1 Solving equations

1.1 Necessary or sufficient

Consider the equation

$$x + 1 = x.$$

To solve it we first square both sides and get

$$x^2 + 2x + 1 = x^2.$$

Subtracting $x^2 + 1$ from both sides gives

$$2x = -1.$$

Finally dividing by 2 results in

$$x = -\frac{1}{2}.$$

The problem is that this is not a solution of the original equation. In fact, the original equation has no solution. The very first step that squares both sides is not reversible since $a^2 = b^2$ does not imply $a = b$.

What we need to learn from this is that solving an equation or a system of equations usually requires two stages. First we try to simplify our equations until we get a few solution candidates. This involves finding consequences of the equations. So we essentially find a necessary condition for a solution. In the second stage we need to verify that our solution candidates are in fact solutions. Sometimes this is not true for some of the solution candidates. The solution set contains those candidates that actually satisfy the equation.

We do not need the second stage if during the first stage we only use reversible steps. Reversible steps produce equivalent equations.

1.2 Is it legal?

Now consider the equation

$$x(x + 1) = x^2.$$

Dividing both sides by x gives

$$x + 1 = x$$

and we now know that this equation has no solution. Again we have a problem since $x = 0$ is clearly a solution of the original equation. The problem is that dividing by a quantity is only allowed if the quantity is not 0.

1.3 Style

We solve the equation $2(x+2) = x$.

Example of what not to do:

$$2(x+2) = x$$
$$2x+4 = x$$
$$x+4 = 0$$
$$x = -4$$

This seems natural and in fact this is how we solve an equation for ourselves on our scratch paper. In spite of this, we do not consider this correct. The first problem is that it is not a sentence but there are more serious issues. It is not clear what the logical relationship between the four equations are. Do we claim that all four are immediately true for some reason? Do we claim that the four equations are equivalent? Perhaps the first follows from the second which follows from the third and so on. In this case the four equations are actually equivalent but what we most likely mean is that the first implies the second which implies the third, and so on. This vagueness is the main reason for the issue mentioned in Section 1.1. Here is the correct solution.

Expanding the parentheses gives

$$2x+4 = x.$$

Subtracting x from both sides results in

$$x+4 = 0.$$

Finally subtracting 4 from both sides implies that the only possibility for the solution is

$$x = -4.$$

Substituting -4 for x in the original equation verifies that this is in fact a solution. So the only solution is $x = -4$.

Instead of verifying at the end that our candidate is in fact a solution, we could have said that our steps are reversible so the solution is $x = -4$. This is true in this particular case but it is very dangerous thing to claim in more complicated cases. Only do this if you are absolutely sure that this is true. The steps for example were not reversible in Section 1.1 because of the squaring of both sides. This is easy to miss though.

Explaining every single step can be a bit painful, so in practice we usually don't show all these minor steps.