WILD POLYOMINO WEAK
(1,2)-ACHIEVEMENT GAMES

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Abstract

A version of polyomino achievement games is studied, in which the first player marks one and the second player marks two cells at each move. A wild polyomino is a finite set of cells that are connected through an edge or through a corner. All wild polyominos on an infinite 2-dimensional rectangular board are characterized to be weak winners or losers.

1. Introduction

Achievement games for polyominos have been introduced by Frank Harary [Gar, Ha1, Ha2, HH, Ha3]. They are generalizations of the well known game Tic-Tac-Toe, where the target shape can be some predetermined set of polyominos. The type of the board can vary as well. It can be a tiling of the plane by triangles [HH, BH3] or hexagons [BH2]. The board can also have higher dimensiona [HW], it can be a Platonic solid [BH1] or the hyperbolic plane [Bod]. For further results of the problem see [HHS, HS1, HS2, HS3].

A *polyomino* or *animal* is a finite set of cells of the infinite chessboard such that both the polyomino and its complement are connected through edges. A *wild polyomino* or *wild animal* is a finite set of cells of the infinite chessboard that is connected through edges or through corners. We only consider polyominos up to congruence, that is, the location of the polyomino on the board is not important.

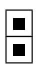
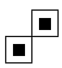
In a *polyomino weak achievement game,* or *A-achievement game,* two players alternately mark previously unmarked cells of the board using their own colors. The first player (the maker) wins if he can mark a set of cells congruent to a given polyomino. The second player (the breaker) wins if she can prevent the maker from achieving the given polyomino. In a $(1,2)$-achievement game [Plu] the first player marks a single cell while the second player marks two cells at each move.

A polyomino is called a (*weak*) *winner* if the first player can always win the weak achievement game with the given polyomino. Otherwise the polyomino is called a *loser.*

In this paper we classify all wild animals as winners or losers in the weak $(1,2)$-achievement game on an infinite rectangular board.

2. Animals with at most three cells

There are three wild animals with fewer than three cells:

P_0  P_1  P_2 

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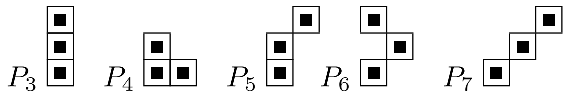
It is clear that all of these are winners. There are five wild animals with three cells:

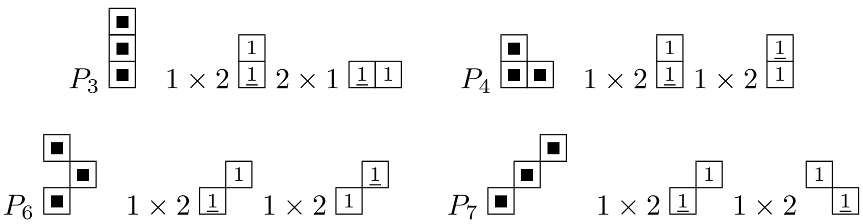


Proposition 2.1. *The animals P_3, P_4, P_6 and P_7 are losers.*

Proof. To show that an animal is a loser we can pave the board twice, using pairs of cells. We choose the two pavings such that every position of the given animal contains a full pair of cells either from one of the pavings or from the other. The existence of such double paving allows the breaker to win by marking the two cells which are the pairs of the maker's mark in the two pavings. None of these cells can already be marked by the maker but it is possible that they are already marked by breaker. In that case the breaker can mark any other cell.

A paving is determined by its fundamental region with dimensions $m \times n$ by shifting the region by vectors in the set $\{(ma, nb) \mid a, b \in \mathbf{Z}\}$. In the figures below, cells containing the same numbers in the fundamental region are the pairs of the paving. A double paving requires two fundamental regions. We also need to know the relative positions of these regions on the board. We indicate this by a special cell in each of the fundamental regions which are on top of each other on the board. The special cells have underlined characters in them.

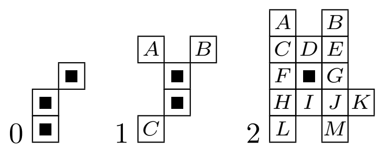
The following figure shows the animals and the fundamental regions of their double pavings:



The reader can easily verify that these double pavings have the required properties. □

Proposition 2.2. *The animal P_5 is a winner.*

Proof. The following is a winning strategy for the maker.

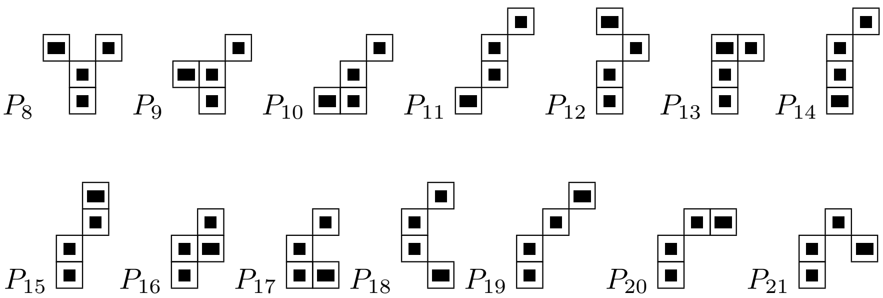


Cells with a solid block represent the marks of the maker. Cells with letters in them represent empty cells. The game starts at situation 2 and ends at situation 0. It is easy to see that no matter how the breaker marks two empty cells of situation 2, the maker can mark one of the cells containing the letter D, G or I and achieve situation 1. It is also clear that if the breaker marks two empty cells of situation 1 then the maker can win by marking the third empty cell. □

Note that P_5 has cells connected by edges and cells connected by corners as well. Perhaps this is the reason a double paving does not exists allowing it to be a winner.

3. Animals with four cells

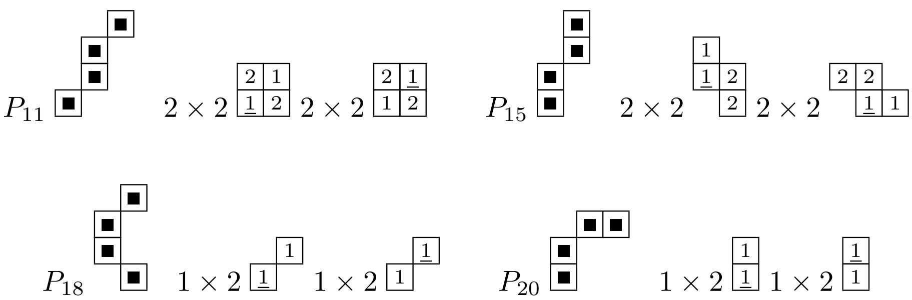
An animal containing a smaller losing animal is a loser itself. Hence the only possibility to find winners with four cells is to consider animals that are created from P_5 by adding a single cell. There are fourteen such animals:



All of these except P_{11}, P_{15}, P_{18} and P_{20} contain a loser of Proposition 2.1 and therefore they are losers themselves.

Proposition 3.1. *The animals P_{11}, P_{15}, P_{18} and P_{20} are losers.*

Proof. As in Proposition 2.1 it suffices to find an appropriate double paving of the board. The following figure shows the animals and the fundamental regions of their double pavings:



□

Since all animals with four cells are losers, there cannot be any winners with more than three cells. So we have the following main result.

Theorem 3.2. *The only winning wild animals in the weak $(2, 1)$-achievement game on the 2-dimensional infinite rectangular board are the animals P_0, P_1, P_2 and P_5.*

There are several questions to be answered. What are the winning pairs or triples of animals? What are handicap numbers of the losers? What are the winners in the (1,2)-achievement game on triangular, hexagonal and higher dimensional boards? What are the winners in the (2,3)-achievement game?

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