FELL BUNDLES OVER r-DISCRETE GROUPOIDS AND INVERSE SEMIGROUPS

Nándor Sieben

Abstract

Fell bundles over r-discrete groupoids and inverse semigroups are closely related. Homogeneous Fell bundles over inverse semigroups are semidirect product bundles of twisted inverse semigroup actions.

This paper is the result of an unfinished project. It is not carefully written, most likely contains errors.

1. introduction

2. Inverse semigroup bundles

Definition 2.1. An inverse semigroup S is *upward-directed* if for all idempotents f and g in S there is an idempotent e such that $f \leq e$ and $g \leq e$.

Note that every unital inverse semigroup is upward-directed.

Definition 2.2. A *Fell bundle* B over a discrete upward-directed inverse semigroup S is the union of the pairwise disjoint Banach spaces $\{B_s\}_{s\in S}$, continuous, linear *inclusion maps* $i_{t,s} : B_t \to B_s$ for all $t \leq s$ and with multiplication and involution such that for all $a, b, c \in B$, $r, s, t \in S$ and $\lambda, \mu \in \mathbf{C}$ we have

(a) $B_r B_s \subset B_{rs}$;
(b) $(\lambda b)(\mu c) = \lambda\mu(bc)$;
(c) $(ab)c = a(bc)$;
(d) $\|bc\| \leq \|b\|\,\|c\|$;
(e) $B_s^* \subset B_{s^*}$;
(f) $(\lambda b)^* = \bar{\lambda} b^*$;
(g) $(bc)^* = c^* b^*$;
(h) $b^{**} = b$;
(i) $\|b^*\| = \|b\|$;
(j) $\|bb^*\| = \|b\|^2$;
(k) $bb^* \geq 0$ in B_{ss^*} if $b \in B_s$;
(l) $i_{r,s} \circ i_{s,t} = i_{r,t}$ if $t \leq r \leq s$;
(m) $i_{f,e}$ is injective if $e \leq f$ in E_S;
(n) $i_{r,s}(b_s)c_t = i_{rt,st}(b_s c_t)$ if $s \leq r$;
(o) $i_{s,t}(b^*) = i_{s^*,t^*}(b)^*$ if $b \in B_t$;
(p) $i_{s,s}$ is the identity map for all $s \in S$.

Note that B_{ss^*} is a C^*-algebra for all $s \in S$ and so condition (k) makes sense. Also note that conditions (e) and (h) imply that $B_s^* = B_{s^*}$ for all $s \in S$.

Lemma 2.3. *If B is a Fell bundle over S then*
(a) $b_s i_{r,t}(c_t) = i_{sr,st}(b_s c_t)$ if $t \leq r$, $b_s \in B_s$ and $c_t \in B_t$;
(b) $i_{r,q}(b_q) i_{t,s}(c_s) = i_{rt,qs}(b_q c_s)$ if $q \leq r$, $s \leq t$, $b_q \in B_q$ and $c_s \in B_s$.

Proof. Part (a) follows from the calculation

$$b_s i_{r,t}(c_t) = (i_{r^*,t^*}(c_t^*) b_s^*)^* = i_{sr,st}(b_s c_t)^{**} = i_{sr,st}(b_s c_t).$$

Part (b) follows using part (a)

$$i_{r,q}(b_q) i_{t,s}(c_s) = i_{rt,qt}(b_q i_{t,s}(c_s)) = i_{rt,qt}(i_{qt,qs}(b_q c_s)) = i_{rt,qs}(b_q c_s).$$

□

It follows from this lemma that if $e \leq f$ then $i_{fe} : B_e \to B_f$ is an isomorphism between C^*-algebras.

Lemma 2.4. *If B is a Fell bundle over S, $a \in B_s$, $ss^* \leq e$, u_j is a bounded approximate identity for B_e, then $\lim_j u_j a = a$.*

Proof.
$$\begin{aligned}\lim_j \|u_j a - a\|^2 &= \lim_j \|(u_j a - a)(u_j a - a)^*\| \\ &= \lim_j \|u_j aa^* u_j - u_j aa^* - aa^* u_j + aa^*\| \\ &= \lim_j \|i_{e,ss^*}(u_j aa^* u_j - u_j aa^* - aa^* u_j + aa^*)\| \\ &= \lim_j \|u_j i_{e,ss^*}(aa^*) u_j - u_j i_{e,ss^*}(aa^*) - i_{e,ss^*}(aa^*) u_j + i_{e,ss^*}(aa^*)\| \\ &= 0\,.\end{aligned}$$
□

Let $B_s \cdot B_t$ denote the closed linear span of $B_s B_t$ in B_{st}. This product has the following properties:

Lemma 2.5. *If B is a Fell bundle over S, then we have*
(a) *$(B_r \cdot B_s) \cdot B_t = B_r \cdot (B_s \cdot B_t)$ for all $r, s, t \in S$;*
(b) *$B_s \cdot B_{s^*}$ is a closed ideal of B_{ss^*} for all $s \in S$;*
(c) *$B_s \cdot B_{s^*} \cdot B_s = B_s$ for all $s \in S$;*
(d) *$B_e \cdot B_f = B_e$ for all $e \leq f$ in E_S;*
(e) *$B_s \cdot B_e = B_s$ for all $s^*s \leq e$ in S;*
(f) *$B_e \cdot B_s = B_s$ for all $ss^* \leq e$ in S.*

Proof. The proof of (a) and (b) is straightforward. To show (c), note that $B_s \cdot B_{s^*} \cdot B_s \subset B_{ss^*s} = B_s$. On the other hand, if u_λ is a bounded approximate identity for the C^*-algebra $B_s \cdot B_{s^*}$ and $a \in B_s$, then
$$\begin{aligned}\lim_\lambda \|u_\lambda a - a\|^2 &= \lim_\lambda \|(u_\lambda a - a)(u_\lambda a - a)^*\| \\ &= \lim_\lambda \|u_\lambda aa^* u_\lambda - u_\lambda aa^* - aa^* u_\lambda + aa^*\| = 0\,.\end{aligned}$$
Hence $a \in (B_s \cdot B_{s^*})B_s \subset B_s \cdot B_{s^*} \cdot B_s$. To show (d) note that $B_e \cdot B_f \subset B_e$. On the other hand, if $a \in B_e$ and u_j is a bounded approximate identity for B_f, then
$$\|auj - a\| = \|i_{fe}(au_j - a)\| = \|i_{fe}(a)u_j - i_{fe}(a)\| \to 0.$$
(e) follows from the calculation:
$$\begin{aligned}B_s &= B_s \cdot B_{s^*} \cdot B_s \subset B_s \cdot B_{s^*s} = B_s \cdot (B_{s^*s} \cdot B_e) \\ &= (B_s \cdot B_{s^*s}) \cdot B_e \subset B_s \cdot B_e \subset B_s.\end{aligned}$$
Finally, (f) follows from (e) taking conjugates. □

Definition 2.6. A Fell bundle B over S is called *saturated* if $B_r \cdot B_s = B_{rs}$ for all $r, s \in S$.

Example 2.7. Let $A = \overline{\bigoplus_{t\in G} A_t}$ be a graded C^*-algebra over the group G, and let $S = \{A_\alpha : \alpha \in S(G)\}$ be the inverse semigroup of closed subspaces of A generated by the A_t's as in [Exe, Theorem 5.4]. Then $\cup\{A_\alpha, \alpha) : \alpha \in S(G)\}$ is a saturated Fell bundle over $S(G)$.

Lemma 2.8. *A Fell bundle B over S is saturated if $B_s \cdot B_{s^*} = B_{ss^*}$ for all $s, t \in S$.*

Proof. If $s, t \in S$ then $B_s \cdot B_t \subset B_{st}$. To show the other direction note that
$$B_s B_t \supset B_s i_{t,s^*st}(B_{s^*st}) = i_{st,ss^*st}(B_s B_{s^*st}) = B_s B_{s^*st}$$
hence
$$B_s \cdot B_t \supset B_s \cdot B_{s^*st} \supset B_s \cdot B_{s^*} \cdot B_{st} = B_{ss^*} \cdot B_{st} = B_{st}$$
by Lemma 2.5(f). □

Proposition 2.9. *If (B, S, β, v) is a Busby-Smith twisted action then the pull-back Banach bundle $r^*B = \{(b, s) \in B \times S : b \in B_{ss^*}\}$ is a saturated Fell bundle over S with product and adjoint defined by*

$$(a, s)(b, t) = (\beta_s(\beta_s^{-1}(a)b)v_{s,t}, st), \qquad (a, s)^* = (\beta_s^{-1}(a^*)v_{s^*,s}^*, s^*)$$

and inclusion maps defined by $i_{ts}(a, s) = (a, t)$ for all $s \leq t$.

Proof. The proof of [Sie, Proposition 3.1] shows that B is a Fell bundle. If $s, t \in S$ then we have

$$\begin{aligned}(r^*B)_s(r^*B)_t &= (B_{r(s)}, s)(B_{r(t)}, t) = (\beta_r(\beta_r^{-1}(B_{r(s)})B_{r(t)})v_{s,t}, st)\\ &= (\beta_s(B_{r(s^*)}B_{r(t)})v_{s,t}, st) = (r^*B)_{st}.\end{aligned}$$

□

Definition 2.10. The Fell bundle constructed in the previous proposition is called the *twisted semidirect product bundle* of the Busby-Smith twisted action.

If B is an inverse semigroup Fell bundle then the set $\Gamma_c(B)$ of finitely supported sections becomes a $*$-algebra if we extend the bundle operations linearly.

Lemma 2.11. *$\Gamma_c(B)$ has an enveloping C^*-algebra.*

Proof. If Π is a representation of $\Gamma_c(B)$ and $f = \sum_s b_s \in \Gamma_c(B)$ then

$$\|\Pi(f)\| = \|\Pi(\sum_s b_s)\| \leq \sum_s \|\Pi(b_s)\|$$

and

$$\begin{aligned}\|\Pi(b_s)\|^2 = \|\Pi(b_s)^*\Pi(b_s)\| &= \|\Pi((b_s)^*(b_s))\|\\ \|\Pi(b_s^*b_s)\| &\leq \|b_s^*b_s\|\end{aligned}$$

since B_{s^*s} is a C^*-algebra. Thus

$$\|\Pi(f)\| \leq \sum_s \|b_s\|.$$

□

Recall [QS] that if $r^*(B)$ is the semidirect product bundle of the action (B, S, β), then the crossed product $B \times_\beta S$ is defined as the completion of $\Gamma_c(r^*B)$ in the greatest C^*-seminorm coming from coherent representations, that is, from representations Π satisfying

$$\Pi(b, e) = \Pi(b, f) \quad \text{whenever } b \in B_e \text{ and } e \leq f.$$

We need a version of coherence for bundles.

Definition 2.12. If B is a Fell bundle then a representation Π of $\Gamma_c(B)$ is called *coherent* if $\Pi(b_s) = \Pi(i_{ts}(b_s))$ for all $b_s \in B_s$ and $s \leq t$.

The kernel of a coherent representation contains the ideal

$$I_B := \operatorname{span}\{b_s - i_{t,s}(b_s) : b_s \in B_s \text{ and } s \leq t\}.$$

Definition 2.13. The C^*-algebra $C^*(B)$ of the bundle B is the enveloping C^*-algebra of the quotient

$$\Gamma_c(B)/I_B.$$

Example 2.14. The set of natural numbers $\mathbf{N}$ with minimum as the multiplication is a semilattice. If $\{A_i\}_{i\in\mathbf{N}}$ are C^*-algebras such that $A_i \subset A_{i+1}$ for all i, then $\cup(A_i, i)$ is a Fell bundle B over $\mathbf{N}$ such that $C^*(B)$ is the direct limit of the A_i's.

Example 2.15. Let ${}_AX_B$ be an imprimitivity bimodule and

$$S = \{s, s^*, e, ss^*, s^*s, 0\}$$

be the inverse semigroup with identity e such that $ss = s^*s^* = 0$. We can build a saturated Fell bundle D over S by letting

$$D_s = X, \quad D_{s^*} = \tilde{X}, \quad D_e = A \oplus B, \quad D_{ss^*} = A, \quad D_{s^*s} = B, \quad D_0 = \{0\}$$

with the obvious multiplication. The inclusion maps are defined by

$$\begin{gathered} i_{e,ss^*}(a) = (a, 0), \quad i_{e,s^*s}(b) = (0, b), \\ i_{ss^*,0}(0) = 0_A, \quad i_{s^*s,0}(0) = 0_B, \\ i_{e,0}(0) = (0, 0). \end{gathered}$$

The map

$$x + \tilde{y} + (a_1, b_1) + b_2 + a_2 + 0 \mapsto \begin{pmatrix} a_1 + a_2 & x \\ \tilde{y} & b_1 + b_2 \end{pmatrix}$$

induces an isomorphism between $C^*(D)$ and the linking algebra of ${}_AX_B$.

Example 2.16. If r^*A is the twisted semidirect product bundle of (A, S, α, w) then $C^*(r^*A) \cong A \times_{\alpha,w} S$.

3. From groupoids to inverse semigroups

In this section we extend the connection between groupoid and inverse semigroup actions found in [QS] to the Fell bundle case. If A is a Fell bundle over the r-discrete groupoid G and S is a full inverse subsemigroup of G^{op} then we can build an *associated* Fell bundle B over S as the union of

$$B_s := \{(f, s) : f \in \Gamma_0(A) \text{ and } f(x) = 0 \text{ for } x \notin s\}$$

with inclusion maps defined by $i_{ts}(f, s) = (f, t)$ for $s \le t$. The operations on B are

$$(f, s)(g, t) = (fg, st) \qquad \text{and} \qquad (f, s)^* = (f^*, s^*),$$

where fg is the convolution product

$$(fg)(x) = \textstyle\sum_y f(y)g(y^{-1}x).$$

Note that the sum above has only one term since there is at most one $y \in s$ such that $r(y) = r(x)$. The norm is defined by

$$\|(f, s)\| = \sup_{x\in s}\|f(x)\|.$$

Example 3.1. Kumjian [Kum, 2.5(iii)] shows that if ${}_AX_B$ is an imprimitivity bimodule then

$$C_0 = A, \quad C_1 = B, \quad C_\partial = X, \quad C_{\partial^{-1}} = \tilde{X}$$

is a Fell bundle over the groupoid $\Delta = \{0, 1, \partial, \partial^{-1}\}$ where $d(\partial) = 0$ and $r(\partial) = 1$. The Fell bundle in Example 2.15 is the Fell bundle associated to C.

Recall from [QS] that if α is an action of an r-discrete groupoid G on an upper semicontinuous C^*-bundle A and S is a full inverse subsemigroup of G^{op} then there is a unique action β of S on $B = \Gamma_0(A)$ such that

$$\beta_s(f)(u) = \begin{cases} \alpha_{us}(f(s^*us)) & \text{if } u \in r(s) \\ 0 & \text{else} \end{cases}$$

for $s \in S$, $f \in B_{d(s)}$ and $u \in G^0$. The crossed products $A \times_\alpha G$ and $B \times_\beta S$, which are the enveloping C^*-algebras of $\Gamma_c(r^*A)$ and $\Gamma_c(r^*B)/I_\beta$ respectively, are isomorphic. We show that r^*B is isomorphic to the Fell bundle associated to r^*A. This gives the motivation to expect that the full C^*-algebra of a groupoid Fell bundle is isomorphic to the C^*-algebra of the associated inverse semigroup Fell bundle.

Example 3.2. [Kum 2.5(v)] If α is an action of an r-discrete groupoid G on an unupper semicontinuous C^*-bundle then

$$r^*A = \{(a, x) \in A \times G : a \in A_{r(x)}\}$$

becomes a Fell bundle called the semidirect product bundle with product and adjoint defined by

$$(a, x)(b, y) = (a\alpha_x(b), xy), \quad (a, x)^* = (\alpha_{x^{-1}}(a^*), x^{-1}).$$

Proposition 3.3. *The semidirect product bundle r^*B of the semigroup action (B, S, β) created from the groupoid action (A, G, α) is isomorphic to the Fell bundle C associated to the semidirect product groupoid Fell bundle r^*A.*

$$\begin{array}{ccccc} & & r^*A & \rightarrow & C \\ & \nearrow & & & \\ (A,G,\alpha) & & & & \uparrow \Pi \\ & \searrow & & & \\ & & (B,S,\beta) & \rightarrow & r^*B \end{array}$$

Proof. We have

$$r^*B = \{(g, s) : g \in B_{ss^*}\} = \{(g, s) : g \in \Gamma_0(A), g(u) = 0 \text{ for } u \notin ss^*\}$$

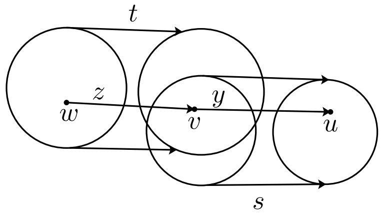
and

$$\begin{aligned} C &= \{(f, s) \mid f \in \Gamma_0(r^*A), f(x) = 0 \text{ for } x \notin s\} \\ &= \{(f, s) \mid f : S \to r^*A, f(y) \in (A_{r(y)}, y) \text{ for } y \in S \text{ and } f(x) = 0 \text{ for } x \notin s\}. \end{aligned}$$

The map $\Pi : r^*B \to C$ defined by $(g, s) \mapsto (f, s)$ where $f(x) = (g(r(x)), x)$ is an isomorphism. We check multiplicativity. If (g, s), $(k, t) \in r^*B$ then

$$\begin{aligned} \Pi((g, s)(k, t)) &= \Pi(\beta_s(\beta_{s^*}(g)k), st) =: (l, st) \\ \Pi(g, s)\Pi(k, t) &=: (f, s)(h, t) = (fh, st). \end{aligned}$$

To show equality we need to show that $l(x) = (fh)(x)$ for all $x \in G$. If $x \notin st$ then both expressions are 0. If $x \in st$ then there are unique $y \in s$ and $z \in t$ such that $x = yz$.



Let $u = r(x) = r(y)$, $w = d(x) = d(z)$ and $v = r(z) = d(y)$. Then

$$\begin{aligned} l(x) &= (\beta_s(\beta_{s^*}(g)k)(r(x)), x) = (\alpha_{us}(\beta_{s^*}(g)k)(s^*us)), x) \\ &= (\alpha_y((\beta_{s^*}(g)k)(v)), x) = (\alpha_y(\alpha_{vs^*}(g(svs^*))k(v)), x) \\ &= (\alpha_y(\alpha_{y^{-1}}(g(u))k(v)), x) = (g(u)\alpha_y(k(v)), x) \\ &= (g(r(y)), y)(k(r(y^{-1}x)), y^{-1}x) = f(y)h(y^{-1}x) \\ &= (fh)(x). \end{aligned}$$

□

To show that the C^*-algebra of a groupoid Fell bundle is isomorphic to the C^*-algebra of the corresponding inverse semigroup Fell bundle, we need the following technical result.

Lemma 3.4. *If B is a Fell bundle over S and C is a sub-bundle of B such that*
(a) C_s is dense in B_s for all $s \in S$;
(b) C_e is an ideal of B_e for all $e \in E_S$;
(c) $C_e B_s \subset C_s$ for all $ss^* \leq e$,
then $C^*(B)$ is the enveloping C^*-algebra of

$$\Gamma_c(C)/(I \cap \Gamma_c(C)).$$

Proof. A representation Π of $\Gamma_c(C)$ is continuous from the pointwise convergence topology to the norm topology and hence has a unique extension $\bar{\Pi}$ of $\Gamma_c(B)$ since if $c \in C_s$ then

$$\|\Pi(c)\|^2 = \|\Pi(c)^*\Pi(c)\| = \|\Pi(c^*c)\| \leq \|c^*c\| = \|c\|^2$$

since Π restricted to the ideal C_{s^*s} of the C^*-algebra B_{s^*s} is contractive.

It remains to show that $\bar{\Pi}$ is coherent if Π is. If $b \in B_s$ and $\{c_j\}$ is a bounded approximate identity for B_s contained in C_s then

$$\bar{\Pi}(b) = \bar{\Pi}(\lim_j c_j b) = \lim_j \Pi(c_j b) = \lim_j \Pi(i_{ts} c_j b) = \bar{\Pi}(i_{ts}(\lim_j c_j b)) = \bar{\Pi}(i_{ts} b).$$

□

Fix a Fell bundle over the r-discrete groupoid G and let B be the associated Fell bundle over the full inverse subsemigroup S of G^{op}. If

$$C_s = \{(f, s) : f \in \Gamma_c(A),\ \operatorname{supp} f \subset s\}$$

then the bundle C satisfies the conditions of the previous lemma.

Proposition 3.5. *The map $\Lambda : \Gamma_c(C) \to \Gamma_c(A)$ defined on the generators by*

$$\Lambda(f, s) = f$$

is a surjective $*$-homomorphism with kernel $I_B \cap \Gamma_c(C)$.

Proof. A standard partition of unity argument shows that Λ is onto. It follows straight from the definition of the associated bundle that Λ is multiplicative and preserves adjoints.

It follows, using the same partition of unity arguments as in the proof of [QS Theorem 7.1 (formula 7.1)], that $\ker \Lambda$ is equal to

$$I = \operatorname{span}\{(f, s) - (f, t) : f \in \Gamma_c(A), \operatorname{supp} f \subset s \cap t\}$$

which is equal to

$$J = \operatorname{span}\{(f, s) - (f, t) : f \in \Gamma_c(A), \operatorname{supp} f \subset s \leq t\}.$$

Since

$$I_B = \operatorname{span}\{(f, s) - i_{t,s}(f, s) : (f, s) \in B_s, s \leq t\}$$

and $i_{t,s}(f, s) = (f, t)$, we have $J = I_B \cap \Gamma_c(C)$. □

Corollary 3.6. *If A is a Fell bundle over the r-discrete groupoid G, S is a full inverse subsemigroup of G^{op} and B is the associated Fell bunde over S, then $C^*(A)$ and $C^*(B)$ are isomorphic.*

Proof. The Λ of Proposition 3.5 factors through an isomorphism between $D := \Gamma_c(C)/(I_B \cap \Gamma_c(C))$ and $\Gamma_c(A)$. By 3.4 $C^*(B)$ is the enveloping C^*-algebra of D. Thus $C^*(B)$ is isomorphic to $C^*(A)$ since $C^*(A)$ is the enveloping C^*-algebra of $\Gamma_c(A)$. □

Note that we have also shown that $\Gamma_c(A)$ has an enveloping C^*-algebra.

4. Homogeneous bundles

We adapt the notion of homogeneous bundles introduced in [Fel] to the semigroup case. If B is a Fell bundle over S and $e \in E_S$ then let

$$B_{|e} = \bigcup\{B_t : t^*t \leq e\} \quad B^{|e} = \bigcup\{B_t : tt^* \leq e\}.$$

Definition 4.1. Let B be a Fell bundle over S and $s \in S$. An *s-multiplier* of B is a pair $m = (m_l, m_r)$ of maps $m_l : B^{|s^*s} \to B$, $a \mapsto ma$ and $m_r : B_{|ss^*} \to B$, $a \mapsto am$ such that

(a) $ma_t \in B_{st}$ for all $a_t \in B_t \subset B^{|s^*s}$;
(b) $a_tm \in B_{ts}$ for all $a_t \in B_t \subset B_{|ss^*}$;
(c) $a(mb) = (am)b$ for all $a \in B_{|ss^*}$, $b \in B^{|s^*s}$;
(d) $m(ab) = (ma)b$ for all $a \in B^{|s^*s}$;
(e) $(ab)m = a(bm)$ for all $b \in B_{|ss^*}$;
(f) $mi_{t,r}(a_r) = i_{st,sr}(ma_r)$ for all $a_r \in B_r$, $B_t \subset B^{|s^*s}$ and $r \leq t$;
(g) $i_{t,r}(a_r)m = i_{ts,rs}(a_rm)$ for all $a_r \in B_r$, $B_t \subset B_{|ss^*}$ and $r \leq t$.

We write $M_s(B)$ for the set of s-multipliers of B and $M(B)$ for the set $\cup_{s\in S} M_s(B)$ of *multipliers* of B. The multiplication notation is motivated by the fact that every element c of the fiber B_s can be viewed as an s-multiplier $c = (c_l, c_r)$ defined by $c_l(a) = ca$ and $c_r(b) = bc$.

Definition 4.2. The *adjoint* of an s-multiplier m is the s^*-multiplier m^* defined by

$$m^*a = (a^*m)^*, \quad bm^* = (mb^*)^*.$$

If $e \in E_S$ then an e-multiplier is a multiplier of the C^*-algebra B_e. If B is a saturated Fell bundle then B_s is a $B_{ss^*} - B_{s^*s}$ imprimitivity bimodule for all s. In this case, an s-multiplier is a multiplier of this imprimitivity bimodule in the sense of [ER].

Definition 4.3. An s-multiplier m is called *unitary* if $mm^* = 1_{M(B_{ss^*})}$ and $m^*m = 1_{M(B_{s^*s})}$, that is, $amm^* = a = mm^*a$ for all $a \in B_{ss^*}$ and $am^*m = a = m^*ma$ for all $a \in B_{s^*s}$. We write $UM_s(B)$ for the set of unitary s-multipliers of B.

The following example is the motivation for the definition of homogeneous bundles.

Example 4.4. Let $B = r^*A$ be the twisted semidirect product bundle of the twisted action (A, S, α, w). We can think of $u_s := (1_{M(A_{ss^*})}, s)$ as an s-multiplier of r^*A by formally applying the bundle multiplication. That is, by definition

$$u_s(a, r) = (\alpha_s(a)w_{s,r}, sr)$$
$$(b, t)u_s = (bw_{t,s}, ts)$$

for all $(a, r) \in B^{|s^*s} = \cup\{(A_{rr^*}, r) : rr^* \leq s^*s\}$ and $(b, t) \in B_{|ss^*} = \cup\{(A_{tt^*}, t) : t^*t \leq ss^*\}$.

If $r \leq s$, and $(b,e) \in B_{|rr^*} \subset B_{|ss^*}$ for some $e \in E_S$ then

$$(b,e)u_r = (bw_{e,r}, er) = (b, er) = (bw_{e,s}, es) = (b,e)u_s,$$

that is, the multipliers $\{u_s\}$ are coherent.

The adjoint of u_s is $u_s^* = (w_{s^*,s}^*, s^*)$, which can be gotten by formally calculating $(1_{M(A_s)}, s)^*$. Since

$$\begin{aligned} u_s u_s^* &= (\alpha_s(w_{s^*,s}^*, s) w_{s,s^*}, ss^*) = (w_{s,s^*}^* w_{s,s^*}, ss^*) \\ &= (1_{M(A_{ss^*})}, ss^*) = 1_{M(B_{ss^*})} \\ u_{s^*} u_s &= (w_{s^*,s}^* w_{s^*,s}, s^*s) = (1_{M(A_{s^*s})}, s^*s) \\ &= 1_{M(B_{s^*s})}, \end{aligned}$$

u_s is a unitary multiplier for all $s \in S$.

If $(a, s^*s) \in B_{s^*s}$ then

$$\begin{aligned} u_s(a, s^*s)u_s^* &= (\alpha_s(a), s)(w_{s^*,s}^*, s^*) = (\alpha_s(a) w_{s,s^*}, ss^*) \\ &= (\alpha_s(a) w_{s,s^*s} w_{ss^*,s}^* w_{s,s^{**}} w_{s,s^*}, ss^*) \\ &= (\alpha_s(a), ss^*), \end{aligned}$$

that is, α_s is adjoining by the unitary u_s.

Similar calculation shows that

$$u_s u_t u_{st}^* = (w_{s,t}, st(st)^*)$$

so the 2-cocycle w also can be gotten from the multipliers $\{u_s\}$.

Definition 4.5. A Fell bundle B over S is called *homogeneous* if there is a unitary s-multiplier u_s for all $s \in S$ such that the set of unitary multipliers is *coherent* in the sense that $u_e = 1_{M(B_e)}$ for all $e \in E_S$ and

$$au_s = au_t$$

whenever $a \in B_{ss^*}$ and $s \leq t$.

Proposition 4.6. *A homogeneous Fell bundle B over S is saturated.*

Proof. It is clear that $B_s \cdot B_{s^*} \subset B_{ss^*}$ for all $s \in S$. On the other hand, if $a \in B_{ss*}$, $m \in UM_s(B)$ and $\{u_j\}$ is a bounded approximate identity for B_{ss^*} then

$$a = \lim_j au_j = \lim_j amm^* u_j \in B_s \cdot B_{s^*}$$

since $am \in B_s$ and $m^* u_i \in B_{s^*}$. The result now follows from Lemma 2.8. □

Note that we did not use the fact that the set of multipliers is coherent.

Let B be a homogeneous Fell bundle over S and fix a coherent set of unitary multiplier $\{u_s\}_{s \in S}$. We are going to build a twisted semigroup action (A, S, α, w) and show that the twisted semidirect product bundle r^*A is isomorphic to B. Let $B|E_S = \cup\{B_e : e \in E_S\}$ be the restriction Fell bundle of B to the idempotent semilattice of S, and let $A = C^*(B|E_S)$. A is the direct limit of the C^*-algebras $\{B_e\}$. If $[\cdot] : B_e \to A$ is the natural inclusion map and $A_e := [B_e]$ then A is the completion of $\cup_e A_e$ with the unique C^* norm and A_e is an ideal of A for all $e \in E_S$. We use the notation $[u_s][a] := [u_s a]$ and $[a][u_s] := [au_s]$. Now define

$$\alpha_s : A_{s^*s} \to A_{ss^*} \text{ by } \alpha_s([a]) = [u_s a u_s^*]$$

and let

$$w_{s,t} = [u_s u_t u_{st}^*].$$

Note that these are well-defined by the coherence of $\{u_s\}$.

It is routine to check that we get a twisted semigroup action this way. For example α is twisted multiplicative since for $a \in A_{(st)^*st}$ we have

$$\begin{aligned}\alpha_s(\alpha_t([a])) &= [u_s u_t a u_t^* u_s^*] \\ &= [u_s u_t u_{st}^* u_{st} a u_{st}^* u_{st} u_t^* u_s^*] \\ &= w_{s,t}\alpha_{st}([a])w_{s,t}^*.\end{aligned}$$

Proposition 4.7. *The map $\phi : r^*A \to B$ defined by $\phi([a], s) = au_s$, $a \in B_{ss^*}$, is a Fell bundle isomorphism.*

Proof. It is clear that ϕ is a norm preserving bijection. We show that ϕ is multiplicative.

$$\begin{aligned}\phi(([a], s)([b], t)) &= \phi(\alpha_s(\alpha_{s^*}([a])[b])w_{s,t}, st) \\ &= \phi([u_s u_{s^*} a u_s b u_s^*]w_{s,t}, st) \\ &= a u_s b u_s^* u_s u_t u_{st}^* u_{st} \\ &= a u_s b u_t = \phi(a, s)\phi(b, t).\end{aligned}$$

Next, we show that ϕ preserves adjoints. If $a \in B_{ss^*}$ then

$$\begin{aligned}\phi(([a], s)^*) &= \phi(\alpha_s^{-1}([a]^*)w_{s^*,s}^*, s^*) \\ &= \phi([u_s^* a^* u_s (u_{s^*} u_s u_{s^*s})^*], s^*) \\ &= u_s^* a^* u_{s^*} u_s^* u_{s^*}^* u_{s^*} = u_s^* a^* \\ &= (au_s)^* = (\phi([a], s))^*.\end{aligned}$$

Finally, we show that ϕ preserves the inclusion maps. If $s \le t$ and $a \in B_{ss^*}$ then

$$\begin{aligned}\phi(i_{t,s}([a], s)) &= \phi([a], t) = i_{tt^*,ss^*}(a)u_t = i_{t,s}(au_t) \\ &= i_{t,s}(au_s) = i_{t,s}(\phi([a], s)).\end{aligned}$$

□

REFERENCES

[ER] S. Echterhoff and I. Raeburn, *Multipliers of imprimitivity bimodules and Morita equivalence of crossed products*, Math. Scand. **76** (1995), 289-309.

[Exe] R. Exel, *Partial actions of groups and actions of inverse semigroups*, Proc. Amer. Math. Soc, **126** (1998), no. 12, 3481–3494.

[Fel] J.M.G. Fell, *An extension of Mackey's method to Banach $*$-algebraic bundles*, Memoirs Amer. Math. Soc. 90, Providence, Rhode Island, 1969.

[Kum] A. Kumjian, *Fell bundles over groupoids*, Proc. Amer. Math. Soc. **126** (1998), no. 4, 1115–1125.

[QS] J. Quigg and N. Sieben, *C^*-actions of r-discrete groupoids and inverse semigroups*, J. Austral. Math. Soc. Ser. A (to appear).

[Sie] N. Sieben, *C^*-crossed products by twisted inverse semigroup actions*, J. Operator Theory **39** (1998), no. 2, 361–393.

DEPARTMENT OF MATEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287-1804
Email: `nandor.sieben@asu.edu`