<u>Arrangements of Hyperplanes</u>

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Bibliography

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[2] On the Algebra Associated with the Geometric Lattice
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[3] Arrangements of Hyperplanes
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[4] Introduction to Arrangements
Peter Orlik

For this project, We will study the formation of topological spaces by hyperplane cuts. We will study sets of, called arrangements of, hyperplanes that are central, that is they have at least one point in common. There is a combinatorial invariant of these topological spaces, known as the Orlik-Solomon algebra. Our main goal is to study this algebra and the invariants of this algebra, and look at some of the implications the Orlik-Solomon algebra has on these spaces. I will be quoting the above [3] often and mostly without reference to it.

It will be necessary for the reader to be familiar with abstract algebra, exterior algebras, and elementary topology and homotopy to get a good understanding of this topic by reading this paper.

This paper is divided up into three sections. The first is a few pages that addresses the preliminaries just mentioned. This can be easily skipped by those familiar with these topics. The second section is an introduction to arrangements; the third is an introduction to the Orlik-Solomon algebra.

1 - Preliminaries

Def'n A nonempty set **V** is said to be a *vector space* over a field **K** if

(i) there exists an operation called addition denoted by + that associates to each pair of vectors x,y in **V** a new vector x+y that is also in **V**.

(ii) there exists an operation called scalar multiplication that associates to an a in **K** and an x in **V** a new vector ax that is also in **V**.

(iii) these operations satisfy the following axioms for all x,y,z in **V** and a,b in **K**.

(1) $x+y=y+x$
(2) $(x+y)+z=x+(y+z)$
(3) There exists an element 0 in **V** such that $0+x=x$
(4) For all x in **V** there exists an element -x in **V** such that $x+(-x)=0$
(5) $a(x+y)= ax+ay$
(6) $(a+b)x = ax+bx$
(7) $a(bx)=(ab)x$
(8) There exists an element 1 in **K** such that $1x=x$

Example

$\mathbf{R}^n$ and $\mathbf{C}^n$ are vector spaces along with regular vector addition and scalar multiplication.

Def'n A *group* (**G**,+) is a nonempty set along with any operation denoted +, that satisfies the following properties

(1) If a,b are in **G** then a+b is in **G**.
(2) If a,b,c are in **G** then $a+(b+c) = (a+b)+c$
(3) There exists 0 in **G** such that If a is in **G** then $a+0=0+a=a$
(4) For each a in **G** there exists a (-a) such that $a+(-a)=0=(-a)+a$.

Furthermore, a group is called an *abelian group* if in addition to above **G** also satisfies
(5) $a+b = b+a$

Example

The rigid motions of the triangle form a group under composition usually denoted D_3, but this is not an abelian group.

The Integers mod 17 under normal addition form an abelian group.

Def'n A *ring* (T,+,*) (sometimes called an associative ring) obeys the following rules.

(1) (T, +) is an ableian group
(2) If a,b are in T then a*b is in T
(3) (a*b)*c = a*(b*c), for all a,b,c in T
(4) there exists a 1 in T such that for all a in T, a*1=a=1*a
(5) a*(b+c)= a*b +a*c and (a+b)*c = a*c + b*c ,for all a,b,c in T

A ring is commutative if also
(6) a*b = b*a for all a,b in T

A ring is a *field* if in addition to (1)-(6), (7) is also satisfied.
(7) for all a in T not equal to 0, there exists an a^{-1} such that $a*a^{-1}=1$

Example

The Integers mod 17, or any prime for that matter, under addition and multiplication of integers forms a field.

R is a field, but $\mathbf{R}^2$ is not.

n x n matricies for a non-commutative ring under matrix addition and matrix multiplication

Def'n A nonempty subset I of a ring T is said to be an *ideal* of T if I is a subgroup under addition , and if a*r and r*a is in I for all a in I and for all r in T.

Example

< p > is an ideal of the integers.

Explanation without building all the required machinery to define a quotient ring, I will try to give you an idea of what they are.

Example The quotient ring of $\mathbf{Z}[x]/<x^2-3>$ has elements of the form a+ bx, where a,b are in **Z**. The way this is seen is that you set $x^2-3=0$ and use this relation in $\mathbf{Z}[x]$, so that each x^2 term can be replaced with 3. Each x^3 can be replaced with a 3x, and so on.

Def'n A ring (associative ring) **A** is said to be an *algebra* over a field **K** if **A** is also a vector space over **K** such that a(ST)=(aS)T=S(aT) for all a in **K**, and for all S,T in **A**.

Another way to say this loosely is that an algebra is a vector space in which you can also multiply vectors, so that the product of two vectors is another vector.

Note that the dot product and cross product can not count as 'multiplication' for algebras.

Example

For any given n, n x n matricies with entries in **R**, and scalars in **R**, and the usual stuff gives an algebra.

An exterior algebra is an algebra like the definition above. It has an anti-commutative multiplication, and it is graded. This means there are levels on which each member of the algebra lies. If you allow me to treat these just as objects then I will give a semi-example.

Polynomials with x, y, z that have the rule xy=-yx. From this rule we get that xx=-xx. Thus anything multiplied by itself is zero. This is an important feature. This setup is graded as well x or 3x+4y are on level one, that is, they are have linear terms. xy is on the second level, and xyz is on the third level. There are no more levels.

Def'n A topology,T , on a set X is any set of subsets of X such that:

1. $X \in T$ and $0 \in T$.
2. T is closed under intersections.
3. T is closed under unions.

Def'n Two topological spaces are equivalent, or homeomorphic, if there exists a continuous bijection between the two spaces such that the inverse is also continuous.

Example

A sphere with two holes, a cylinder, an annulus, and a disc with one hole are all homeomorphic. A sphere and a torus are not homeomorphic.

Two spaces are homotopy equivalent if one can be continuously deformed into the other. We cannot give a quick definition of homotopy equivalent. It should be pointed out here that if two spaces are homeomorphic then they are homotopy equivalent. The converse, though, is not neccesarily true. In topology, spaces that are homotopy equivalent, but not homeomorphic, are somewhat interesting.

Example

The pinched torus, and the one point union of the sphere and the circle are homotopy equivalent, but not homeomorphic.

2- Arrangements
We start out with a few fun filled facts and examples about arrangements.

Def'n Let **K** be a field. Let **V** be a vector space over **K** of dimension L. A hyperplane **H** in **V** is a vector subspace of dimension L-1. An arrangement **A** is a finite set of hyperplanes in **V**.

Def'n Q=Q(**A**) is called the defining polynomial of **A**, where **H** is in **A**, if **H** is in the kernel of **Q**.

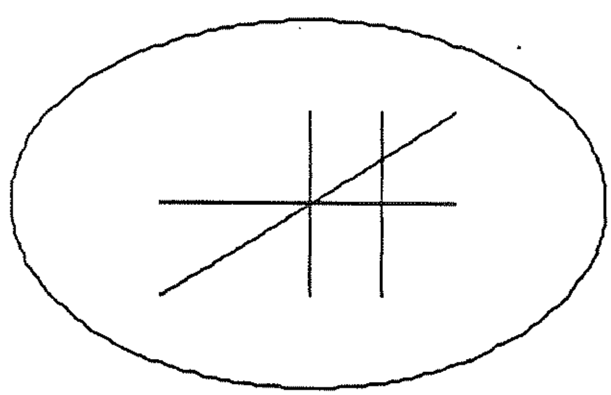
Example Let L=3. Let **Q**=xyz(x-y)(x-z). The arrangement corresponding to **Q** is $\{H_i\}$, where H_1 is the plane x=0, H_2 is the plane y=0, H_4 is the plane x=y, and so on.

Note we will only be dealing with central arrangements.

Def'n An arrangement is a central arrangement if $\bigcap H_i \neq 0$ otherwise it is an affine arrangement. Its no suprise the center of **A** is $\bigcap H_i$. The center of **A** always contains the origin.

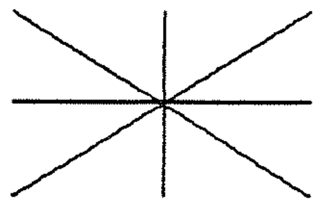
Let **A** $=\{H_1, \ldots, H_n\}$., L=3. To draw several planes in $\mathbf{R}^3$ would be messy so we use the projective plane. Let $\mathbf{P}^2$ be the plane z=1. $S_i = H_i \cap \mathbf{P}^2$. So instead of drawing the H_i in $\mathbf{R}^3$, we draw S_i in $\mathbf{P}^2$. Remember the line at infinity (i.e. the intersection of the planes z=1 and z=0 will be this annoying circle in the next example).

Example Let **Q**= xyz(x-y)(x-z). The 'drawing' for **A** defined by **Q** is:



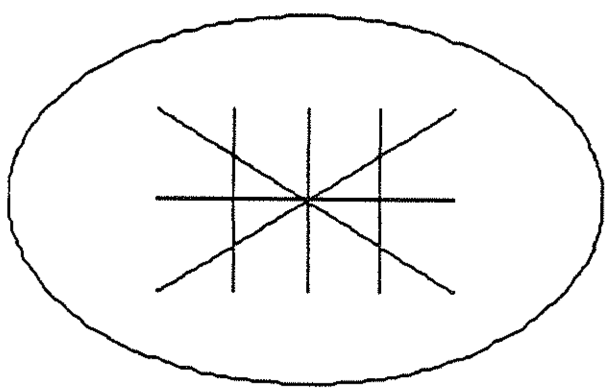
Def'n The matriod of **A** is defined to be the set of dependent hyperplane sub-arrangements. The circuits of **A** are defined to be the set of minimal dependent sets. Minimal means that if I take any hyperplane out of the set, it is no longer dependent. You may be confused with the notion of 'dependence' and 'independence' here, but don't be. These notions are just the plain old linear algebra ideas with vectors. An example that shows this comes up shortly.

Example Let L=2. Let **A** be defined by **Q**=xy(x-y)(x+y), and let the hyperplanes (lines) be labelled 1 thru 4 respectively. **A** looks like:



The matroid of **A** is {1234,123,124,234,134}, the circuits of **A** are {123,124,234,134}. These arrangements aren't very exiting when L=2.

Example Let L=3, **Q**=xyz(x+y)(x-y)(x+z)(x-z). The drawing of **A** given by **Q** is:



where

1={1,0,0}
2={0,1,0}
3={0,0,1}
4={1,1,0}
5={1,-1,0}
6={1,0,-1}
7={1,0,1}

The matroid of **A** is given by the dependent rows, the circuits are the set of 3 row vectors that are dependent, and the set of 4 row vectors such that no 3 of them are dependent.

You can see by comparing the picture and the circuits that three lines in $\mathbf{P}^2$ (Actually planes in $\mathbf{R}^3$) are dependent iff they are concurrent (they meet at a point).

__Def'n__ The variety of **A** is $\cup \mathbf{H}_i$. The complement of **A** is $V \backslash \cup \mathbf{H}_i$.

__Def'n__ The cardinality of an arrangement is the number of hyperplanes in the arrangement, denoted $|A|$.

__Def'n__ Let **A** be an arrangement and let $\mathbf{L}(\mathbf{A})=\mathbf{L}$ be the set of non-empty intersections of elements of **A**. Define a partial order on **L** by $X \leq Y \Leftrightarrow Y \subseteq X$

This poset gives rise to a geometric lattice with reverse inclusion. I will give a few examples in a bit. This is sometimes called the lattice of flats.

__Def'n__ Two arrangements **A** and **B** are lattice equivalent or **L**-equivalent if there is an order preserving bijection between the two lattices.

It is known that if two arrangements are not **L**-equivalent then the corresponding complement spaces of the complexified arrangements are not homeomorphic. These are exactly the spaces that we will be studying in detail.

__Example__

Consider the Arrangement A in $\mathbf{R}^3$ defined by $\mathbf{Q}(x)=xyz(x-y)(x+y)(x-z)$. The geometric lattice, or the Hasse diagram, of this arrangement is figure 1. Where 136 is the subspace of $\mathbf{R}^3$ obtained by intersecting the first, third and sixth hyperplanes, and so forth. 0 is the empty intersection restriction of $\mathbf{R}^3$ which is just $\mathbf{R}^3$.

Associated with this poset is the Mobius function $\mu : L \to R$ such that $\mu(0)=1$ and $\sum_{0 \subseteq Y \subseteq X} \mu(Y) = 0$. Once we start talking algebra, this will become more important. For now we define $a(k) = \sum_{rank(X)=k} \mu(X)$ and observe these for the previous example. (figure 2)

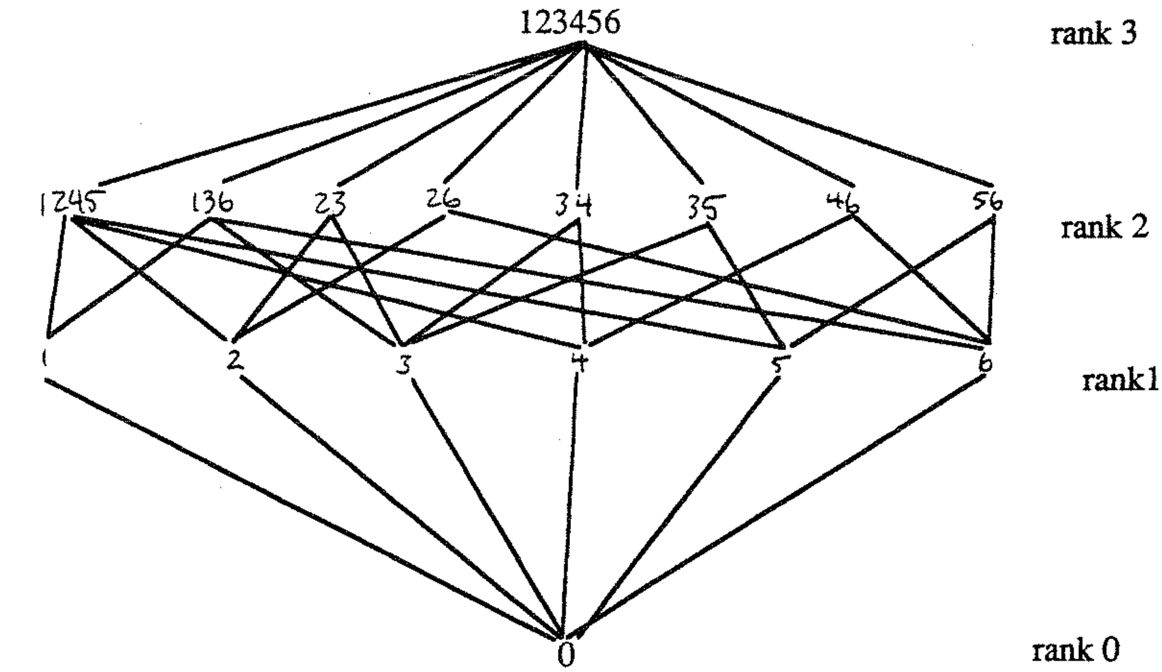


figure 1

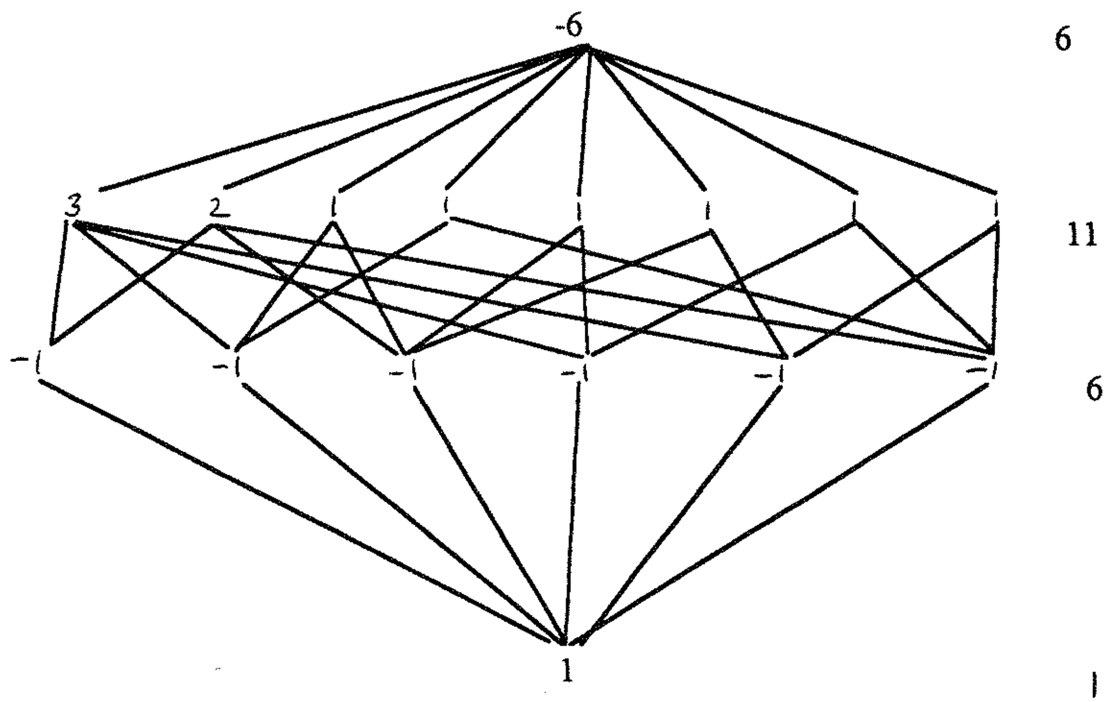


figure 2

Def'n Let **A** be an arrangement with intersection poset **L** and mobius function μ. Let t be indeterminate. The Poincare polynomial of **A** is given by $\pi(\mathbf{A},t)=\sum_{X\in L}\mu(X)(-t)^{rank(X)}$.

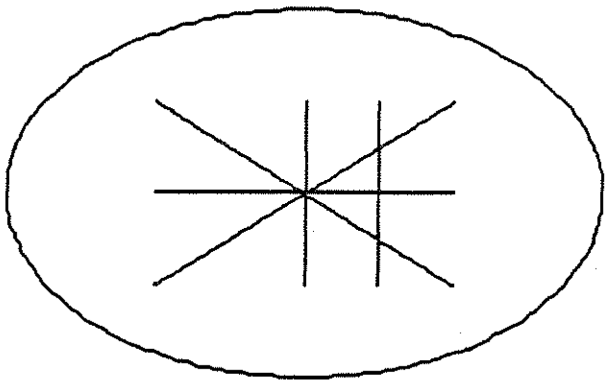
$\pi(\mathbf{A},t)=1+6t+11t^2+6t^3$ for the previous example.

We now construct the Orlik-Solomon algebra A=A(**A**). The spaces we will be studying are the complement spaces of complexified real central hyperplane arrangements. In this case the Orlik-Solomon algebra is isomorphic to the cohomology algebra. As a result, A is a homotopy type invariant of these spaces. The spaces of special interest are the spaces that have arrangements that are not **L**-equivalent, but have isomorphic algebras (i.e. A-equivalent). These spaces will have the features of not being homeomorphic, but possible homotopy equivalent.

3- Construction of the Orlik-Solomon Algebra

Def'n Let **A** be an arrangement over **K**. Let $E^1 = \oplus e_H, H \in A$, and let $E = E(A) = \wedge(E^1)$ be the exterior algebra of E^1. Note that E^1 has a **K** basis consisting of elements e_H in one to one correspondence with the hyperplanes of **A**. Write $e_1 \wedge e_2$ as e_{12}. The algebra is graded. If $|A| = n$, then $E = \oplus E^i$ where i goes from 0 to n.

Example Q= xyz(x+y)(x-y)(x-z)



$E^0 = K$ has dimension 1
$E^1 = \oplus K e_i$, i goes from 1 to 6, has dimension 6
$E^2 = \oplus K e_{ij}$, $1 \le i < j \le 6$, has dimension 15
$E^3 = \oplus K e_{ijk}$, $1 \le i < j < k \le 6$, has dimension 20
$E^4 = \oplus K e_{ijkl}$, $1 \le i < j < k < l \le 6$, has dimension 15
$E^5 = \oplus K e_{ijklm}$, $1 \le i < j < k < l < m \le 6$, has dimension 6
$E^6 = K e_{123456}$, has dimension 1

$E = \oplus E^i$, where i goes from 1 to 6

Things in **E** look like linear combinations of the 64 basis elements with coefficients coming out of **K**.

In general, if you have an arrangement with n hyperplanes then the dimension of E^p= n choose p.

Def'n Define a k-linear map $\partial: E \to E$ by $\partial(1)=0$, $\partial(e_i)=1$, and for $p>=2$, $\partial(e_{H1}, \ldots, e_{Hp}) = \Sigma (-1)^{k-1} e_{H1} \ldots e_{Hk-1} e_{Hk+1} \ldots e_{Hp}$, where k ranges from 1 to p, and for all $H_1, \ldots, H_p$ in **A**.

Def'n Given a p-tuple of hyperplanes S=(H_1, ... ,H_p), write $| S |$=p, $\bigcap S = H_1 \cap ... \cap H_p$, $e_S = e_{H_1} ... e_{H_p} \in H$. Since A is central $\bigcap S \in L, \forall S$.
If p=0 we agree that S=() the empty tuple, e_S=1, and $\bigcap S$=V. Since the rank function on **L** is just the codimension, then it is clear that rank($\bigcap S$)≤$| S |$.

Def'n Call S independent if rank($\bigcap S$)=$| S |$ and dependent if rank($\bigcap S$)≤$| S |$.

Notice that this agrees with our earlier use of independent and independent. In fact, S is dependent iff it is in the matroid. The matroid construction is what we do before we figure out A(**A**).

Def'n Let A be an arrangement. Let I=I(**A**) be the ideal of E generated by $\partial e_S$for all dependent S∈**S**.

I is a graded ideal. $I = \oplus_{p=0}^{n} I_p$.
which in turn makes A a graded algebra. $A = \oplus_{p=0}^{n} A_p$.

In order for the corresponding algebras of two different arrangements to be there are some size resrictions of the dimensions of the algebras that can be obtained from the lattice of flats. This gives us a few invariants of the algebra itself. The first is one we are already familiar with.

Th'm The Poincare polynomial is an invariant of the algebras.

i.e. $\pi(A_1,t)=\sum_{p\geq 0}(\dim A^p)t^p=\pi(A_2,t)$.

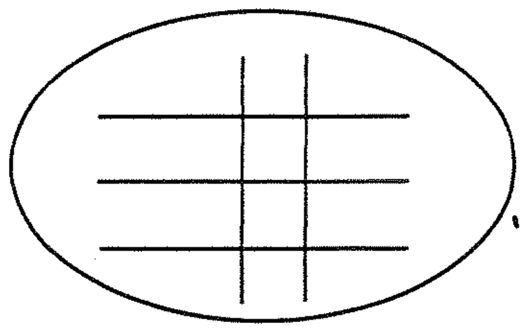
This result implies a more easily noticeable invariant. Since the dimension of A^1 is just the number of hyperplanes in the corresponding arrangment, the two algebras will be isomorphic only if the number of hyperplanes in their respective arrangements are equal.

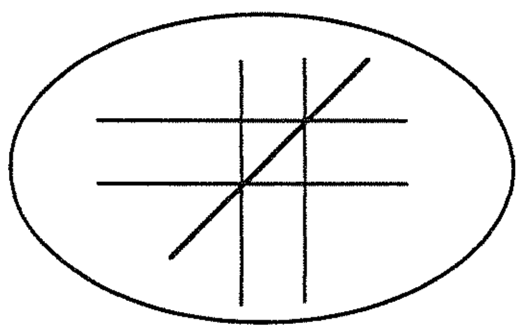
There are two other known invariants which were introduced in [2]. These are also dimension type restrictions. The first is the number

$$\Psi = \sum_{\substack{X \in L \\ rank(X)=2 \\ |X| \geq 3}} \binom{|X|}{3}.$$

For the second, we introduce the map $\Delta : E^1 \otimes I^2 \to E^3$ where $e_i \otimes \partial e_J \longmapsto e_i \partial e_J$ where J is dependent and the size of J is 3.
$\Phi = nullity(\Delta)$ is an invariant of A.

Example
Let **A** and **A'** be the corresponding arrangements in $\mathbf{R}^3$ given by these pictures.



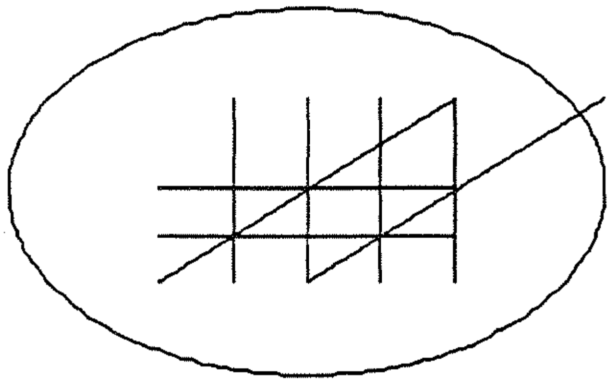


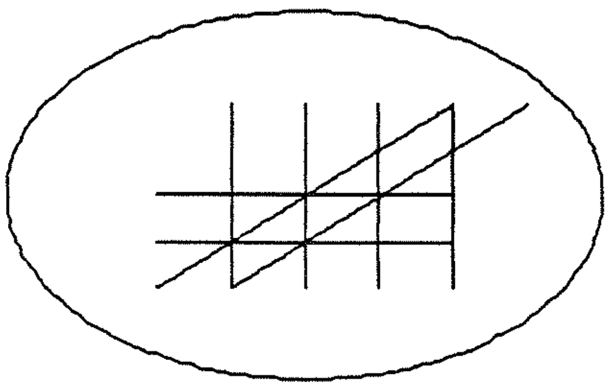
$\pi = (1+t)(1+2t)(1+3t) = \pi'$
$\Psi = 5$ and $\Psi' = 4$.
This tells us that the corresponding algebras are not isomorphic, and furthermore that the complement spaces are not homotopy equivalent.

It has not been proven yet that $\pi = \pi'$, $\Psi = \Psi'$, and $\Phi = \Phi'$ doesn't imply that $A \cong A'$ in $\mathbf{R}^3$.

Example

Let A and A' be the corresponding arrangements in $\mathbf{R}^3$ given by these pictures.





The respective invariants coincide, so it is possible that the algebras are isomorphic. More interestingly, it has been shown that the corresponding complement spaces in $\mathbf{C}^3$ are not homotopy equivalent the old fasion way. This is an interesting example because of its situation. If we were to find an isomorphism between the corresponding algebras, then this would be the first example of this type where the algebras are isomorphic and the 'spaces' are not homotopy equivalent. If we were to find that the algebras were not isomorphic then it would show that the three invariants do not imply that the algebras being isomorphic. This is the task at hand now.