# POLYOMINOES WITH MINIMUM SITE-PERIMETER AND FULL SET ACHIEVEMENT GAMES

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Abstract

The site-perimeter of a polyomino is the number of empty cells connected to the polyomino by an edge. A formula for the minimum site-perimeter with a given cell size is found. This formula is used to show the effectiveness of a simple random strategy in polyomino set achievement games.

## 1. Introduction

Achievement games for polyominoes have been introduced by Frank Harary [Gar, Har]. They are generalizations of the well known game Tic-Tac-Toe, where the target shape can be some predetermined set of polyominoes. The type of the board can vary as well. It can be a tiling of the plane by triangles [BH3] or hexagons [BH2]. The game board can be a Platonic solid [BH1] or the hyperbolic plane [Bod]. A comprehensive investigation of these possibilities can be found in [Bod]. Many more abstract generalizations were studied in [EH, SD, Sie].

A *rectangular board* is the set of *cells* that are the translations of the unit square $[0,1]\times[0,1]$ by vectors of $\mathbf{Z}^2$. Informally, a rectangular board is the infinite chessboard. Two cells are called *adjacent* if they share a common edge. Adjacent cells are also called *neighbors*.

A *polyomino* or *animal* is a finite set of cells of the rectangular board in which the cells are connected through adjacent cells. Note that we allow holes in our polyominoes. We only consider polyominoes up to congruence (combination of translations, rotations and reflections). The number of cells $s(P)$ of a polyomino $P$ is called the *size* of $P$.

The set $E(P)$ of empty cells adjacent to any cell of $P$ is called the *exterior boundary* of $P$. The *site-perimeter* of $P$ is the number of elements $e(P) = |E(P)|$ in $E(P)$. For example the site-perimeter of the animal with a single cell is 4. The site-perimeter is also called *exterior perimeter*.

In a *polyomino set* $(p,q)$*-achievement game* two players alternately mark $p$ and $q$ previously unmarked cells of the board using their own colors. If $p$ or $q$ is not 1 then the game is often called *biased*. The player who marks a polyomino congruent to one of a given set of polyominoes wins the game. In a *weak set achievement game* the second player (the *breaker*) only tries to prevent the first player (the *maker*) from achieving one of the polyominoes.

In this paper we study *polyomino weak full set* $(1,q)$*-achievement games* where the set of target polyominoes is the set $\mathcal{F}_s$ of all polyominoes of size $s$. In this game the maker can follow the strategy of marking a random cell adjacent to one of his earlier marks. We investigate when this strategy is effective in Section 2.

The answer depends on how small the site-perimeter of the polyominoes in $\mathcal{F}_s$ can be. This question is interesting on its own right. The site-perimeter plays an important role in percolation theory. It is also used as a fixed parameter when counting the number of polyominoes [DGV, BR].

It turns out that it is easier to find the maximum size of an animal with a given site-perimeter than to find the minimum site-perimeter with a given size. We find all polyominoes with maximum size and fixed site-perimeter in Sections 3 and 4. This allows us to find the minimum site-perimeter with a given size in Section 5.

The maximum size of an animal with a given bond-perimeter is given in [PO]. A related question is solved in [SS] using the number of solvent contacts instead of the site-perimeter. I thank Steve Wilson and the referee of the paper for their help during the writing of this paper.

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## 2. The random neighbor strategy

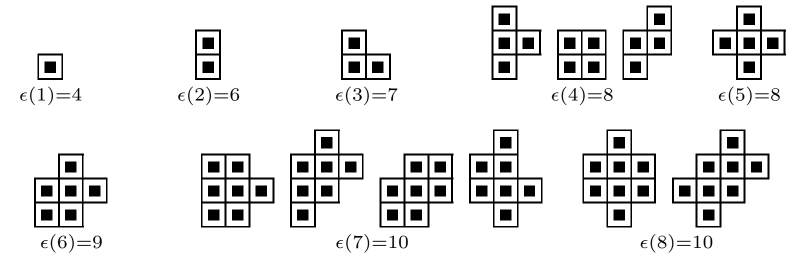
In the full set $(1, q)$-achievement game the maker can follow the strategy of randomly marking a cell adjacent to one of his earlier marks. We call this the *random neighbor strategy.* If the maker is able to follow this strategy for $s$ turns then he can mark an animal of size $s$ and win the $\mathcal{F}_s$ achievement game. Of course it is possible that this strategy fails after $r < s$ turns because the whole exterior boundary $E(P_r)$ of the animal $P_r \in \mathcal{F}_r$ built from the maker's $r$ marks is already marked by the breaker. This will not happen though if the total number $rq$ of marks by the breaker is smaller than the smallest possible site-perimeter of $P_r$.

**Definition 2.1.** We use the notation $\epsilon(s) = \min\{e(P) \mid P \in \mathcal{F}_s\}$ for the *minimum site-perimeter* in $\mathcal{F}_s$.

The proof of the following proposition is clear from the discussion above.

**Proposition 2.2.** *The random neighbor strategy is successful in the full set* $(1, q)$*-achievement game for* $s$ *turns if* $rq < \epsilon(r)$ *for all* $r < s$.

It suffices to require that the inequality holds for $r = s - 1$. We are going to prove this stronger result in Proposition 5.5 after we find a formula for $\epsilon(s)$. For small $s$, this value can be found using a full search. The following figure shows every animal $P \in \mathcal{F}_s$ satisfying $e(P) = \epsilon(s)$ for $s \leq 8$.



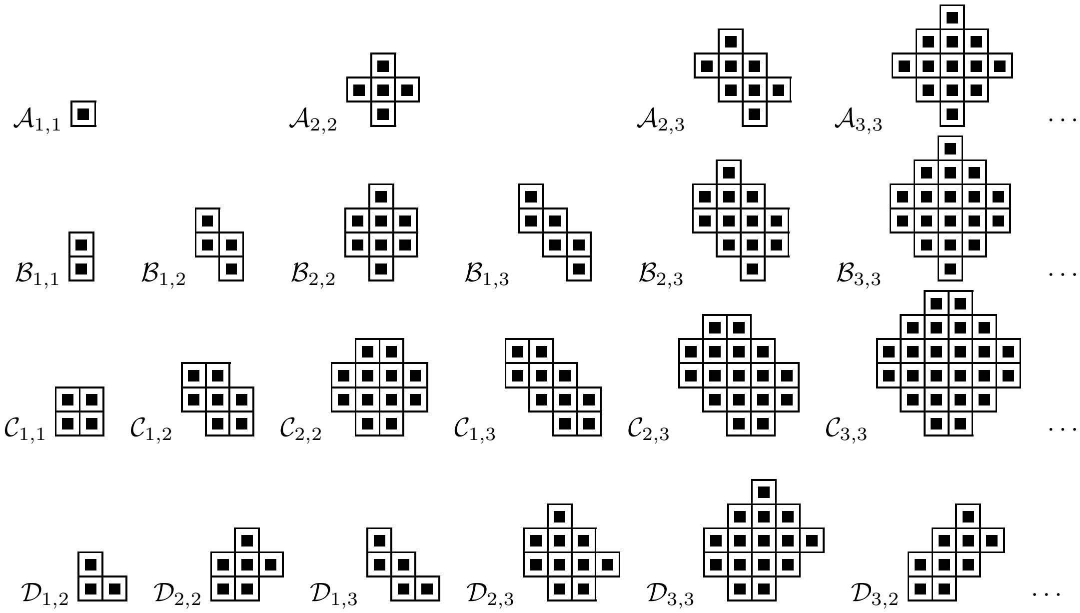

Note that $\epsilon(4) = \epsilon(5)$ and $\epsilon(7) = \epsilon(8)$. Also note that some of the listed animals in $\mathcal{F}_4$ and $\mathcal{F}_7$ can be constructed from animals in $\mathcal{F}_5$ and $\mathcal{F}_8$ respectively by deleting one cell. The listed animals in $\mathcal{F}_i$ for $i \in \{1, 2, 3, 5, 6, 8\}$ have maximal size among animals with the same site-perimeter. Animals with this property have another special property. In the next section we study this property.

## 3. Saturated animals

In this section we study animals that cannot be made larger without increasing their site-perimeter.

**Definition 3.1.** A cell $x \in E(P)$ is *admissible* to the animal $P$ if $e(P \cup \{x\}) \leq e(P)$. $P$ is called *saturated* if it has no admissible cells. We use the notation $\mathcal{S}$ for the set of saturated animals.

**Example 3.2.** It is easy to check that adding a cell to any of the animals in the following four parametrized families increases their site-perimeter and so they are all saturated.




Every listed polyomino is the collection of all cells with center points strictly inside a rectangle standing on its corner. Let $(x_1, x_2)$ denote the coordinates of the center point of the cell $x$. Representatives of the polyominoes can be defined by

$$\begin{aligned}
\mathcal{A}_{k,n} &= \{x \mid -n < x_2 - x_1 < n \text{ and } n-1 < x_2 + x_1 < n + 2k - 1\} \\
\mathcal{B}_{k,n} &= \{x \mid -n < x_2 - x_1 < n + 1 \text{ and } n-1 < x_2 + x_1 < n + 2k\} \\
\mathcal{C}_{k,n} &= \{x \mid -n - 1 < x_2 - x_1 < n + 1 \text{ and } n-1 < x_2 + x_1 < n + 2k + 1\} \\
\mathcal{D}_{k,n} &= \{x \mid -n < x_2 - x_1 < n \text{ and } n-2 < x_2 + x_1 < n + 2k - 1\}.
\end{aligned}$$

In each family, the first and second indices in the notation determine the number of upper left and upper right corners respectively. The families are distinguished by the number of cells in their top and bottom rows, and their left and right columns. For example, the members of family $\mathcal{B}$ have one cell in their top and bottom rows, and two cells in their left and right columns.

Families $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are central symmetric so we can assume that the first index is not larger than the second. Family $\mathcal{D}$ is not central symmetric. Note that $\mathcal{A}_{1,n}$ is not an animal for any $n \geq 2$ and that $\mathcal{D}_{k,1}$ does not exist. Thus the families are

$$\mathcal{A} = \{\mathcal{A}_{k,n} \mid 2 \leq k \leq n\} \cup \{\mathcal{A}_{1,1}\}, \quad \mathcal{D} = \{\mathcal{D}_{k,n} \mid 1 \leq k \text{ and } 2 \leq n\},$$

$$\mathcal{B} = \{\mathcal{B}_{k,n} \mid 1 \leq k \leq n\}, \quad \mathcal{C} = \{\mathcal{C}_{k,n} \mid 1 \leq k \leq n\}.$$

Our next goal is to show that these are the only saturated animals, that is, $\mathcal{S} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.

**Lemma 3.3.** *If $x \in E(P)$ then $e(P \cup \{x\}) = e(P) - 1 + |E(\{x\}) \setminus (P \cup E(P))|$.*

*Proof.* When we add the cell $x$ to the animal $P$, the site-perimeter is decreased by 1 since $x \in E(P)$. At the same time the site-perimeter is increased by the number of cells that are adjacent to $x$ but not in $P \cup E(P)$. □

**Lemma 3.4.** *If $P \subseteq Q$ and $x \in E(Q)$ is admissible to $P$ then $x$ is also admissible to $Q$.*

*Proof.* The result follows from the calculation below using Lemma 3.3 and the fact that $P \cup E(P) \subseteq Q \cup E(Q)$.

$$\begin{aligned} e(Q \cup \{x\}) &= e(Q) - 1 + |E(\{x\}) \setminus (Q \cup E(Q))| \\ &\leq e(Q) - e(P) + e(P) - 1 + |E(\{x\}) \setminus (P \cup E(P))| \\ &= e(Q) - e(P) + e(P \cup \{x\}) \\ &\leq e(Q). \end{aligned}$$
□

**Proposition 3.5.** *If $P_i$ is saturated for all $i \in I$ then so is $P = \bigcap_{i \in I} P_i$.*

*Proof.* Suppose that $P$ is not saturated and so has an admissible cell $x \in E(P)$. Since $x \notin P$, there must be a $j \in I$ such that $x \notin P_j$. Then $x \in E(P_j)$ and so $x$ is admissible to $P_j$ by Lemma 3.4, but this is a contradiction. □

**Definition 3.6.** The *saturation* of the animal $P$ is the saturated animal $\overline{P} = \bigcap\{Q \mid P \subseteq Q \text{ and } Q \text{ is saturated}\}$.

Note that every animal is contained in a saturated animal that is congruent to $\mathcal{A}_{n,n}$ for a large enough $n$. So $\overline{P}$ is well defined, in fact, $\overline{P}$ is the minimum saturated animal containing $P$. It is clear from the definition that if $P \subseteq Q$ then $\overline{P} \subseteq \overline{Q}$.

**Proposition 3.7.** *If $x$ is admissible to $P$ then $x \in \overline{P}$.*

*Proof.* Assume that $x$ is admissible to $P$ but $x \notin \overline{P}$. Then $x \in E(\overline{P})$ since $x \in E(P)$ and $P \subseteq \overline{P}$. Hence $x$ is admissible to $\overline{P}$ by Lemma 3.4 which is a contradiction. □

The following is an easy consequence.

**Corollary 3.8.** *Let $P_0 = P$. If there is a cell $x_i$ admissible to $P_i$ then recursively define $P_{i+1} = P_i \cup \{x_i\}$. Then $\overline{P} = P_k$ for some $k$.*

**Corollary 3.9.** *$e(\overline{P}) \leq e(P)$ for all animal $P$.*

*Proof.* In the process described in Corollary 3.8 we have $e(\overline{P}) = e(P_k) \leq e(P_{k-1}) \leq \cdots \leq e(P_0) = e(P)$. □

**Proposition 3.10.** *Let $\mathcal{S}_1 = \{\mathcal{A}_{1,1}\}$. Recursively define $\mathcal{S}_{n+1} = \{\overline{P \cup \{x\}} \mid P \in \mathcal{S}_n$ and $x \in E(P)\}$. Then $\mathcal{S} = \bigcup_{i=1}^{\infty} \mathcal{S}_i$.*

*Proof.* We only need to check that $\mathcal{S} \subseteq \bigcup_{i=1}^{\infty} \mathcal{S}_i$. Let $Q \in \mathcal{S}$, $x_1 \in Q$ and $Q_1 = \{x_1\}$. If $Q_i \neq Q$ then recursively pick $x_{i+1} \in Q \cap E(Q_i)$ and define $Q_{i+1} = \overline{Q_i \cup \{x_{i+1}\}} \subseteq Q$. Then $Q_i \in \mathcal{S}_i$ for all $i$ and $Q = Q_k$ for some $k$. □

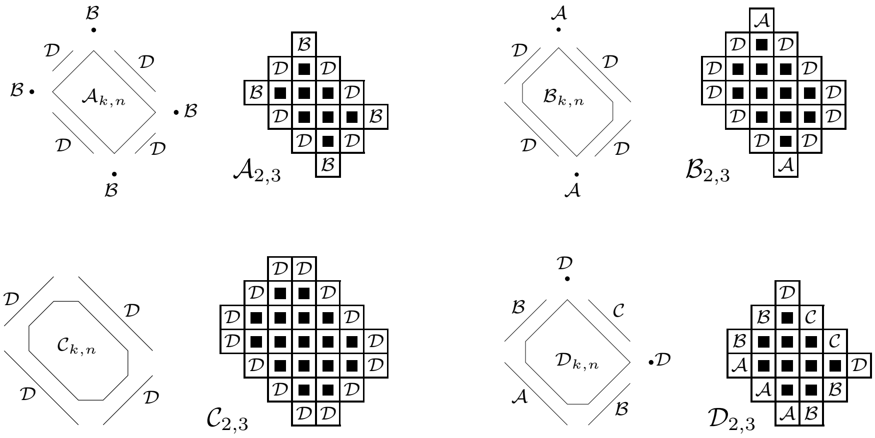
**Lemma 3.11.** *If an animal contains the three full cells but not the empty cell as shown in one of the two figures below, then the empty cell is admissible to the animal.*

*Proof.* In each case adding the empty cell to the animal decreases the site-perimeter by 1 and may increase it by at most 1. □

These two admissibility rules are used in the next result to find the saturation of any animal gotten from the animals in Example 3.2 by adding one extra cell.

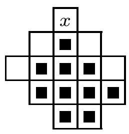
**Proposition 3.12.** *If $P \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ and $x \in E(P)$ then $\overline{P \cup \{x\}} \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.*

*Proof.* Let $Q = P \cup \{x\}$. The following pictures summarize what the saturation of $Q$ is going to be.

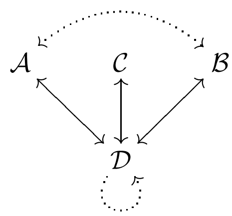



If $x$ is one of the cells symbolized by a line then $\overline{Q}$ is $P$ together with the cells of that line. If $x$ is one of the cells symbolized by a dot then $\overline{Q}$ is $P \cup \{x\}$ together with the cells of the two lines joining to the dot representing $x$. The letters next to lines and dots and also in the empty cells show the type of $\overline{Q}$. For example if $P = \mathcal{D}_{2,3}$ and $x$ is the empty cell on the top of $P$ then $\overline{Q} = \mathcal{D}_{3,3} \in \mathcal{D}$ as shown in the following figure. The empty cells need to be added to $Q$ to get $\overline{Q}$.

$$\overline{\mathcal{D}_{2,3} \cup \{x\}} = \mathcal{D}_{3,3}$$




The following graph shows the possible type changes from $P$ to $\overline{Q}$. Solid lines indicate that $e(\overline{Q}) = e(P) + 1$ while dotted lines indicate that $e(\overline{Q}) = e(P) + 2$.




Note that getting from type $\mathcal{A}$ to type $\mathcal{D}$ is not possible if $P = \mathcal{A}_{1,1}$. This is the only case when a transition shown on the graph is not possible. □

**Theorem 3.13.** *Every saturated animal is in one the families in Example 3.2, that is, $\mathcal{S} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.*

*Proof.* The result follows from Propositions 3.10 and 3.12. □

**Definition 3.14.** Let $\mathcal{G}_e$ be the set of animals with site-perimeter $e$. We use the notation $\sigma(e) = \max\{s(P) \mid P \in \mathcal{G}_e\}$ for the *maximum cell size* in $\mathcal{G}_e$.

The following proposition is the reason for studying saturated animals.

**Proposition 3.15.** *If $P$ is in $\mathcal{G}_e$ with $s(P) = \sigma(e)$ then $P$ is saturated.*

*Proof.* If $P$ is not saturated then $1 < s(P) < s(\overline{P})$. By Corollary 3.9, we have $e(P) \geq e(\overline{P})$. By the type change graph at the end of the proof of Proposition 3.12, we can change $\overline{P}$ into other saturated animals $\overline{P} = Q_0, Q_1, \ldots, Q_l$ along the solid arrows such that $e(Q_{i+1}) = e(Q_i) + 1$ for each $i$ and $e(Q_l) = e(P)$. This is a contradiction since $Q_l \in \mathcal{G}_e$ but $s(Q_l) \geq s(\overline{P}) > s(P)$. □

Note that $\mathcal{B}_{1,2} \in \mathcal{G}_8$ is saturated but $s(\mathcal{B}_{1,2}) = 4 \neq \sigma(8)$ since $\mathcal{A}_{2,2} \in \mathcal{G}_8$ with $s(\mathcal{A}_{2,2}) = 5$. In the next section we find those saturated animals that make the size maximal in $\mathcal{G}_e$.

## 4. Animals of fixed perimeter and maximal size

It is easy to calculate the size and the site-perimeter of saturated animals. The following table shows these values:

| $P$ | $\mathcal{A}_{k,n}$ | $\mathcal{B}_{k,n}$ | $\mathcal{C}_{k,n}$ | $\mathcal{D}_{k,n}$ |
|---|---|---|---|---|
| $e(P)$ | $2(n+k)$ | $2(n+k+1)$ | $2(n+k+2)$ | $2(n+k)+1$ |
| $s(P)$ | $nk+(n-1)(k-1)$ | $2nk$ | $n(k+1)+k(n+1)$ | $(2n-1)k$ |

From this we can calculate the maximum size for a given site-perimeter.

**Theorem 4.1.** *For $e \in \{4, 6, 7, 8, \ldots\}$ we have $\sigma(e) = \lfloor e^2/8 - e/2 + 1 \rfloor$.*

*Proof.* Any animal in $\mathcal{G}_e$ with maximal size is saturated, so we only need to consider saturated animals. For a fixed site-perimeter $e$ we can express the parameter $k$ for the families in Example 3.2 in terms of $e$ and $n$, and hence $s(P)$ in terms of $e$ and $n$. The resulting expression for the size is quadratic. The maximum value is taken at the closest integer to the vertex. The location of this integer depends on the congruence class of $e$ modulo 4. We summarize the calculation in the tables below. The site-perimeter in the families $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ is always even. So for even $e$ such that $e \geq 4$ we have the following.

| $P$ | | $\mathcal{A}_{k,n}$ | $\mathcal{B}_{k,n}$ | $\mathcal{C}_{k,n}$ |
|---|---|---|---|---|
| $k$ | | $e/2-n$ | $e/2-n-1$ | $e/2-n-2$ |
| $s(P)$ | | $-2n^2+en-e/2+1$ | $-2n^2+(e-2)n$ | $-2n^2+(e-4)n+e/2-2$ |
| vertex | | $e/4$ | $e/4-1/2$ | $e/4-1$ |
| $e \equiv 0\ (4)$ | optimal $n$ | $*$ $e/4$ | $e/4$ | $e/4-1$ |
| | optimal $k$ | $e/4$ | $e/4-1$ | $e/4-1$ |
| | max $s(P)$ | $e^2/8-e/2+1$ | $e^2/8-e/2$ | $e^2/8-e/2$ |
| $e \equiv 2\ (4)$ | optimal $n$ | $*$ $e/4+1/2$ | $*$ $e/4-1/2$ | $e/4-1/2$ |
| | optimal $k$ | $e/4-1/2$ | $e/4-1/2$ | $e/4-3/2$ |
| | max $s(P)$ | $e^2/8-e/2+1/2$ | $e^2/8-e/2+1/2$ | $e^2/8-e/2-1/2$ |

The site-perimeter in the family $\mathcal{D}$ is always odd so for odd $e$ such that $e \geq 7$ we have the following.

<table>
<tr><th colspan="2">$P$</th><th colspan="2">$\mathcal{D}_{k,n}$</th></tr>
<tr><td colspan="2">$k$<br>$s(P)$<br>vertex</td><td colspan="2">$e/2 - n - 1/s$<br>$-2n^2 + en - e/2 + 1/2$<br>$e/4$</td></tr>
<tr><td rowspan="3">$e \equiv 1\ (4)$</td><td>optimal $n$</td><td>$*$</td><td>$e/4 - 1/4$</td></tr>
<tr><td>optimal $k$</td><td></td><td>$e/4 - 1/4$</td></tr>
<tr><td>max $s(P)$</td><td></td><td>$e^2/8 - e/2 + 3/8$</td></tr>
<tr><td rowspan="3">$e \equiv -1\ (4)$</td><td>optimal $n$</td><td>$*$</td><td>$e/4 + 1/4$</td></tr>
<tr><td>optimal $k$</td><td></td><td>$e/4 - 3/4$</td></tr>
<tr><td>max $s(P)$</td><td></td><td>$e^2/8 - e/2 + 3/8$</td></tr>
</table>

Now we just need to pick the family that gives the maximum size. For $e$ divisible by 4 this is family $\mathcal{A}$. For $e \equiv 2\ (4)$ this is either family $\mathcal{A}$ or $\mathcal{B}$. For odd $e$ only family $\mathcal{D}$ is possible. These choices are denoted by a star in the table. Thus

$$\sigma(e) = \begin{cases} e^2/8 - e/2 + 1 & \text{if } e \equiv 0\ (4) \\ e^2/8 - e/2 + 1/2 & \text{if } e \equiv 2\ (4) \\ e^2/8 - e/2 + 3/8 & \text{if } e \equiv 1\ (2) \end{cases}$$

which simplifies to $\lfloor e^2/8 - e/2 + 1 \rfloor$. Note that family $\mathcal{C}$ never gives the maximum. □

**Lemma 4.2.** *The function $\sigma$ is strictly increasing.*

*Proof.* The value $\rho(5)$ is undefined and $\rho(4) = 1 < 2 = \rho(6)$. For $e \geq 6$ we have

$$\begin{aligned} \sigma(e+1) - \sigma(e) &> (e^2/8 - e/2 + 3/8) - ((e+1)^2 - (e+1)/2 + 1) \\ &= e/4 - 1 > 0. \end{aligned}$$

□

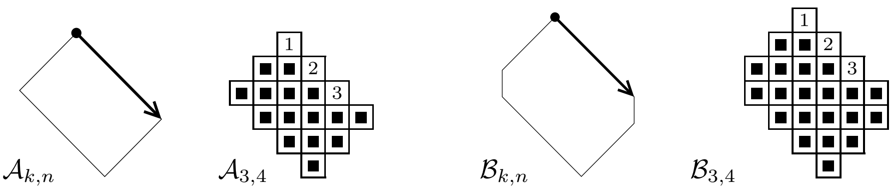
**Corollary 4.3.** *If $\sigma(e) < s$ then $\epsilon(s) > e$.*

*Proof.* For a contradiction suppose that $\epsilon(s) \leq e$. Then there is a $P$ such that $s(P) = s$ and $e(P) \leq e$. Hence $\sigma(e(P)) \geq s$. This is a contradiction since $\sigma$ is increasing and so $\sigma(e(P)) \leq \sigma(e) < s$. □

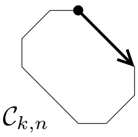
## 5. Animals with minimum site-perimeter

In this section we use our formula for $\sigma$ to find a formula for $\epsilon$. The main difficulty is that we do not know whether $\epsilon$ is increasing. We are going to show that the animals realizing the minimum site-perimeter can be gotten from saturated animals be peeling off some of the cells along an edge. The following lemma is an easy consequence of Lemma 3.11.

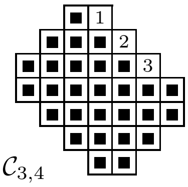
**Lemma 5.1.** *The site-perimeter of a saturated animal with parameter $(k, n)$ does not change if we delete $l \in \{1, \ldots, n-1\}$ cells starting at the corner indicated by a dot, going along the edge indicated by the arrow. For example in the animals below we can delete any of the set of cells $\{1\}$, $\{1, 2\}$ or $\{1, 2, 3\}$ without changing the site-perimeter.*



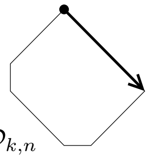

$\mathcal{A}_{k,n}$ $\mathcal{A}_{3,4}$ $\mathcal{B}_{k,n}$ $\mathcal{B}_{3,4}$



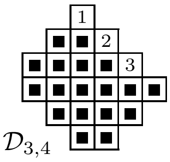
$\mathcal{C}_{k,n}$




$\mathcal{C}_{3,4}$



$\mathcal{D}_{k,n}$




$\mathcal{D}_{3,4}$

**Proposition 5.2.** *If $s \in \mathbf{N}$ then $\epsilon(s)$ is the smallest number $e \in \{4, 6, 7, 8, \ldots\}$ such that $\sigma(e) \geq s$.*

*Proof.* If $\sigma(e) < s$ then $\epsilon(s) > e$ by Corollary 4.3. Let $e$ be minimal with $\sigma(e) \geq s$. It suffices to show that there is an animal $P$ with $s(P) = s$ and $e(P) = e$. We need to consider four cases.

If $e \equiv 0$ (4) then $e - 1 \equiv 1$ (2). Hence we have

$$\sigma(e) = s(\mathcal{A}_{e/4,e/4}) = e^2/8 - e/2 + 1$$
$$\sigma(e-1) = (e-1)^2/8 - (e-1)/2 + 3/8$$

and so $\sigma(e) - \sigma(e-1) = e/4$. This means that $s > \sigma(e-1) = \sigma(e) - e/4$. Hence $P$ can be created by peeling off cells from $\mathcal{A}_{e/4,e/4}$ along an edge. Note that we must have $e \geq 8$ and so $e/4 \geq 2$ which means that peeling off cells does not create a disconnected set of cells.

If $e \equiv 2$ (4) then $e - 1 \equiv 1$ (2). Hence we have

$$\sigma(e) = s(\mathcal{A}_{e/4-1/2,e/4+1/2}) = e^2/8 - e/2 + 1/2$$
$$\sigma(e-1) = (e-1)^2/8 - (e-1)/2 + 3/8$$

and so $\sigma(e) - \sigma(e-1) = e/4 - 1/2$. This means that $s > \sigma(e-1) = \sigma(e) - (e/4 - 1/2)$. Hence $P$ can be created by peeling off cells from $\mathcal{A}_{e/4-1/2,e/4+1/2}$ along the longer edge. Note that we must have $e \geq 6$ and so $e/4 + 1/2 \geq 2$ which means that peeling off cells does not create a disconnected set of cells.

If $e \equiv -1$ (4) then $e - 1 \equiv 2$ (4). Hence we have

$$\sigma(e) = s(D_{e/4-3/4,e/4+1/4}) = e^2/8 - e/2 + 3/8$$
$$\sigma(e-1) = (e-1)^2/8 - (e-1)/2 + 1/2$$

and so $\sigma(e) - \sigma(e-1) = e/4 - 3/4$. This means that $s > \sigma(e-1) = \sigma(e) - (e/4 - 3/4)$. Hence $P$ can be created by peeling off cells from $D_{e/4-3/4,e/4+1/4}$ along the longer edge.

If $e \equiv 1$ (4) then $e - 1 \equiv 0$ (4). Hence we have

$$\sigma(e) = s(D_{e/4-1/4,e/4-1/4}) = e^2/8 - e/2 + 3/8$$
$$\sigma(e-1) = (e-1)^2/8 - (e-1)/2 + 1$$

and so $\sigma(e) - \sigma(e-1) = e/4 - 5/4$. This means that $s > \sigma(e-1) = \sigma(e) - (e/4 - 5/4)$. Hence $P$ can be created by peeling off cells from $D_{e/4-1/4,e/4-1/4}$ along an edge. $\square$

**Theorem 5.3.** $\epsilon(s) = \lceil 2 + \sqrt{8s-4} \rceil$.

*Proof.* The functions

$$\sigma_\sharp(e) = e^2/8 - e/2 + 1$$
$$\sigma_\flat(e) = e^2/8 - e/2 + 3/8$$

are strictly increasing and $\sigma_\flat(e) \leq \sigma(e) \leq \sigma_\sharp(e)$ for $e \geq 4$. Let $x = \lceil \sigma_\sharp^{-1}(s) \rceil = \lceil 2 + \sqrt{8s-4} \rceil$. Since both $\sigma(x)$ and $s$ are integers and

$$\sigma(x) \geq \sigma_\flat(x) \geq \sigma_\flat(\sigma_\sharp^{-1}(s)) \geq \sigma_\sharp(\sigma_\sharp^{-1}(s)) - 5/8 = s - 5/8$$

we must have $\sigma(x) \geq s$. We also have

$$\sigma(x-1) \leq \sigma_\sharp(x-1) < \sigma_\sharp(\sigma_\sharp^{-1}(s)) = s$$

so $\epsilon(s) = x$ by Proposition 5.2. $\square$

**Lemma 5.4.** *For $s \geq 2$ we have $\epsilon(s+1) - \epsilon(s) \leq 1$.*

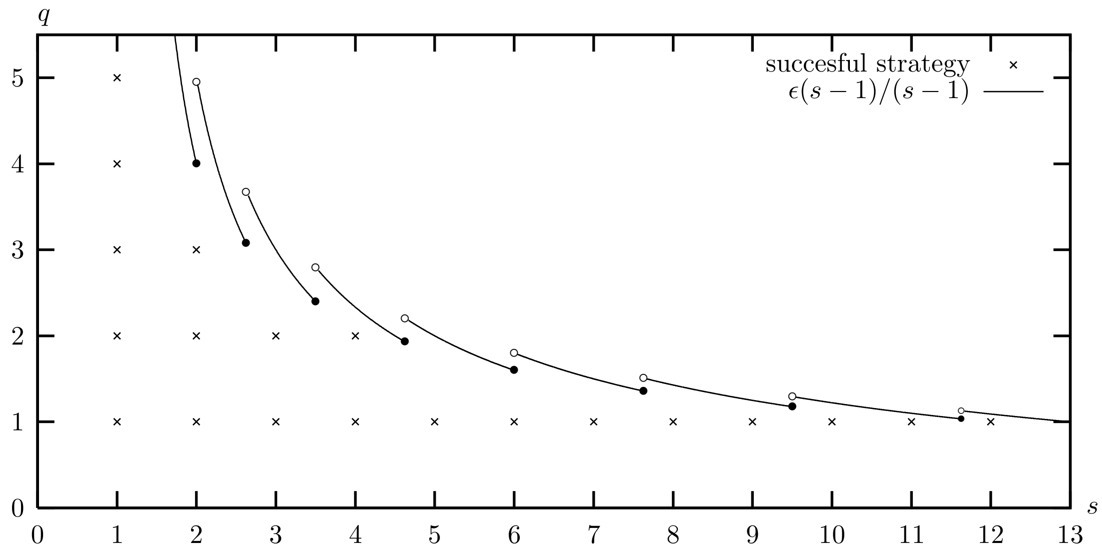
*Proof.* It is easy to see that $\left(2 + \sqrt{8(x+1) - 2}\right) - \left(2 + \sqrt{8x - 2}\right) \leq 1$ when $x \geq 65/32$ so for $s \geq 3 > 65/32$ we must have $\epsilon(s+1) - \epsilon(s) \leq 1$ since $a - b \leq 1$ implies $\lceil a \rceil - \lceil b \rceil \leq 1$. For $s = 2$ we have $\epsilon(2+1) - \epsilon(2) = 7 - 6 = 1$. □

Note that the result does not hold for $s = 1$ since $\epsilon(2) - \epsilon(1) = 6 - 4 = 2$.

**Proposition 5.5.** *The random neighbor strategy is successful in the full set $(1, q)$-achievement game for $s > 1$ turns if $(s-1)q < \epsilon(s-1)$.*

*Proof.* The result follows from Proposition 2.2 and Lemma 5.4. □

Note that $\epsilon(11) = 12$ and $\epsilon(12) = 12$. Thus the maker can achieve $\mathcal{F}_s$ up to $s \leq 12$ in the $(1,1)$-achievement game using the random neighbor strategy. It is clear that in the $(1,1)$-achievement game the maker can mark an arbitrarily large animal by marking below or to the right of his previous mark. This strategy is not the random neighbor strategy though. The following figure shows the list of $(s, q)$ pairs for which the $(1, q)$-achievement game is successful for $s$ terms together with the graph of the function defined by $\epsilon(s-1)/(s-1)$. Note that $(1, q)$ is in this list for any $q$.




Similar questions could be asked about triangular, hexagonal, higher dimensional rectangular or other playing boards. Something interesting happens if the cells of the board are the edges of the infinite chessboard; $\epsilon$ is no longer an increasing function.

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