = 2^{n-1}$
2. $\rho(K_n) = 2$
3. $\rho(W_n) = 4$
4. $\rho(K_{m,n}) = 4$
5. $\rho(Q_n) = 2^n$
6. $\rho(Petersen) = 5$
7. $\rho(Lemke) = 8$

Recall

1. $\pi(P_n) = 2^{n-1}$
2. $\pi(K_n) = n$
3. $\pi(W_n) = n$
4. $\pi(K_{m,n}) = m + n$
5. $\pi(Q_n) = 2^n$
6. $\pi(Petersen) = 10$
7. $\pi(Lemke) = 8$

Question

What property of G makes $\pi(G) = \rho(G)$?

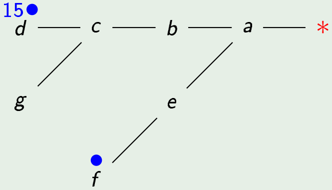
Theorem

$(p_1, \ldots, p_m)$ length sequence of maximum path partition for T

$$\rho(T) = 2^{p_1} + \sum_{i=2}^{m} 2^{p_i - 1} - m + 1$$

Example

$\rho(T) = (2^4 - 1) + (2^1 - 1) + (2^0 - 1) + 1 = 17$



$(4, 2, 1)$ length sequence

No Cycle Lemma

Definition

Transition digraph:

pebbling move $(v, v \to u)$ adds the edges: $v \Longrightarrow u$

strict rubbling move $(v, w \to u)$ adds the edges: $v \longrightarrow u \longleftarrow w$

Rubbling sequence (multiset) acyclic if no cycle in graph

Theorem

TFAE

1. v *is reachable from pebble distribution* p
2. $\exists$ *multiset* S *of moves such that* $p_S \geq 0$ *and* $p_S(v) \geq 1$
3. $\exists$ *acyclic multiset* R *such that* $p_R \geq 0$ *and* $p_R(v) \geq 1$
4. v *is reachable from* p *through an acyclic rubbling sequence*

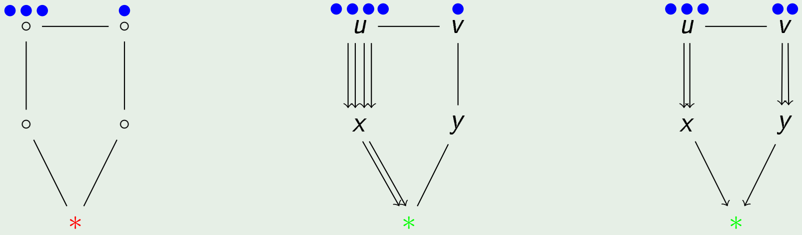
Cycle graphs

Theorem

$\rho(C_{2k}) = 2^k$ *and* $\rho(C_{2k+1}) = \lfloor \frac{7\cdot 2^{k-1}-2}{3} \rfloor + 1.$

Example

$\rho(C_5) = 5$

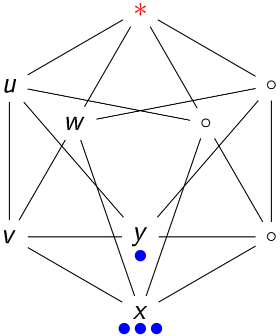


Graham's conjecture fails for rubbling

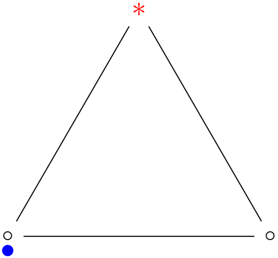
Example

$\rho(C_3 \square C_3) = 5 > 4 = 2 \cdot 2 = \rho(C_3)\rho(C_3)$

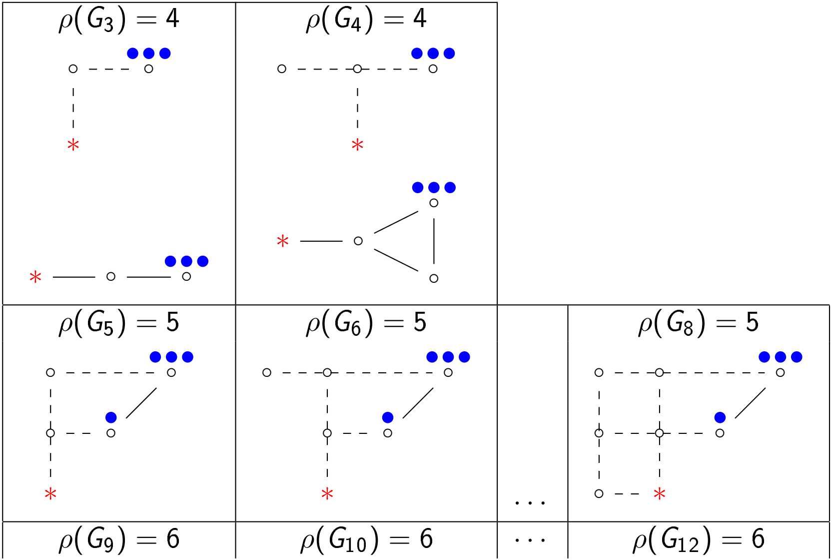
$C_3 \square C_3$



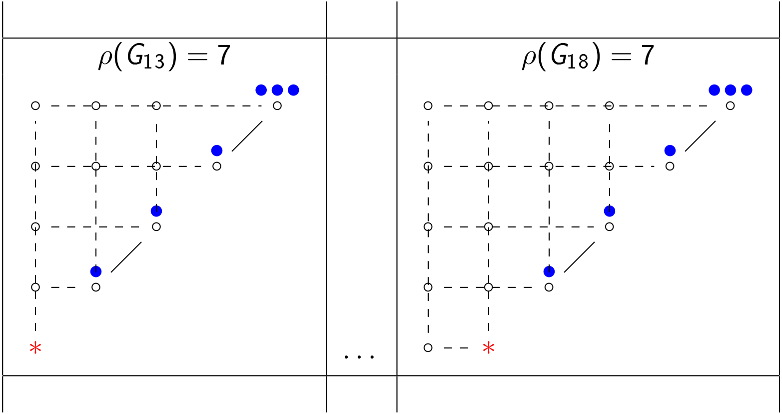
C_3



Diameter two graph family $\rho(G_n) = \lfloor \sqrt{2n-1} \rfloor + 2$



Diameter two graph family $\rho(G_n) = \lfloor \sqrt{2n-1} \rfloor + 2$



Diameter two graph family $\rho(G_n) = \lfloor \sqrt{2n-1} \rfloor + 2$

Definition

$f(n,k) := \max\{\rho(G) : \mathrm{diam}(G) = k \text{ and } |V(G)| = n\}$

Fact

$f(n,2) = \rho(G_n)$ *for* $n \in \{3, \ldots, 9\}$

Question

Is $f(n,2) = \rho(G_n)$ for $n \geq 10$?

Question

$f(n,k) =?$

Recall

$\mathrm{diam}(G) = 2$ implies $\pi(G) \in \{n, n+1\}$

Optimal rubbling $\rho_{\text{opt}}(G)$

Definition

Minimum number of pebbles in a chosen pebble distribution from which any vertex is reachable.

Example

$\rho_{\text{opt}}(P_6) = 4$

$\pi_{\text{opt}}(P_6) = 4$

Optimal rubbling

Theorem

1. $\rho_{opt}(P_n) = \lceil \frac{n+1}{2} \rceil$
2. $\rho_{opt}(C_n) = \lceil \frac{n}{2} \rceil$
3. $\rho_{opt}(K_n) = 2$
4. $\rho_{opt}(W_n) = 2$
5. $\rho_{opt}(K_{m,n}) = 3$
6. $\rho_{opt}(Petersen) = 4$

Recall

1. $\pi_{\text{opt}}(P_n) = \lceil \frac{2n}{3} \rceil$
2. $\pi_{\text{opt}}(C_n) = \lceil \frac{2n}{3} \rceil$
3. $\pi_{\text{opt}}(K_n) = 2$
4. $\pi_{\text{opt}}(W_n) = 2$
5. $\pi_{\text{opt}}(K_{m,n}) = 3$
6. $\pi_{\text{opt}}(Petersen) = 4$

Question

$\rho_{\text{opt}}(Q_2) = 2,\ \rho_{\text{opt}}(Q_3) = 3,\ \rho_{\text{opt}}(Q_4) = 4,\ \rho_{\text{opt}}(Q_5) = 6,$
$\rho_{\text{opt}}(Q_n) = ?$

Optimal rubbling

- Lisa Danz: Optimal t-rubbling of complete m-ary trees (REU project with Gallian, Duluth)
- László Papp: Optimal rubbling of caterpillars (thesis with Katona, Budapest)
- László Papp: $\rho_{\text{opt}}(P_{3k+r}\square P_2) = 2k + 1 + \lceil \frac{r}{3} \rceil$ Optimal rubbling of prisms and ladders (project with Katona, Budapest)

Question

$\rho_{\text{opt}}(P_n\square P_m) = ?$, $\rho_{\text{opt}}(C_n\square C_m) = ?$

Question

How does optimal rubbling change if we only allow strict rubbling moves?

Cover rubbling

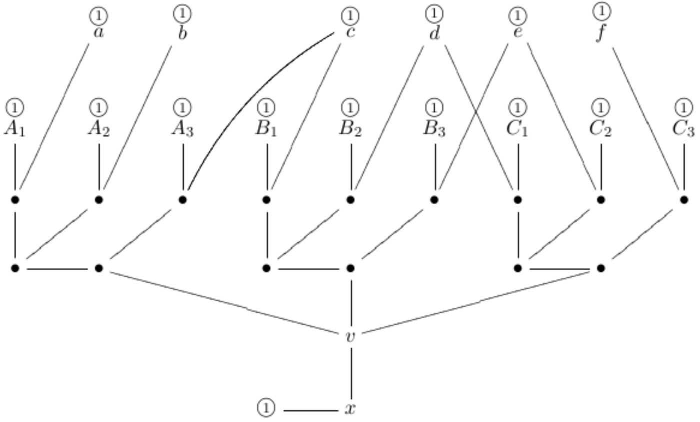
Conjecture

The cover pebbling and cover rubbling numbers are the same for all graphs. Stacking Lemma remains true.

Complexity

Theorem

Decision problem whether a vertex is reachable from a given configuration is NP-complete.



Conjecture

Finding the rubbling number is Π_2^P-hard.

A natural approach for a proof would be to use the corresponding pebbling results.

Question

How to reduce pebbling to rubbling?

t-rubbling number $\rho_t(G)$

Definition

Minimum number of pebbles to make any vertex reachable with t pebbles no matter how the pebbles are distributed.

Theorem

$\rho_t(T) = t2^{p_1} + \sum_{i=2}^{m} 2^{p_i - 1} - m + 1$ *for a tree* T.

Question

What is $\rho_t(G)$ for $G = C_n$ and other simple graphs?

Conjecture

$t \mapsto \rho_t(G)$ is linear for $t \geq t_0$.
Is there a t_0 that works for all G?

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