

RUBBLING AND OPTIMAL RUBBLING OF GRAPHS

CHRISTOPHER BELFORD AND NÁNDOR SIEBEN

ABSTRACT. A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed. In this new move one pebble is removed at vertices v and w adjacent to a vertex u and an extra pebble is added at vertex u . A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using rubbling moves. The rubbling number of a graph is the smallest number m needed to guarantee that any vertex is reachable from any pebble distribution of m pebbles. The optimal rubbling number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable. We determine the rubbling and optimal rubbling number of some families of graphs and we show that Graham's conjecture does not hold for rubbling numbers.

1. INTRODUCTION

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. How many pebbles will guarantee that every vertex is reachable, or that all vertices are reachable at the same time? How can we place the smallest number of pebbles such that every vertex is reachable? For a comprehensive list of references for the extensive literature see the survey papers [5, 6].

In the current paper we propose the study of an extension of pebbling called *rubbling*. In this version we also allow a move that removes a pebble from the vertices v and w that are adjacent to a vertex u , and adds a pebble at vertex u . We find rubbling versions of some of the well known pebbling tools such as the transition digraph, the No Cycle Lemma, squishing and smoothing. We use these tools to find rubbling numbers and optimal rubbling numbers for some families of graphs including paths, trees, complete graphs, complete bipartite graphs, wheels and cycles. We also show that Graham's conjecture does not hold for rubbling numbers.

Our techniques are similar to those used in the pebbling literature, but they are not the same. Some rubbling results require completely different tools, some require more efforts than their pebbling counterparts. Some graphs have equal pebbling and rubbling numbers, some have a much smaller rubbling number than pebbling number. It seems intriguing to understand what graph properties are responsible for these differences, in particular, what property forces the pebbling and the rubbling number to be the same. Rubbling also seems to be connected to fractional pebbling. Developing the theory of rubbling may introduce new tools and deeper understanding of pebbling.

Date: 10/11/2008.

1991 Mathematics Subject Classification. 05C99.

Key words and phrases. pebbling, optimal pebbling, rubbling.

2. PRELIMINARIES

Let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \rightarrow \mathbf{Z}$ where $p(v)$ is the number of pebbles placed at v . A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v \in V(G)} p(v)$. We are going to use the notation $p(v_1, \dots, v_n, *) = (a_1, \dots, a_n, q(*))$ to indicate that $p(v_i) = a_i$ for $i \in \{1, \dots, n\}$ and $p(w) = q(w)$ for all $w \in V(G) \setminus \{v_1, \dots, v_n\}$.

Definition 2.1. Consider a pebble function p on the graph G . If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \rightarrow u)$ removes two pebbles at vertex v and adds one pebble at vertex u to create a new pebble function

$$p_{(v, v \rightarrow u)}(v, u, *) = (p(v) - 2, p(u) + 1, p(*)).$$

If $\{w, u\} \in E(G)$ and $v \neq w$ then the *strict rubbling move* $(v, w \rightarrow u)$ removes one pebble each at vertices v and w and adds one pebble at vertex u to create a new pebble function

$$p_{(v, w \rightarrow u)}(v, w, u, *) = (p(v) - 1, p(w) - 1, p(u) + 1, p(*)).$$

A *rubbling move* is either a pebbling move or a strict rubbling move.

Note that the rubbling moves $(v, w \rightarrow u)$ and $(w, v \rightarrow u)$ are the same. Also note that the resulting pebble function might not be a pebble distribution even if p is.

Definition 2.2. A *rubbling sequence* is a finite sequence $s = (s_1, \dots, s_k)$ of rubbling moves. The pebble function gotten from the pebble function p after applying the moves in s is denoted by p_s .

The concatenation of the rubbling sequences $r = (r_1, \dots, r_k)$ and $s = (s_1, \dots, s_l)$ is denoted by $rs = (r_1, \dots, r_k, s_1, \dots, s_l)$.

Definition 2.3. A rubbling sequence (s_1, \dots, s_n) is *executable* from the pebble distribution p if $p_{(s_1, \dots, s_i)}$ is nonnegative for all $i \in \{1, \dots, n\}$. A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbling sequence s such that $p_s(v) \geq 1$. The *rubbling number* $\rho(G)$ of a graph G is the minimum number m such that every vertex of G is reachable from any pebble distribution of size m .

A vertex is reachable if a pebble can be moved to that vertex using rubbling moves with actual pebbles without ever running out of pebbles. Changing the order of moves in an executable rubbling sequence s may result in a sequence r that is no longer executable. On the other hand the ordering of the moves has no effect on the resulting pebble function, that is, $p_s = p_r$. This justifies the following definition.

Definition 2.4. Let S be a multiset of rubbling moves. The pebble function gotten from the pebble function p after applying the moves in S in any order is denoted by p_S .

3. RUBBLING TREES

The pebbling number of trees was found in [2]. We modify Chung's argument to find the rubbling number of trees. Let v be a vertex of a tree G . Let \vec{G} be the digraph gotten from G by directing the edges towards v . A *path partition* of \vec{G} is an ordered partition $\mathcal{P} = (P_1, \dots, P_m)$ of the edges of \vec{G} into directed paths so that $p_i \geq p_{i+1}$ where p_i is the length of P_i for all i . We call (p_1, \dots, p_m) the *length sequence* of \mathcal{P} . A path partition of G is a path partition of \vec{G} for some vertex v of G . A path partition \mathcal{P} *majorizes* another path partition \mathcal{P}' if $(p_1, \dots, p_m) \geq (p'_1, \dots, p'_{m'})$ in the lexicographic order. A path partition

is *v*-*maximum* if it majorizes all path partitions of \overrightarrow{G} . A path partition is *maximum* if it majorizes all path partitions of G .

For $k \in \mathbf{N}$ and $v \in V(G)$ let $\rho(G, v, k)$ be the minimum number m such that for every pebble distribution p on G with size m there is an executable rubbling sequence s with $p_s(v) \geq k$. Note that $\rho(G) = \max\{\rho(G, v, 1) \mid v \in V(G)\}$. Also note that $\rho(G, v, k+1) - 1$ is the maximum size of a pebble distribution on G from which at most k pebbles can be moved to vertex v .

Proposition 3.1. *Let v be a vertex of the tree G and (p_1, \dots, p_m) be the length sequence of a v -maximum path partition \mathcal{P} of \overrightarrow{G} . Then $\rho(G, v, k) = k2^{p_1} + \sum_{i=2}^m 2^{p_i-1} - m + 1$ for all $k \geq 1$.*

Proof. We use induction on the number of vertices of G . The formula clearly works when $|V(G)| = 1$. For the inductive step let $\{v_1, \dots, v_n\}$ be the set of vertices adjacent to v . The removal of v from \overrightarrow{G} creates a digraph that is the disjoint union of the directed trees $\overrightarrow{G}_1, \dots, \overrightarrow{G}_n$. The path partition \mathcal{P} induces a maximum path partition of \overrightarrow{G}_i with length sequence $(p_{i,1} - 1, p_{i,2}, \dots, p_{i,m_i})$ for all i . With this notation, the multisets $\{p_1, \dots, p_m\}$ and $\{p_{1,1}, \dots, p_{1,m_1}, \dots, p_{n,1}, \dots, p_{n,m_n}\}$ are equal. We can assume without loss of generality that $p_1 = p_{1,1}$. Let k_i be the number of pebbles reaching v_i from G_i . Then

$$\rho(G, v, k) = \max \left\{ \sum_{i=1}^n (\rho(G_i, v_i, k_i + 1) - 1) \mid \left\lfloor \frac{k_1 + \dots + k_n}{2} \right\rfloor < k \right\} + 1$$

and so by the inductive hypothesis

$$\rho(G, v, k) = \max \left\{ \sum_{i=1}^n \left((k_i + 1)2^{p_{i,1}-1} + \sum_{j=2}^{m_i} 2^{p_{i,j}-1} - m_i \right) \mid k_1 + \dots + k_n \leq 2k - 1 \right\} + 1.$$

Since $2^a + 2^b \geq 2^{a-1} + 2^{b+1}$ for all integers satisfying $a > b$, the maximum occurs when $k_1 = 2k - 1$ and $k_2 = \dots = k_n = 0$. So

$$\begin{aligned} \rho(G, v, k) &= 2k2^{p_{1,1}-1} + \sum_{j=2}^{m_1} 2^{p_{1,j}-1} - m_1 + \sum_{i=2}^n (2^{p_{i,1}-1} + \sum_{j=2}^{m_i} 2^{p_{i,j}-1} - m_i) + 1 \\ &= k2^{p_{1,1}} + \sum_{j=2}^{m_1} 2^{p_{1,j}-1} + \sum_{i=2}^n \sum_{j=1}^{m_i} 2^{p_{i,j}-1} - \sum_{i=1}^n m_i + 1 \\ &= k2^{p_1} + \sum_{i=2}^m 2^{p_i-1} - m + 1. \end{aligned}$$

□

Proposition 3.2. *Let (p_1, \dots, p_m) be the length sequence of a maximum path partition of G . Then $\rho(G) = 2^{p_1} + \sum_{i=2}^m 2^{p_i-1} - m + 1$.*

Proof. The result follows from the previous proposition and the fact that $2^a + 2^b \geq 2^{a-1} + 2^{b+1}$ for all integers satisfying $a > b$. □

The pebbling number of G is $\pi(G) = \sum_{i=1}^m 2^{p_i} - m + 1$ by [2]. The following is an important special case of Proposition 3.2.

Proposition 3.3. *The rubbling number of the path P_n with n vertices is $\rho(P_n) = 2^{n-1}$.*

Note that the pebbling number of P_n is also $\pi(P_n) = 2^{n-1}$. As another application of Proposition 3.2, we can find the rubbling number of a complete binary tree.

Proposition 3.4. *The rubbling number of the complete binary tree B_h with height h is $\rho(B_h) = 4^h + (h - 3)2^{h-1} + 2$.*

Proof. The length sequence of a maximum path partition is

$$(2h, \overbrace{h-1, h-1}^2, \overbrace{h-2, \dots, h-2}^{2^2}, \dots, \overbrace{1, \dots, 1}^{2^{h-1}}).$$

The result now follows from the calculation below

$$\begin{aligned} \rho(B_h) &= 2^{2h} - 1 + 2(2^{h-2} - 1) + \dots + 2^{h-2}(2 - 1) + 2^{h-1}(2^0 - 1) + 1 \\ &= 4^h + (h - 2)2^{h-1} - (1 + 2 + \dots + 2^{h-2}) + 1 \\ &= 4^h + (h - 3)2^{h-1} + 2. \end{aligned}$$

□

4. THE TRANSITION DIGRAPH AND THE NO CYCLE LEMMA

Definition 4.1. Given a multiset S of rubbling moves on G , the *transition digraph* $T(G, S)$ is a directed multigraph whose vertex set is $V(G)$, and each move $(v, w \rightarrow u)$ in S is represented by two directed edges (v, u) and (w, u) . The transition digraph of a rubbling sequence $s = (s_1, \dots, s_n)$ is $T(G, s) = T(G, S)$, where $S = \{s_1, \dots, s_n\}$ is the multiset of moves in s . Let $d_{T(G, S)}^-$ represent the in-degree and $d_{T(G, S)}^+$ the out-degree in $T(G, S)$. We simply write d^- and d^+ if the transition digraph is clear from context.

The transition digraph only depends on the rubbling moves and the graph but not on the pebble distribution or on the order of the moves. It is possible that $T(G, S) = T(G, R)$ even if $S \neq R$. If $T(G, S) = T(G, R)$ then $p_S = p_R$, so the effect of a rubbling sequence on a pebble function only depends on the transition digraph. In fact we have the following.

Lemma 4.2. *If p is a pebble function on G and S is a multiset of rubbling moves then*

$$p_S(v) = p(v) + d^-(v)/2 - d^+(v)$$

for all $v \in V(G)$.

Proof. The three terms on the right hand side represent the original number of pebbles, the number of pebbles arrived at v and the number of pebbles moved away from v . □

We are often interested in the value of $q_R(v) - p_S(v)$. The function Δ defined in the following lemma is going to simplify our notation. The three parameters of Δ represent the change in the number of pebbles, the change in the in-degree and the change in the out-degree. The proof is a trivial calculation.

Lemma 4.3. *Define $\Delta(a, b, c) = a + b/2 - c$. Then*

$$q_R(v) - p_S(v) = \Delta(q(v) - p(v), d_{T(G, R)}^-(v) - d_{T(G, S)}^-(v), d_{T(G, R)}^+(v) - d_{T(G, S)}^+(v)).$$

If the rubbling sequence s is executable from a pebble distribution p then we must have $p_s \geq 0$. This motivates the following terminology.

Definition 4.4. A multiset S of rubbling moves on G is *balanced* with a pebble distribution p at vertex v if $p_S(v) \geq 0$. We say S is *balanced* with p if S is balanced with p at all $v \in V(G)$, that is, $p_S \geq 0$. We say that a rubbling sequence s is balanced with p if the multiset of moves in s is balanced with p .

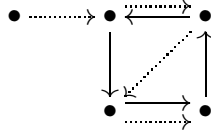


FIGURE 4.1. Arrows of $T(G, Q)$. The solid arrows belong to C .

S is trivially balanced with a pebble distribution at v if $d_{T(G,S)}^+(v) = 0$. The balance condition is necessary but not sufficient for a rubbling sequence to be executable. The pebble distribution $p(u, v, w) = (1, 1, 1)$ on the cycle C_3 is balanced with $s = ((u, u \rightarrow v), (v, v \rightarrow w), (w, w \rightarrow u))$, but s is not executable. The problem is caused by the cycle in the transition digraph. The goal of this section is to overcome this difficulty.

Definition 4.5. A multiset of rubbling moves or a rubbling sequence is called *acyclic* if the corresponding transition digraph has no directed cycles. Let S be a multiset of rubbling moves. An acyclic multiset $R \subseteq S$ is called an *untangling* of S if $p_R \geq p_S$.

Proposition 4.6. *Every multiset of rubbling moves has an untangling.*

Proof. Let S be the multiset of rubbling moves. Suppose that $T(G, S)$ has a directed cycle C . Let Q be the multiset of elements of S corresponding to the arrows of C , see Figure 4.1. We show that $p_R \geq p_S$ where $R = S \setminus Q$. If $v \in V(C)$ then there is an $a \leq -1$ such that

$$p_R(v) - p_S(v) = \Delta(0, -2, a) = -1 - a \geq 0.$$

If $v \in V(G) \setminus V(C)$ then there is an $a \leq 0$ such that

$$p_R(v) - p_S(v) = \Delta(0, 0, a) \geq 0.$$

We can repeat this process on R until we eliminate all the cycles. This can be finished in finitely many steps since every step decreases the number of edges in R . The resulting multiset is an untangling of S . \square

Note that a multiset of moves can have several untanglings. Also note that if a pebble distribution p is balanced with S and R is an untangling of S then $p_R \geq p_S \geq 0$ and so p is also balanced with R .

Proposition 4.7. *If the pebble distribution p on G is balanced with the acyclic multiset S of rubbling moves then there is a sequence s of the elements of S such that s is executable from p .*

Proof. First note that if the pebble distribution q on G is balanced with the multiset R of rubbling moves and $t = (v, w \rightarrow u) \in R$ such that $d_{T(G,R)}^-(v) = 0 = d_{T(G,R)}^-(w)$ then t is executable from q . If $v \neq w$ then $q(v) \geq d_{T(G,R)}^+(v) \geq 1$ and $q(w) \geq d_{T(G,R)}^+(w) \geq 1$. If $v = w$ then $q(v) \geq d^+(v) \geq 2$. In both cases t is executable from q .

We define s recursively. Let $R_1 = S$. Since R_1 is acyclic, we must have a move $s_1 = (v_1, w_1 \rightarrow u_1) \in R_1$ such that $d_{T(G,R_1)}^-(v_1) = 0 = d_{T(G,R_1)}^-(w_1)$. Then s_1 is executable from p . Let $R_i = R_{i-1} \setminus \{s_{i-1}\}$. Then R_i is acyclic so we must have a move $s_i = (v_i, w_i \rightarrow u_i) \in R_i$ such that $d_{T(G,R_i)}^-(v_i) = 0 = d_{T(G,R_i)}^-(w_i)$. Then $p_{(s_1, \dots, s_{i-1})}$ is balanced with R_i since $(p_{(s_1, \dots, s_{i-1})})_{R_i} = p_S \geq 0$ and so s_i is executable from $p_{(s_1, \dots, s_{i-1})}$. The sequence $s = (s_1, \dots, s_{|S|})$ is an ordering of the elements of S that is executable from p . \square

The following is the rubbling version of the No-Cycle Lemma for pebbling [3, 7, 8].

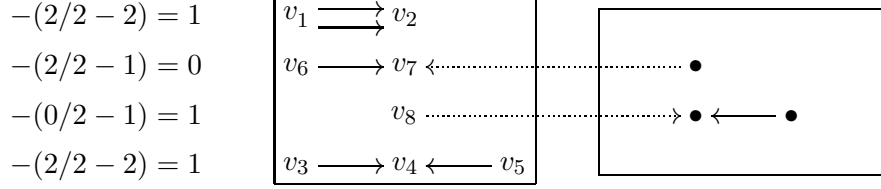


FIGURE 5.1. Arrows in $T(G, S)$ representing the possible types of rubbling moves in E . The vertices in the same box are equivalent. The solid arrows connect equivalent vertices. The calculation on the left shows the change in $\sum_i (\frac{1}{2}d^-(v_i) - d^+(v_i))$ after the removal of one of the rubbling moves.

Lemma 4.8. (No Cycle) *Let p be a pebble distribution on G and $v \in V(G)$. The following are equivalent.*

- (1) v is reachable from p .
- (2) There is a multiset S of rubbling moves such that S is balanced with p and $p_S(v) \geq 1$.
- (3) There is an acyclic multiset R of rubbling moves such that R is balanced with p and $p_R(v) \geq 1$.
- (4) v is reachable from p through an acyclic rubbling sequence.

Proof. If v is reachable from p then there is an executable sequence s of rubbling moves. The multiset S of rubbling moves of s is balanced with p and $p_S(v) \geq 1$. So (1) implies (2). If S satisfies (2) then an untangling R of S satisfies (3). Suppose R satisfies (3). By Proposition 4.7, there is an executable ordering r of the moves of R . This r is acyclic and v is reachable through r since $p_r(v) = p_R(v) \geq 1$. So (3) implies (4). Finally, (4) clearly implies (1). \square

Corollary 4.9. *If a vertex is reachable from a pebble distribution p on G then it is also reachable by a rubbling sequence in which no move of the form $(v, a \rightarrow u)$ is followed by a move of the form $(u, b \rightarrow v)$.*

5. BASIC RESULTS

It is clear from the definition that for all graphs G we have $\rho(G) \leq \pi(G)$ where π is the pebbling number. For the pebbling number we have $2^{\text{diam}(G)} \leq \pi(G)$. This is also true for the rubbling number.

Proposition 5.1. *If the graph G has diameter d then $2^d \leq \rho(G)$.*

Proof. Let v_0 and v_d be vertices at distance d . Let $p(v_0, *) = (m, 0)$ be a pebble distribution from which v_d is reachable through the rubbling sequence s . We now build a quotient rubbling problem. Let $[v]$ be the equivalence class of v in the partition of the vertices of G according to their distances from v_0 . The quotient simple graph H is isomorphic to P_{d+1} with leaves $[v_0] = \{v_0\}$ and $[v_d]$. Let $q([v]) = \sum_{w \in [v]} p(w)$ for all $[v] \in V(H)$ and note that $q([v_0], *) = (m, 0)$. The rubbling sequence s induces a multiset R of rubbling moves on H . We construct this R from the multiset S of rubbling moves of s . Let E be the multiset of moves of S of the form $(v, w \rightarrow u)$ where $v \in [u]$ or $w \in [u]$. Define R to be the multiset of moves of the form $([v], [w] \rightarrow [u])$ where $(v, w \rightarrow u)$ runs through the elements of $S \setminus E$.

We show that R is balanced with q . Figure 5.1 shows the possible types of moves in E . The removal of any of these moves does not decrease the value of $\sum_{v_i \in [v]} (\frac{1}{2}d^-(v_i) - d^+(v_i))$ and so

$$q_R([v]) = \sum_{v_i \in [v]} p_{S \setminus E}(v_i) \geq \sum_{v_i \in [v]} p_S(v_i) \geq 0$$

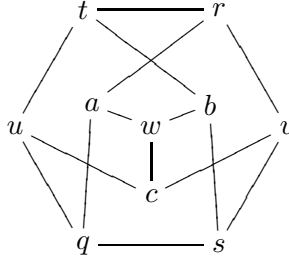


FIGURE 5.2. The Petersen graph P .

since p is balanced with S .

We also have $q_R([v_d]) \geq 1$ since v_d is reachable and so $p_S(v_d) \geq 1$. Thus $[v_d]$ is reachable from q and so the result now follows from Proposition 3.3. \square

For the pebbling number we have $\pi(G) \geq |V(G)|$. This inequality does not hold for the rubbling number as we can see in the next result.

Proposition 5.2. *We have the following values for the rubbling number:*

- a. $\rho(K_n) = 2$ for $n \geq 2$ where K_n is the complete graph with n vertices;
- b. $\rho(W_n) = 4$ for $n \geq 5$ where W_n is the wheel with n vertices;
- c. $\rho(K_{m,n}) = 4$ for $m, n \geq 2$ where $K_{m,n}$ is a complete bipartite graph;
- d. $\rho(Q_n) = 2^n$ for $n \geq 1$ where Q_n is the n -dimensional hypercube;

Proof. a. A single pebble is clearly not sufficient but any vertex is reachable with two pebbles using a single move.

b. If we have 4 pebbles then we can move 2 pebbles to the center using two moves. Then any other vertex is reachable from the center in a single move. On the other hand $\rho(W_n) \geq 2^{\text{diam}(W_n)} = 2^2 = 4$.

c. It is easy to see that from any pebble distribution of size 4 any vertex is reachable in at most 3 moves. On the other hand we have $\rho(K_{m,n}) \geq 2^{\text{diam}(K_{m,n})} = 2^2 = 4$.

d. We know [2] that $\pi(Q_n) = 2^n$. The result now follows from the inequality $2^n = 2^{\text{diam}(Q_n)} \leq \rho(Q_n) \leq \pi(Q_n) = 2^n$. \square

The pebbling numbers of these graphs are $\pi(K_n) = n$, $\pi(W_n) = n$, $\pi(K_{m,n}) = m + n$ and $\pi(Q_n) = 2^n$.

Proposition 5.3. *The rubbling number of the Petersen graph P is $\rho(P) = 5$.*

Proof. Consider Figure 5.2. It is easy to see that vertex w is not reachable from the pebble distribution $p(r, s, *) = (3, 1, 0)$ and so $\rho(P) > 4$. To show that $\rho(P) \leq 5$, assume that a vertex is not reachable from a pebble distribution p of size 5. Since P is vertex transitive, we can assume that this vertex is w . Then we must have

$$p(a) + p(b) + p(c) + \left\lfloor \frac{p(q) + p(r)}{2} \right\rfloor + \left\lfloor \frac{p(s) + p(t)}{2} \right\rfloor + \left\lfloor \frac{p(u) + p(v)}{2} \right\rfloor \leq 1,$$

otherwise we could make the total number of pebbles at vertices a, b and c more than 2 after which w is reachable. This forces $p(a) = p(b) = p(c) = 0$ and two of the remaining terms to be 0 as well. So by symmetry we can assume that the last term is 1 and all the other terms are 0. Then we must have $p(u) + p(v) = 3$ and $p(q) + p(r) = 1 = p(s) + p(t)$. A simple case analysis shows that w is reachable from this p , which is a contradiction. \square

We know from [5] that the pebbling number of the Petersen graph is $\pi(P) = 10$.

6. SQUISHING

The following terms are needed for the rubbling version of the Squishing Lemma of [1]. A *thread* in a graph is a path containing vertices of degree 2. A pebble distribution is *squished* on a thread P if all the pebbles on P are placed on a single vertex of P or on two adjacent vertices of P . A pebble distribution can be made squished using squishing moves. A *squishing move* removes one pebble from each of two vertices on a thread and puts two pebbles on some vertex between them on the thread.

Lemma 6.1. *Let P be a thread in G . If vertex $x \notin V(P)$ is reachable from the pebble distribution p then x is reachable from p through a rubbling sequence in which there is no strict rubbling move of the form $(v, w \rightarrow u)$ where $u \in V(P)$.*

Proof. Let S be an acyclic multiset of rubbling moves balanced with p such that $p_S(x) \geq 1$. Let E be the multiset of strict rubbling moves of S of the form $(v, w \rightarrow u)$ where $u \in V(P)$.

If $e = (v, w \rightarrow u) \in E$ then we have $d_{T(G, S \setminus \{e\})}^+(u) = d_{T(G, S)}^+(u) = 0$ since S is acyclic and so $S \setminus \{e\}$ is balanced with p at u . It is clear that $p_{S \setminus \{e\}}(y) \geq p_S(y)$ for all $y \in V(G) \setminus \{u\}$ and so $S \setminus \{e\}$ is balanced with p . We still know that $S \setminus \{e\}$ is acyclic and $p_{S \setminus \{e\}}(x) \geq 1$, so induction shows that $R = S \setminus E$ is balanced with p .

By Proposition 4.7, there is an ordering r of the elements of R that is executable from p . Then v is reachable through r since $p_r(v) = p_S(v) \geq 1$. \square

The following is the rubbling version of the Squishing Lemma for pebbling [1].

Lemma 6.2. (Squishing) *If vertex v is not reachable from a pebble distribution with size n then there is a pebble distribution r of size n that is squished on each thread not containing v such that v is not reachable from r either.*

Proof. The result follows from the proof of [1, Lemma 4] and Lemma 6.1. \square

7. RUBBLING C_n

The Squishing Lemma allows us to find the rubbling numbers of cycles. The pebbling numbers $\pi(C_{2k}) = 2^k$ $\pi(C_{2k+1}) = 2 \lfloor \frac{2^{k+1}}{3} \rfloor + 1$ were determined in [10, 1].

Proposition 7.1. *The rubbling number of an even cycle is $\rho(C_{2k}) = 2^k$.*

Proof. It is well known [10] that $\pi(C_{2k}) = 2^k$. The first result now follows since

$$2^k = 2^{\text{diam}(C_{2k})} \leq \rho(C_{2k}) \leq \pi(C_{2k}) = 2^k.$$

\square

Proposition 7.2. *The rubbling number of an odd cycle is $\rho(C_{2k+1}) = \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor + 1$.*

Proof. Let C_{2k+1} be the cycle with consecutive vertices

$$x_k, x_{k-1}, \dots, x_1, v, y_1, y_2, \dots, y_k, x_k.$$

First we show that $\rho(C_{2k+1}) \leq \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor + 1$. Let p be a pebble distribution on C_{2k+1} from which not every vertex is reachable. It suffices to show that p contains at most $\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$ pebbles. By symmetry, we can assume that v is the vertex that is not reachable from p . By the Squishing Lemma, we can assume that p is squished on the thread with consecutive vertices $y_1, \dots, y_k, x_k, \dots, x_1$.

First we consider the case when all the pebbles are at distance k from v , that is, $p(x_k, y_k, *) = (a, b, 0)$. By symmetry, we can assume that $0 \leq a \leq b$. Then we must have

$$(7.1) \quad \left\lfloor \frac{a}{2} \right\rfloor + b \leq 2^k - 1,$$

otherwise we could move $\lfloor \frac{a}{2} \rfloor$ pebbles from vertex x_k to vertex y_k and then reach v from b_k . Hence $\frac{a}{2} < \lfloor \frac{a}{2} \rfloor + 1 \leq 2^k - 1 - b + 1 = 2^k - b$ and so

$$(7.2) \quad a + 2b \leq 2^{k+1} - 1.$$

We also must have

$$(7.3) \quad \left\lfloor \frac{b - 2^{k-1}}{2} \right\rfloor + a \leq 2^{k-1} - 1,$$

otherwise we could move $\lfloor \frac{b - 2^{k-1}}{2} \rfloor$ pebbles from vertex y_k to vertex x_k after which x_1 is reachable from x_k and y_1 is reachable from y_k , and so v would be reachable by the move $(x_1, y_1 \rightarrow v)$. Hence $\frac{b - 2^{k-1}}{2} < \lfloor \frac{b - 2^{k-1}}{2} \rfloor + 1 \leq 2^{k-1} - 1 - a + 1 = 2^{k-1} - a$ and so

$$(7.4) \quad b + 2a \leq 2^k + 2^{k-1} - 1.$$

Adding (7.2) and (7.4) gives

$$3(a + b) \leq 2^{k+1} - 1 + 2^k + 2^{k-1} - 1 = 7 \cdot 2^{k-1} - 2,$$

which shows that $|p| = a + b \leq \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$.

Now we consider the case when some pebbles are closer to v than k , that is, $p(x_i, x_{i+1}, *) = (b, a, 0)$ with $b \geq 1$ and $a \geq 0$ for some $1 \leq i < k$. Then we must have $\lfloor \frac{a}{2} \rfloor + b \leq 2^i - 1 \leq 2^{k-1} - 1$ otherwise v is reachable. Hence

$$\begin{aligned} |p| &= a + b \leq a - \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{2} \right\rfloor + b \\ &\leq \left\lfloor \frac{a}{2} \right\rfloor + 1 + 2^{k-1} - 1 \leq 2^{k-1} - 1 - b + 1 + 2^{k-1} - 1 \\ &= 2 \cdot 2^{k-1} - 2 < \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor. \end{aligned}$$

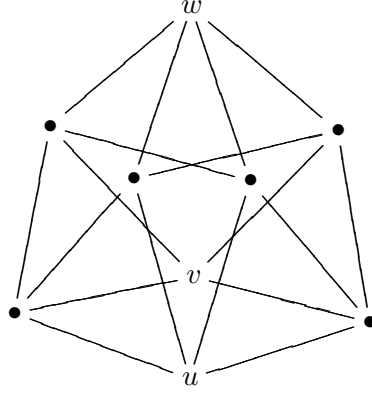
Now we show that we can always distribute $\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$ pebbles so that v is unreachable and so $\rho(C_{2k+1}) \geq \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor + 1$. Let $a = \lfloor \frac{2^k}{3} \rfloor$ and $b = \lfloor \frac{5 \cdot 2^{k-1}}{3} \rfloor$. It is easy to check that

$$a = \begin{cases} \frac{2^k - 2}{3}, & k \text{ odd} \\ \frac{2^k - 1}{3}, & k \text{ even} \end{cases}, \quad b = \begin{cases} \frac{5 \cdot 2^{k-1} - 2}{3}, & k \text{ odd} \\ \frac{5 \cdot 2^{k-1} - 1}{3}, & k \text{ even} \end{cases}, \quad \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor = \begin{cases} \frac{7 \cdot 2^{k-1} - 4}{3}, & k \text{ odd} \\ \frac{7 \cdot 2^{k-1} - 2}{3}, & k \text{ even} \end{cases}$$

and so $a + b = \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$. We show that v is unreachable from the pebble distribution $p(x_k, y_k, *) = (a, b, 0)$.

It is easy to see that a and b satisfy (7.2) and (7.4). Suppose that v is reachable from p , that is, there is an acyclic multiset S of rubbling moves that is balanced with p satisfying $p_S(v) \geq 1$. The balance condition at v shows that $d^-(v) \geq 2$. Hence S must have at least one of $(x_1, y_1 \rightarrow v)$, $(x_1, x_1 \rightarrow v)$ or $(y_1, y_2 \rightarrow v)$.

First assume that $(x_1, y_1 \rightarrow v) \in S$. The argument used in the proof of Proposition 3.3 shows that then $T(G, S)$ has at least 2^{i-1} arrows from x_i to x_{i-1} and from y_i to y_{i-1} for all $i \in \{2, \dots, k\}$. Since S is acyclic, any arrow in $T(G, S)$ pointing to x_k must come from y_k . So the balance condition at x_k requires m arrows from y_k to x_k satisfying $2^{k-1} \leq a + \frac{m}{2}$. The balance condition at y_k gives $2^{k-1} + m \leq b$. Combining the two inequalities gives $2^k + 2^{k-1} \leq b + 2a$ which contradicts (7.4).

FIGURE 8.1. The product graph $C_3 \square C_3$.

Next assume that $(y_1, y_1 \rightarrow v) \in S$. Then $T(G, S)$ has at least 2^i arrows from y_i to y_{i-1} for all $i \in \{2, \dots, k\}$. The balance condition at y_k requires m arrows from x_k to y_k satisfying $2^k \leq b + \frac{m}{2}$. We must have $d^-(x_k) = 0$, otherwise there is a directed path from v to x_k which is impossible since S is acyclic. The balance condition at x_k gives $m \leq a$. Combining the two inequalities gives $2^{k+1} \leq a + 2b$ which contradicts (7.2).

Similar argument shows that $(x_1, x_1 \rightarrow v) \in S$ is also impossible. \square

8. GRAHAM'S CONJECTURE

The *Cartesian product* $G \square H$ of the graphs G and H has vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = \{(v_1, w_1), (v_2, w_2)\} \mid (v_1 = v_2 \text{ and } \{w_1, w_2\} \in E(H)) \text{ or } (w_1 = w_2 \text{ and } \{v_1, v_2\} \in E(G))\}$.

Graham's conjecture $\pi(G \square H) \leq \pi(G)\pi(H)$ generated a lot of interest but it is still unresolved. We know from [4] that the inequality holds for the optimal pebbling number.

Proposition 8.1. $\rho(C_3 \square C_3) > 4$.

Proof. Using the notation of Figure 8.1, we show that w is not reachable from the pebble distribution $p(u, v, *) = (3, 1, 0)$. All the pebbles in p are of distance 2 from w . We have only 4 pebbles, so the only possibility to reach w is to use pebbling moves that decrease the distance of the pebbles from w . This is impossible since u and v do not have a common neighbor vertex that is at distance 1 from w . \square

It is not hard to see that $\rho(C_3 \square C_3) = 5$. Note that $\rho(C_3 \square C_3) > 4 = \rho(C_3)\rho(C_3)$ so Graham's conjecture does not hold for rubbling numbers.

9. OPTIMAL RUBBLING

Optimal pebbling was studied in [10, 9, 4, 1]. In this section we investigate the optimal rubbling number of certain graphs.

Definition 9.1. The *optimal rubbling number* $\rho_{\text{opt}}(G)$ of a graph G is the minimum number m for which there is a pebble distribution of size m from which every vertex of G is reachable.

Proposition 9.2. *We have the following values for the optimal rubbling number:*

- $\rho_{\text{opt}}(K_n) = 2$ for $n \geq 2$ where K_n is the complete graph with n vertices;
- $\rho_{\text{opt}}(W_n) = 2$ for $n \geq 5$ where W_n is the wheel with n vertices;
- $\rho_{\text{opt}}(K_{m,n}) = 3$ for $m, n \geq 3$ where $K_{m,n}$ is the complete bipartite graph;
- $\rho_{\text{opt}}(P) = 4$ where P is the Petersen graph.

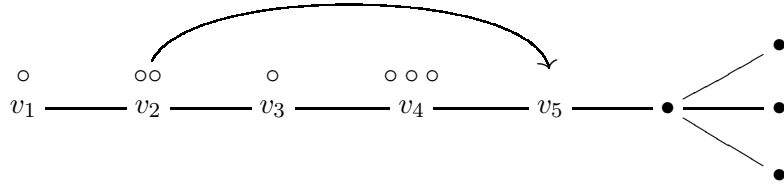


FIGURE 9.1. Visualization of a single rolling move with $i = 2$ and $n = 5$. An arrow indicates the transfer of a single pebble

Proof. a. Not every vertex of K_n is reachable from a distribution of size 1 since $n \geq 2$. On the other hand any vertex is reachable by a single move from any distribution of size 2.

b. Again, not every vertex of W_n is reachable from a distribution of size 1. On the other hand, every vertex is reachable from the distribution that has 2 pebbles at the center of W_n .

c. Let A and B be the natural partition of the vertex set of $K_{m,n}$. Let p be a pebble distribution of size 2. If p places both pebbles on vertices in A then there is a vertex in A that is not reachable from p . If p places both pebbles on vertices in B then there is a vertex in B that is not reachable from p . If p places one pebble on a vertex in A and one pebble on a vertex in B then both A and B have vertices that are unreachable from p . On the other hand any vertex is reachable in at most two moves from a pebble distribution that places one pebble on a vertex in A and two pebbles on a vertex in B .

d. Every vertex is reachable from the pebble distribution that has 4 pebbles on any of the vertices. We show that 3 pebbles are not sufficient to make every vertex reachable using the notation of Figure 5.2. By symmetry, we can assume that a pebble is placed on vertex w and a second pebble is placed on w , a or q . A simple case analysis shows that in all three cases it is impossible to place the third pebble to make each vertex reachable. \square

The optimal pebbling numbers of these graphs are $\pi_{\text{opt}}(K_n) = 2$, $\pi_{\text{opt}}(W_n) = 2$, $\pi_{\text{opt}}(K_{m,n}) = 3$ and $\pi_{\text{opt}}(P) = 4$.

Smoothing was used in [1] to study optimal pebbling numbers. A *smoothing move* removes two pebbles from a vertex v containing at least three pebbles and adds one pebble at each neighbor of v . A smoothing move is only allowed if v has at least three pebbles. Rolling moves serve the same purpose for rubbling as the smoothing moves for pebbling. We want to restrict the set of possible pebble distributions we need to consider, to determine the value of the optimal rubbling number.

Definition 9.3. Let v_1, \dots, v_n be the consecutive vertices of a path such that the degree of v_1 is 1 and the degrees of v_2, v_3, \dots, v_{n-1} are all 2. The subgraph induced by $\{v_1, \dots, v_n\}$ is called an *arm* of the graph. Let p be a pebble distribution such that $p(v_i) \geq 2$ for some $i \in \{1, \dots, n-1\}$, $p(v_n) = 0$, and $p(v_j) \geq 1$ for all $j \in \{1, \dots, n-1\}$. A *single rolling move* creates a new pebble distribution q by taking one pebble from v_i and placing it on v_n , that is $q(v_i, v_n, *) = (p(v_i) - 1, 1, p(*))$. See Figure 9.1.

Lemma 9.4. *Let q be a pebble distribution on G gotten from the pebble distribution p by applying a single rolling move from v_i to v_n on the arm with vertices v_1, \dots, v_n . If vertex $u \in G$ is reachable from p then u is also reachable from q .*

Proof. If u is a vertex of the arm then it is clearly reachable from q so we can assume that u is not on the arm. Let S be an acyclic multiset of rubbling moves balanced with p such that $p_S(u) \geq 1$. Let P be a maximum length directed path in $T(G, S)$ starting at v_i and not going further than v_n . Then P has consecutive vertices $v_i = v_{n_0}, v_{n_1}, \dots, v_{n_k}$ on the arm. Let R be the multiset containing the elements of S without the moves corresponding to the arrows of P . We show that R is balanced with q and so u is reachable from q since

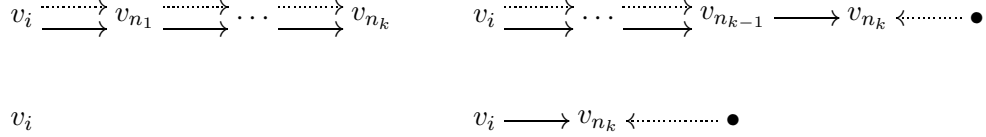


FIGURE 9.2. Four possible configurations for $T(G, S \setminus R)$. The solid arrows represent the arrows of P .

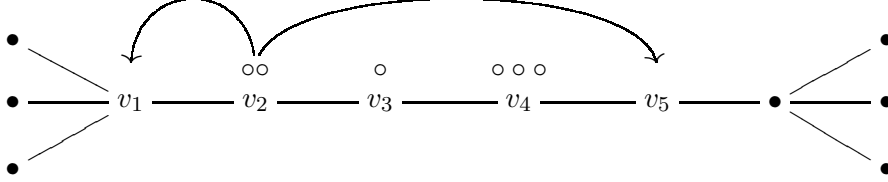


FIGURE 9.3. Visualization of a double rolling move with $i = 2$ and $n = 5$. An arrow indicates the transfer of a single pebble.

$q_R(u) = p_S(u) \geq 1$. Figure 9.2 shows the possible configurations for $T(G, S \setminus R)$. If $n_k = n$ then

$$q_R(v_{n_k}) = p_S(v_{n_k}) + \Delta(1, -2, 0) = p_S(v_{n_k}) \geq 0,$$

while if $n_k \neq n$ then $d_{T(G,S)}^+(v_{n_k}) = 0$ and so

$$q_R(v_{n_k}) = p_S(v_{n_k}) + \Delta(0, -2, 0) \geq p_S(v_{n_k}) - 1 \geq 0.$$

So R is balanced with q at v_{n_k} . If $d_{T(G,S)}^+(v_{n_0}) = 0$ then $n_0 = n_k$, otherwise there is an $a \in \{-1, -2\}$ such that

$$q_R(v_{n_0}) = p_S(v_{n_0}) + \Delta(-1, 0, a) \geq p_S(v_{n_0}) \geq 0$$

and so R is balanced with q at v_{n_0} . If $0 < j < k$ then there is an $a \in \{-1, -2\}$ such that

$$q_R(v_{n_j}) = p_S(v_{n_j}) + \Delta(0, -2, a) \geq p_S(v_{n_j}) \geq 0$$

and so R is balanced with q at v_{n_j} . It is clear that R is balanced with q at every other vertex. \square

Definition 9.5. Let v_1, \dots, v_n be the consecutive vertices of a path such that the degrees of v_2, v_3, \dots, v_{n-1} are all 2. Let p be a pebble distribution such that $p(v_1) = 0 = p(v_n)$, $p(v_i) \geq 2$ for some $i \in \{2, \dots, n-1\}$ and $p(v_j) \geq 1$ for all $j \in \{2, \dots, n-1\}$. A *double rolling move* creates a new pebble distribution q by taking two pebbles from v_i and placing one pebble on v_1 and one pebble on v_n , that is $q(v_i, v_1, v_n, *) = (p(v_i) - 2, 1, 1, p(*))$. See Figure 9.3.

Lemma 9.6. *Let q be a pebble distribution on G gotten from the pebble distribution p by applying a double rolling move from vertex v_i to vertices v_1 and v_n on the path with consecutive vertices v_1, \dots, v_n . If vertex $u \in G$ is reachable from p then u is also reachable from q .*

Proof. If $u \in \{v_1, \dots, v_n\}$ then it is clearly reachable from q so we can assume that $u \notin \{v_1, \dots, v_n\}$. Let S be an acyclic multiset of rubbing moves balanced with p such that $p_S(u) \geq 1$. Let P be a maximum length directed path in $T(G, S)$ starting at v_i and not going further than v_1 or v_n . Then P has consecutive vertices $v_i = v_{n_0}, v_{n_1}, \dots, v_{n_k} \in \{v_1, \dots, v_n\}$. Let R be the multiset containing the elements of S without the moves corresponding to the arrows of P . An argument similar to the one in the proof of Lemma 9.4 shows that R is balanced with q at every vertex except maybe at v_i . If $n_k = n_0$ or the arrow (v_{n_0}, v_{n_1}) in P

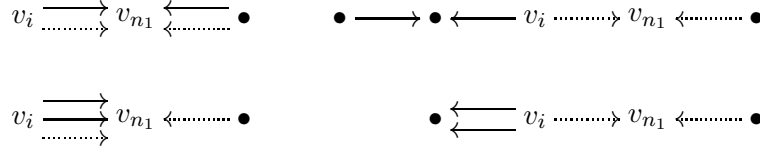


FIGURE 9.4. The four possible configurations for $T(G, S \setminus \tilde{R})$. The solid arrows represent the moves corresponding to the arrows of \tilde{P} . The dotted arrows represent the moves corresponding to the arrows of P .

corresponds to a pebbling move, then R is balanced with q at v_i as well. Then u is reachable from q since $q_R(u) = p_S(u) \geq 1$.

So we can assume that (v_{n_0}, v_{n_1}) corresponds to a strict rubbling move and that $k = 1$. Let \tilde{P} be a maximum length path in $T(G, R)$. Since $k = 1$, the length of \tilde{P} is either 0 or 1. If this length is 0, then q is balanced with R at v_i since $d_{T(G,R)}^+(v_i) = 0$ and we are done. If the length of \tilde{P} is 1, then let \tilde{R} be the multiset containing the elements of R without the moves corresponding to the arrows of \tilde{P} . Figure 9.4 shows the possibilities for $T(G, S \setminus \tilde{R})$. It is easy to check that \tilde{R} is balanced with q in each case. Thus u is reachable from q since $q_{\tilde{R}}(u) \geq p_S(u)$. \square

Rolling moves make it possible to find the optimal rubbling number of paths and cycles. The optimal pebbling number $\pi_{\text{opt}}(P_n) = \lceil \frac{2n}{3} \rceil = \pi_{\text{opt}}(C_n)$ was determined in [10, 1].

Proposition 9.7. The optimal rubbling number of the path is $\rho_{\text{opt}}(P_n) = \lceil \frac{n+1}{2} \rceil$.

Proof. Let P_n be the path with consecutive vertices v_1, \dots, v_n . It is clear that every vertex is reachable from the pebble distribution

$$p(v_i) = \begin{cases} 1, & i \text{ is odd or } i = n \\ 0, & \text{else} \end{cases}$$

which has size $\lceil \frac{n+1}{2} \rceil$.

Now assume that there is a pebble distribution of size $\lceil \frac{n+1}{2} \rceil - 1$ from which every vertex of P_n is reachable. Let us apply all available rolling moves (single or double). The process ends in finitely many steps since a rolling move reduces the number of pebbles on vertices with more than one pebble by at least one. If there is a vertex with more than one pebble and a vertex with no pebbles, then a rolling move is available. The number of pebbles is not larger than the number of vertices, so the resulting pebble distribution q has at most one pebble on each vertex. Every vertex of P_n still must be reachable from q by Lemma 9.6.

The only moves executable directly from q are strict rubbling moves. By the No Cycle Lemma we can assume that every vertex is reachable by a sequence of moves in which a strict rubbling move $(x, y \rightarrow z)$ is not followed by a move of the form $(z, z \rightarrow x)$ or $(z, z \rightarrow y)$. So we can assume that every vertex is reachable through strict rubbling moves. Then we must have $q(v_1) = 1 = q(v_n)$ otherwise v_1 or v_n is not reachable.

A pigeon hole argument shows that there must be two neighbor vertices u and w such that $q(u) = 0 = q(w)$. To avoid the existence of such u and w , we would need to place at least $\lfloor \frac{n-2}{2} \rfloor$ pebbles on the vertices v_2, \dots, v_{n-1} but it is easy to see that the $\lceil \frac{n+1}{2} \rceil - 3$ pebbles available for this purpose are not sufficient.

Then neither u nor w is reachable from q , which is a contradiction. \square

Proposition 9.8. The optimal rubbling number of the cycle is $\rho_{\text{opt}}(C_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 3$.

n	2	3	4	5
$\rho_{\text{opt}}(B_n)$	2	4	6	
$\rho_{\text{opt}}(Q_n)$	2	3	4	6

TABLE 1. Rubbling values without a known general formula.

Proof. Let C_n be the cycle with consecutive vertices v_1, \dots, v_n . It is clear that every vertex is reachable from the pebble distribution

$$p(v_i) = \begin{cases} 1, & i \text{ is odd} \\ 0, & \text{else} \end{cases}$$

which has size $\lceil \frac{n}{2} \rceil$.

Now assume that there is a pebble distribution of size $\lceil \frac{n}{2} \rceil - 1$ from which every vertex of C_n is reachable. Let us apply all available double rolling moves. The process ends in finitely many steps since a double rolling move reduces the number of pebbles on vertices with more than one pebble by two. If there is a vertex with more than one pebble and two vertices with no pebbles, then a double rolling move is available. The number of pebbles is smaller than the number of vertices, so the resulting pebble distribution q has at most one pebble on each vertex. Every vertex of C_n still must be reachable from q .

The only moves executable directly from q are strict rubbling moves. The No Cycle Lemma implies that we can assume that every vertex is reachable through strict rubbling moves. A pigeon hole argument shows that there must be two neighbor vertices u and w such that $q(u) = 0 = q(w)$. But then neither u nor w is reachable from q which is a contradiction. \square

10. FURTHER QUESTIONS

There are plenty of unanswered questions. We list a few of them.

- What is the optimal rubbling number of the complete binary tree B_n and the hypercube Q_n . It is fairly easy to get answers for small n with a computer. The known values are listed in Table 1.
- The *cover rubbling number* of a graph G is the minimum number m such that for every pebble distribution p on G with size m there is an executable rubbling sequence s with $p_s(v) \geq 1$ for all $v \in V(G)$. The cover pebbling number is defined analogously. Is the cover rubbling number the same as the cover pebbling number for every graph? The answer might depend on whether the cover pebbling theorem of [11] can be generalized for rubbling.
- We have $\pi(P_n) = \rho(P_n)$, $\pi(Q_n) = \rho(Q_n)$ and it is easy to check that $\pi(L) = 8 = \rho(L)$ where L is the Lemke graph [6]. This is not always the case though. Is it possible to characterize those graphs for which the pebbling and the rubbling numbers are the same? When is the rubbling number significantly smaller than the pebbling number?

REFERENCES

1. David P. Bunde, Erin W. Chambers, Daniel Cranston, Kevin Milans, and Douglas B. West, *Pebbling and optimal pebbling in graphs*, J. Graph Theory **57** (2008), no. 3, 215–238.
2. Fan R. K. Chung, *Pebbling in hypercubes*, SIAM J. Discrete Math. **2** (1989), no. 4, 467–472.
3. Betsy Crull, Tammy Cundiff, Paul Feltman, Glenn H. Hurlbert, Lara Pudwell, Zsuzsanna Szaniszlo, and Zsolt Tuza, *The cover pebbling number of graphs*, Discrete Math. **296** (2005), no. 1, 15–23.
4. Hung-Lin Fu and Chin-Lin Shiue, *The optimal pebbling number of the complete m -ary tree*, Discrete Math. **222** (2000), no. 1-3, 89–100.

5. Glenn Hurlbert, *A survey of graph pebbling*, Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999), vol. 139, 1999, pp. 41–64.
6. ———, *Recent progress in graph pebbling*, Graph Theory Notes N. Y. **49** (2005), 25–37.
7. Kevin Milans and Bryan Clark, *The complexity of graph pebbling*, SIAM J. Discrete Math. **20** (2006), no. 3, 769–798 (electronic).
8. David Moews, *Pebbling graphs*, J. Combin. Theory Ser. B **55** (1992), no. 2, 244–252.
9. ———, *Optimally pebbling hypercubes and powers*, Discrete Math. **190** (1998), no. 1-3, 271–276.
10. Lior Pachter, Hunter S. Snevily, and Bill Voxman, *On pebbling graphs*, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), vol. 107, 1995, pp. 65–80.
11. Jonas Sjöstrand, *The cover pebbling theorem*, Electron. J. Combin. **12** (2005), Note 22, 5 pp. (electronic). MR MR2180807 (2007f:05091)

NORTHERN ARIZONA UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS, FLAGSTAFF AZ 86011-5717, USA

E-mail address: `cbelford@gmail.com`

E-mail address: `nandor.sieben@nau.edu`