RESONANCE ZONES AND ARNOL'D TONGUES IN SEVERAL CIRCLE MAPS

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In this paper we consider some properties of circle maps, i.e. maps $F : S^1 \to S^1$ having the form

$$x \longrightarrow \langle x + f(x) \rangle \qquad (1)$$

where we use angle brackets to denote the fractional part of the result. In this fashion the function sends the unit-circumference circle onto itself. For this paper we will require $f : \mathbf{R} \to \mathbf{R}$ to be smooth and 1-periodic so that there is no jump discontinuity on the circle. In addition we require that F be a diffeomorphism (a differentiable function with differentiable inverse) and that F be order preserving, i.e. for any $x,y,z \in S^1$ such that $x<y<z$ in the cyclic order, $F(x)<F(y)<F(z)$.

Before proving some general results about arbitrary circle maps, we will consider a special case which is simple but of great interest, the two parameter family of maps having the form

$$x \longrightarrow \langle x + w \rangle \qquad (2)$$

where w is a real parameter and the same conditions are imposed on f as above. The parameter w is called the <u>rotation parameter</u> because the map is simply rotation about the circle by the angle $2\pi w$. Clearly if w is rational and $w=\frac{p}{q}$ then every point of S^1 has period q under (2) and completes exactly p revolutions during its minimal period. The map (2) is much more interesting, however, when w is irrational, for then the orbit of every point in S^1 under (2) is dense on S^1, which we state in the following

Fact: If $F : S^1 \to S^1$ with $F(x) = \langle x + w \rangle$ for some irrational w then the orbit of every $x_0 \in S^1$ is dense on S^1.

Proof: For some $x_0 \in S^1$, consider the set $O = \{F^n(x_0)\}$, the full orbit of x_0 under F. We show that **O** is dense in S^1.

It is clear that the points of **O** are distinct, for otherwise there would be a periodic orbit for some (and thus every) x_0 under (2). It is also clear that **O** must have an accumulation point since S^1 is closed and bounded. Thus for any $\varepsilon > 0$, there must be natural numbers k and m such that $|F^{k+m}(x_0) - F^m(x_0)| < \varepsilon$. But F is a rigid rotation, so the length of an interval of S^1 does not change under it and it must be the case that $|F^k(x_0) - x_0| < \varepsilon$. In fact it must be true for *all* m that $|F^{k+m}(x_0) - F^m(x_0)| < \varepsilon$ so $x_0, F^k(x_0), F^{2k}(x_0), \ldots$ is a sequence of points ε apart. Since ε was arbitrary, this sequence is dense in S^1.

Thus far we have discussed only where points are mapped to by F, not by what route they arrive there. But the route, not merely the destination, is the most descriptive property of the dynamics, so we need a tool to study it. To this end we define the lift of a particular family of maps, of which circle maps of the form (1) are a subset.

Def'n : Let F be a continuous function such that $F : \mathbf{T}^n \to \mathbf{T}^n$, where $\mathbf{T}^n$ is the standard n-torus $S^1 \times S^1 \times \ldots \times S^1$. Then there exists some continuous function $G : \mathbf{R}^n \to \mathbf{R}^n$ such that $F \bullet \Pi = \Pi \bullet G$, where $\bullet$ denotes functional composition and $\Pi : \mathbf{R}^n \to \mathbf{T}^n$ denotes the standard projection map $\Pi(x) = x \bmod \mathbf{Z}^n$. G is called the lift of F.

Perhaps this is easier to understand intuitively, however. In order to study the orbit of a point under a circle map more closely, we need to 'unfold' the orbit so that we can see it as an increasing sequence of points, not merely a collection of points around the circle. The lift defines an increasing function of this type that, when projected down to S^1, becomes the circle map in question. We are justified in discussing *the* lift of a function because lifts are clearly unique up to an additive constant, i.e. for A,B lifts of some circle map F, A=B+c for some $c \in \mathbb{Z}$.

Now that we have the lift firmly in hand, an additional definition will allow us to define perhaps the most important property of circle maps. Recall the following

Def'n : For any function f sending its domain into its domain, the <u>n^{th} iterate</u> f^n of f is $f(f(...(x)...))$. (n times)

Now we can make the following important definition, one which will concern us for much of the rest of this paper.

Def'n : Let F be the lift of a circle map of the form (1). We define the <u>Rotation Number</u> $\rho(F)$ to be

$$\rho(F) = \lim_{n \to \infty} \frac{1}{n} F^n(x_0)$$

for some $x_0 \in S^1$. We immediately answer several of the most obvious questions about $\rho(F)$ via the following

Theorem : The quantity $\rho(F)$ always exists and moreover is independent of the starting point x_0 in the definition.

Proof [1]: Consider the total distance traveled by some point x_0 under k applications of F and denote it by d_k. We can write it as

$$d_k = F^k(x_0) - x_0 = \sum_{i=0}^{k-1} f(F^i(x_0)).$$

Now we note that for $x_1 \leq x_2 \leq x_1 + 1$ under d_k we must have $d_k(x_1) \leq d_k(x_2) \leq d_k(x_1) + 1$. By this observation and the 1-periodicity of f we derive the inequality

$$\left| d_k(x_1) - d_k(x_2) \right| \leq 1 \qquad (3)$$

for all $x_1, x_2 \in S^1$.

Now let m_k be the integer for which

$$m_k \leq d_k(0) < m_k + 1.$$

Then it follows immediately from (3) and this inequality that

$$\left| \frac{d_k(x)}{k} - \frac{m_k}{k} \right| < \frac{2}{k}.$$

for all x. We also see that $\frac{d_{kl}}{kl}$ is the arithmetic mean of the numbers $\frac{d_k(x_{ki})}{k}$ where $x_{ki} = F^{ki}(x)$ for $i \in \{0, \ldots, l-1\}$, so that

$$\left| \frac{a_{kl}(x)}{kl} - \frac{m_k}{k} \right| < \frac{2}{k}.$$

Now if we define the set of intervals

$$\sigma_k = \left[\frac{m_k - 2}{k}, \frac{m_k + 2}{k} \right]$$

we see immediately that $\frac{a_{kl}(x)}{kl} \in \sigma_k$ for all l. Additionally, $\frac{a_{kl}(x)}{kl}$ is in both σ_k and σ_l so the intervals σ_k intersect each other and have lengths converging to zero as k approaches infinity and thus must have a common point which is precisely the rotation number. Now we have proved that the limit defined as the rotation number exists and is independent of the choice of point. ∎

With this valuable theorem complete we can prove an interesting

Corollary : Let F be a lift of a circle map as above. Then $\rho(F)$ is rational just in case some iterate of the circle map has a fixed point.

Proof : Assume the q^{th} iterate of the circle map has a fixed point on the circle. Then under q iterations of F this point is translated by an integral distance p. Since the rotation number may be calculated by starting at any point, we start at the fixed point and $\rho(F) = \frac{p}{q}$.

Conversely, let $\rho(F) = \frac{p}{q} \in \mathbf{Q}$. If it is true for all x that $d_q(x) > p$, we must have $d_q(x) > p + \varepsilon$ for some $\varepsilon > 0$, thus $\rho(F) > \frac{p}{q}$, a contradiction. We finish by achieving a similar contradiction in the case $d_q(x) < p$. ■

All of these facts may be proved without defining the lift [2] but the procedure necessarily involves some sort of record keeping. In effect one must count the number of revolutions while they occur and keep track of them, so one may as well define the lift, in which the number of revolutions is recorded as part of the function.

Fixed and periodic points are always of interest when we study the dynamics of a system. Above we have a fact about such points and when they occur in the systems we are studying. Since each rational number is a separate case of this fact we will define a set of interesting maps which have a correspondence to an arbitrary rational.

Def'n : The $\underline{\rho_0}$ Set is the set

$$A_{\rho_0} = \{F(x) : \rho(F) = \rho_0\}.$$

When $\rho_0 = \frac{p}{q} \in \mathbf{Q}$ we call A_{ρ_0} the $\frac{p}{q}$ Resonance Horn (for reasons which will become clear shortly).

Now we may move into an example which we will discuss at length, the family of nonlinear extensions of (2) having the form

$$x \longrightarrow x + w + a\, f(x) \qquad (4)$$

where we assume that the function is the lift of a corresponding circle map and that, in addition to the usual conditions f have zero average, i.e.and finally that f have zero average, i.e.

$$\int_0^1 f(x)dx \ = 0.$$

Now w still gives the rotation parameter and we refer to the parameter a as the nonlinearity parameter. Extensive work has been done on this family and we present briefly some of these results.

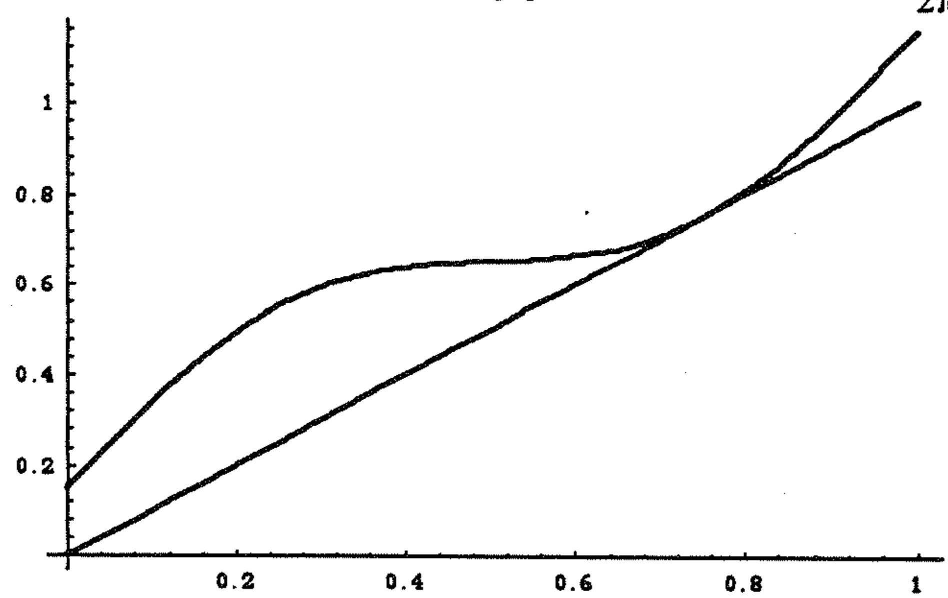
Arnol'd [3] has shown that for $\rho_0 \in \mathbf{Q}$ the ρ_0 set is horn shaped in the (a,w) parameter space (hence our reference to it as such earlier). We will make this statement more precise in a moment. Hall [4] showed that for irrational ρ_0 the ρ_0 set is a line having first derivative zero at a=0. It is clear from considering the dynamics of the system that the ρ_0 sets must never intersect each other; the intersection of sets would imply that the limit in the definition of the rotation number converged to two discrete values.

Hall uses the requirement that f have zero average to show that for every rational $\frac{p}{q}$ there exist two curves in parameter space which begin at $\frac{p}{q}$ and define the outline of the $\frac{p}{q}$ resonance horn. For sufficiently small a these curves are confined to opposite sides of the ray $w = \frac{p}{q}$, i.e. if we let $w_1(a)$ and $w_2(a)$ be the two curves then $w_1(a) \le \frac{p}{q} \le w_2(a)$. In fact the separation of these two curves is related to the magnitude of the Fourier coefficients of f, and the separation is increasing with respect to a. Arnol'd also derives perturbation approximations for several low-order resonance horns, i.e. those with small q.

We make a simple observation about the $\frac{p}{q} = 0$ or fixed point resonance horn in the following

Fact : For maps of the form (3) the $\frac{p}{q}$ resonance horn always has linear bounds.

Proof : The bounds of the horn correspond to solutions of the system of equations $\begin{bmatrix} F(x) = x \\ F'(x) = 1 \end{bmatrix}$ or equivalently $\begin{bmatrix} af(x) = -w \\ f'(x) = 0 \end{bmatrix}$. These are the conditions for the map to be exactly tangent to the line $x=F(x)$ as shown in the following plot of $x \longrightarrow x + .15 + \frac{.15}{2\pi}\sin(2\pi x)$.



Interesctions(as opposed to strict tangency) are the condition for points on the inside of the horn. It is clear from the 1-periodicity, continuity and zero average of f that $F'(x) = 1$ must have at least 2 solutions and from this set of solutions there must be a pair for which f(x) takes on its maximal and minimal values, which must be greater and less than zero respectively. Then when we substitute these values f_{max} and f_{min} into the first equation we are left with two linear functions which bound the resonance horn, namely

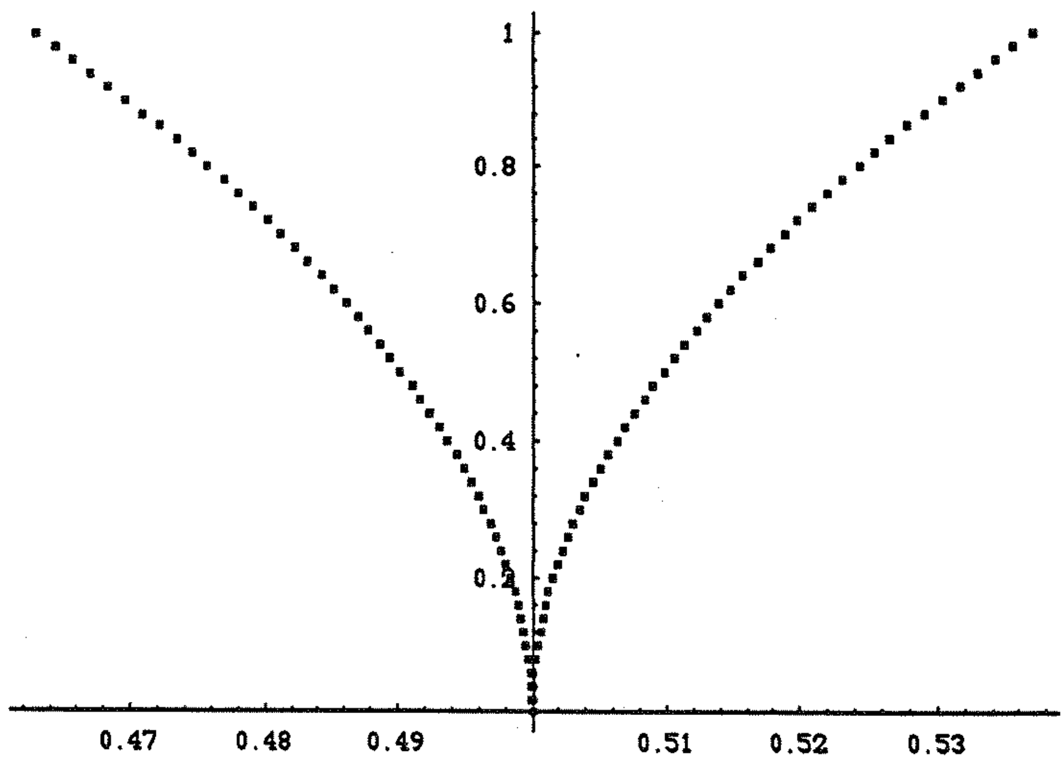
$$a = \frac{-w}{f_{max}} \qquad \text{and} \qquad a = \frac{w}{|f_{min}|}\,. \quad \blacksquare$$

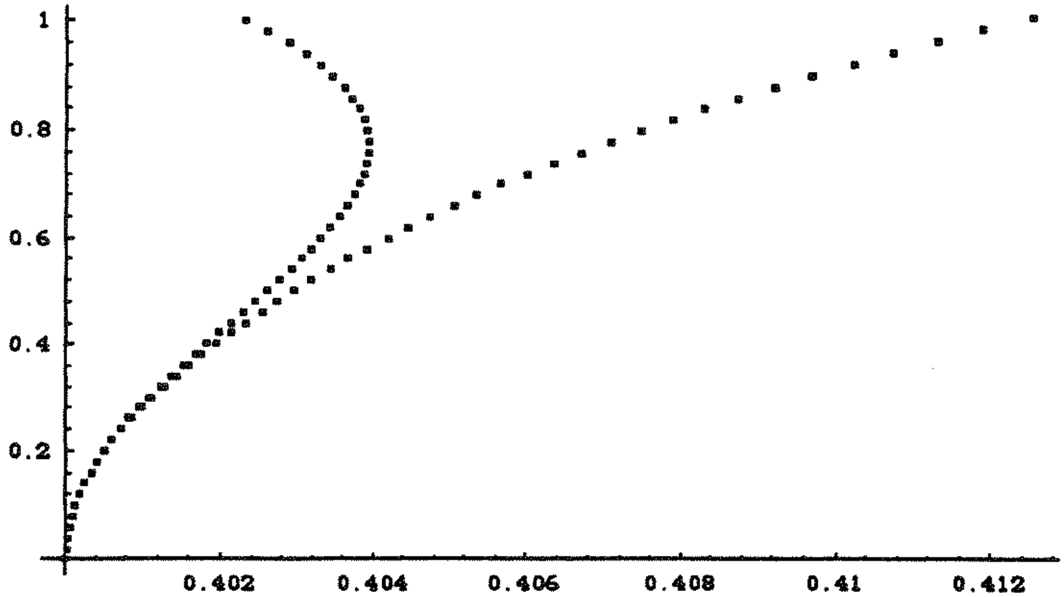
Shortly we will present some plots of resonance horns in parameter space but first we solidify the connection between these maps, flows, and dynamical systems. Since we have been careful to preserve the invertibility of the map by insisting that it be an orientation-preserving diffeomorphism it is clear that any circle map is a return map or Poincare section for some flow on $\mathbf{T}^2$. Any point inside the $\frac{p}{q}$ resonance horn corresponds to the existence of a stable and unstable $\frac{p}{q}$ - cycle in the flow on the torus. As we move toward the bounds of the horn these two cycles become closer together until at the bounding curves the two cycles join to form a single saddle-node cycle. The correspondence to dynamical systems is also easy because any flow on T^n corresponds to some system of n weakly coupled oscillators. In our case, the system of two weakly coupled oscillators gives rise to an invariant attracting 2-torus, on which we see that the return map is a circle map. When the coupling is stronger the invariant torus may break down but the dynamics can sometimes be modeled by a nonivertible circle map.

Now we can discuss a more specific example from the family (4) of maps. Consider the function

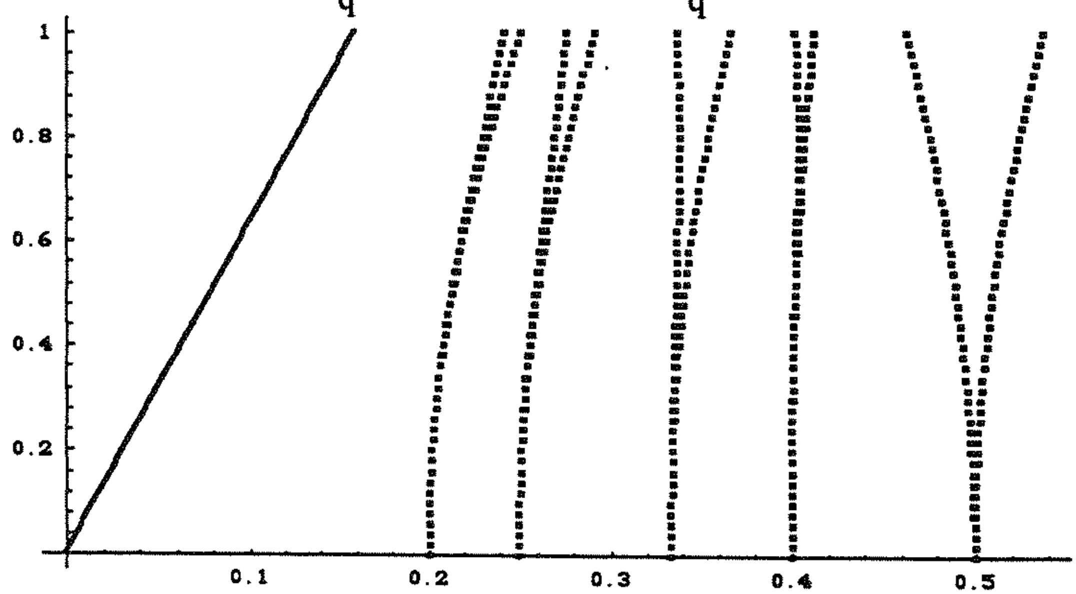
$$x \longrightarrow x + w + \frac{a}{2\pi}\sin(2\pi x)$$

which is of the correct form and has the required properties in f. Much work has been done on this function, enough that it is often referred to as the 'canonical circle map' rather than the more descriptive title 'sine circle map.' It is necessary that $a \in [0,1)$ for the preservation of the diffeomorphism condition; if $a > 1$ then the invertibility of the map fails. We also impose the constraint $w \in [0,1)$, but this is more or less arbitrary, as is the restriction that a not be negative. Later we will see why these constraints do not alter the dynamics of the system or change the parameter-space portrait.





Here are plots of, respectively, the resonance horns $\frac{p}{q} = \frac{1}{2}$ and $\frac{2}{5}$ for the canonical circle map in which the axes of the plot correspond to w and a in the circle map. These plots are greatly exaggerated along the w axis so that the shape of the defining curves is clearer. Now we show a plot of all the canonical map's resonance horns up to q = 5. The individual horns can be identified by their starting point because at a=0 the map becomes a rigid rotation and the rotation number is trivially w; thus the $\frac{p}{q}$ horn begins at w = $\frac{p}{q}$.



It is necessary to note at this point that Hall has showed that all resonance horns open at finite positive angle, which is not obvious from the plots above. In fact the result is more interesting when we consider that there is a horn for every value of $\frac{p}{q}$ and that the horns must become narrower as q increases We say that the above plot shows *all* the resonance horns up to $q = 5$ because the parameter-space portrait is symmetric about $w = \frac{1}{2}$, as we now demonstrate.

Consider the transformation $\left\{\begin{matrix} x & \longrightarrow & -x \\ w & \longrightarrow & -w \end{matrix}\right\}$. This sends F(x) to its negative value, but it also sends all three parts of the definition of F(x) to their negative values, so the map is preserved. Because we are working with a circle, -w is the same as 1-w, which is equivalent to a reflection about $w = \frac{1}{2}$. The transformation of x has no effect on the parameter portrait (the picture of the resonance horns in parameter space). Now we generalize to describe the effects of symmetry in f(x) on the nature of the parameter portrait for general maps of form (4).

Recall that there are three distinct types of symmetry for a one-periodic function : even, for which $f(-x) = f(x)$, odd, for which $f(-x) = -f(x)$ and a third type for which $f(x+\frac{1}{2}) = -f(x)$. The third type occurs when f is even about one point and odd about another but this property does not define the symmetry. We must point out that these symmetries are defined up to translation along x, i.e. if a function is even about $x = \frac{1}{10}$ we still say that it is even in the sense necessary for this analysis.

We have already seen the symmetry associated with odd functions; it was the reason why the parameter portrait of the canonical map was symmetric about $w = \frac{1}{2}$. When f is even we see that the map is preserved by the transformation $\begin{Bmatrix} x \longrightarrow -x \\ w \longrightarrow -w \\ a \longrightarrow -a \end{Bmatrix}$.

This has the effect of reflecting the top half-plane of the parameter space onto the bottom half-plane while also reflecting about $w = \frac{1}{2}$.

Finally when f has the third type of symmetry we use the transformation $\begin{Bmatrix} x \longrightarrow x+\frac{1}{2} \\ a \longrightarrow -a \end{Bmatrix}$ to see that the picture must be reflected across the w axis. Here then is the reason why we did not plot $a < 0$ for the canonical map, because sine contains all three of the above symmetries at various points in its period and so the parameter portrait is reflected in all of the above ways.

To motivate our next example of a circle map, we need to discuss the concept of genericity for periodic functions. In our family (4) we specify the map by changing f(x), which we always require to be 1-periodic. Now every one-periodic function can be written as a Fourier series

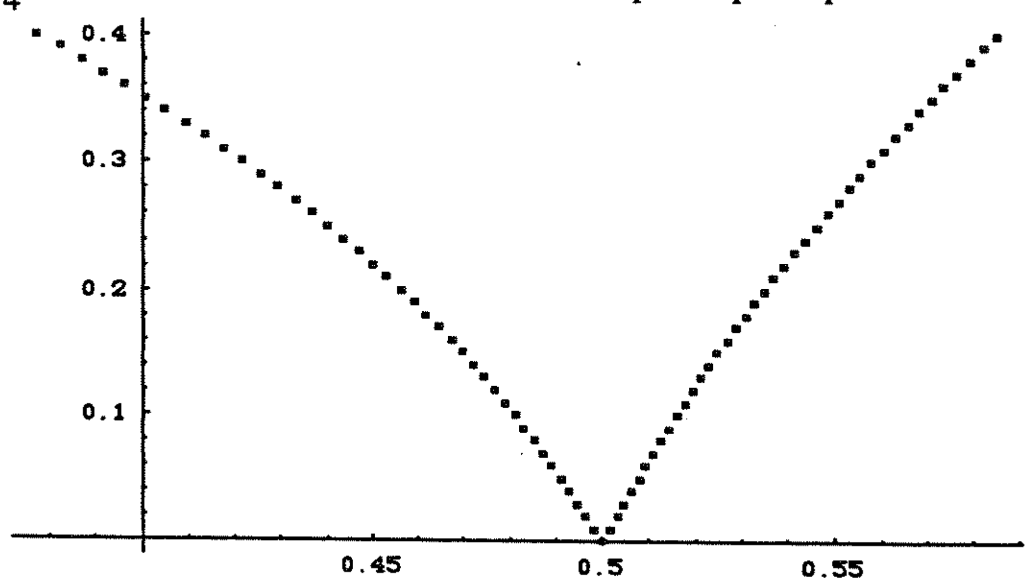
$$f(x) = \sum_{n=0}^{\infty} a_n \cos(2\pi x) + b_n \sin(2\pi x)$$

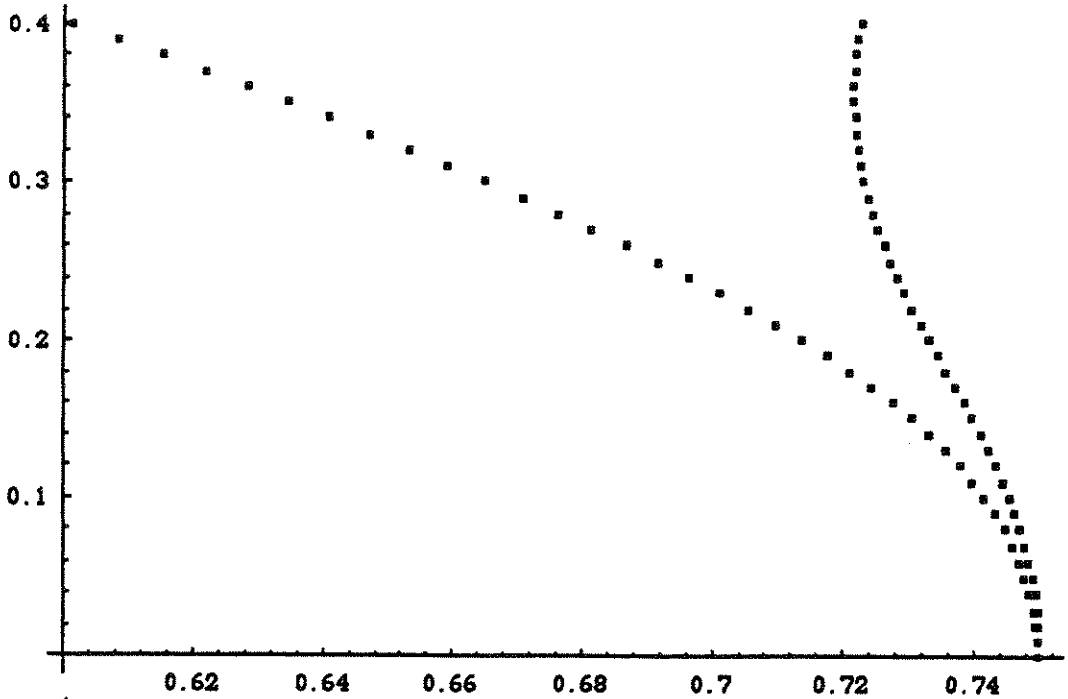
with an infinite number of coefficients, some number of which may be zero. In the sine map, $f(x) = \sin(2\pi x)$ has only one nonzero coefficient in its Fourier expansion, and thus it belongs to a much smaller class of maps than the class having infinitely many nonzero Fourier coefficients. Since a periodic function *may* have infinitely many nonzero coefficients, one with only finitely many such is clearly a special class of function. We therefore say that a periodic function with infinitely many nonzero Fourier coefficients is more generic than one with only finitely many. We wish to study generic functions because they provide insight into a much larger class of functions.

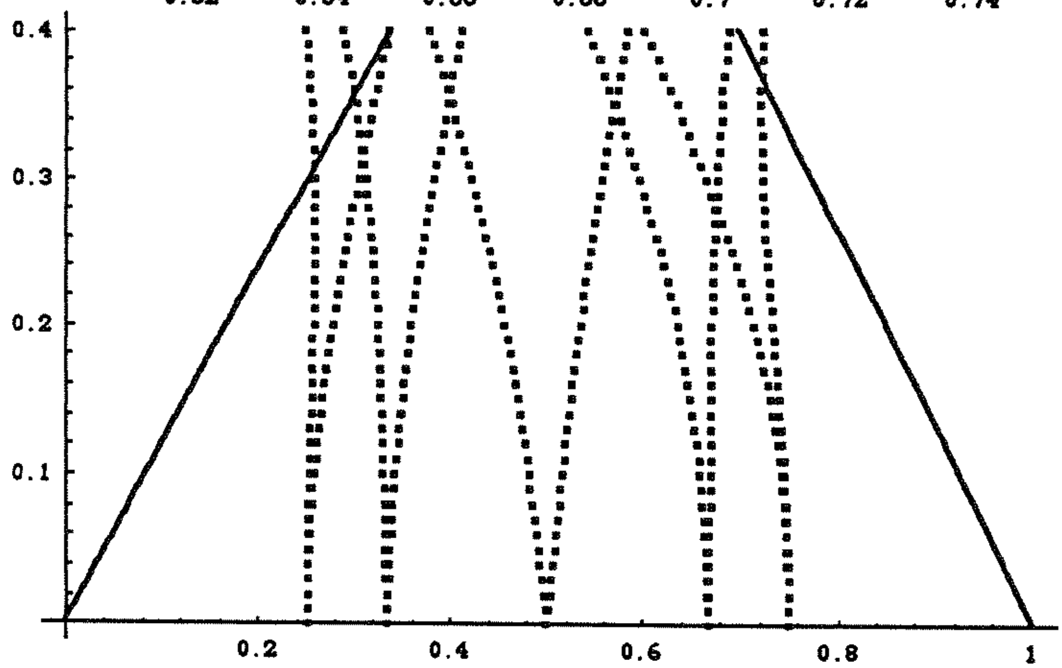
To this end, we study a member of the class (4) having the above property. Let $f(x) = \dfrac{1}{1-\frac{1}{2}\sin(2\pi x)} - \dfrac{2\sqrt{3}}{3}$ or, in expanded form

$$f(x) = (1 - \frac{2\sqrt{2}}{3}) + \frac{1}{2}\sin(2\pi x) + \frac{1}{2^2}\sin^2(2\pi x) + \ldots$$

The constant is subtracted to ensure that f has zero average as required. It can be shown by expansion of $\sin^n(2\pi x)$ for general n that this function does indeed have infinitely many nonzero Fourier coefficients. One immediate and striking feature of this map is that its fixed point resonance horn is not symmetric about w = 0, as we show in the following pictures. These are the resonance horns $\frac{p}{q} = \frac{1}{2}$, $\frac{3}{4}$, and then all the resonance horns up to up to q=4.



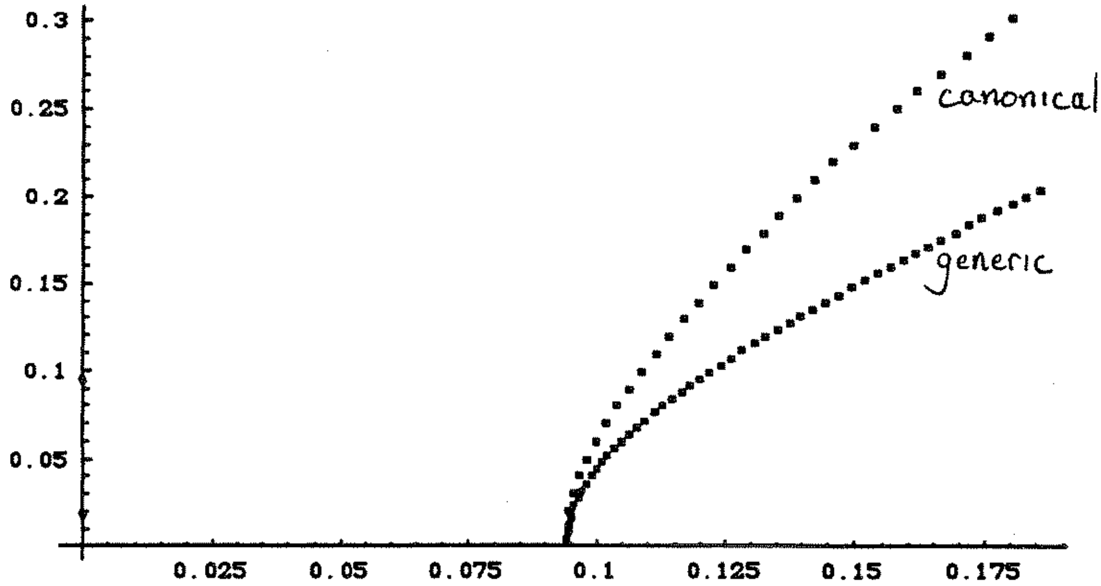




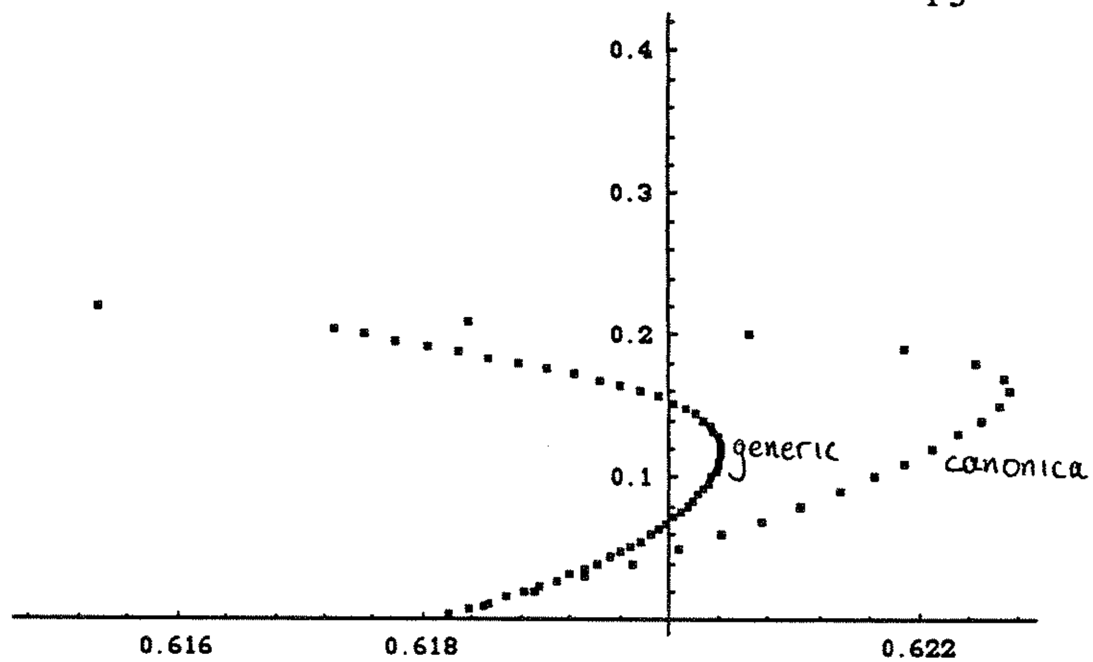
The complexity of the more generic function prohibits plotting any resonance horns for q > 4 on our machine. Note the asymmetry present in these plots. It is easy to see that the two defining curves of the $\frac{p}{q}=\frac{1}{2}$ horn are not the same up to reflection as they were in the canonical map's parameter portrait, and in the above picture we see immediately that the resonance horns $\frac{p}{q}=\frac{1}{4}$ and $\frac{p}{q}=\frac{3}{4}$ are not the same shape, which we proved them to be for the canonical map. Also we can explain the obvious intersection of lines in the last plot by noting that this map is only a diffeomorphism for values of a up to approximately .19. For larger values of a there may be several rotation numbers.

One other feature which we can calculate for both maps and compare is the width of the resonance horns as a function of a. The horns for the canonical map should open roughly as some multiple of a^q for the $\frac{p}{q}$th horn and as a linear function of a for the more generic map. Having obtained data for resonance horns up to q = 5 and q = 4 for these maps and computing a best-fit polynomial to the width data, we find that this is in fact the case.

Since we would like to compare as much as possible between these two maps we also plot some irrational rotation number lines. (Recall Hall's result that the ρ_0 set any irrational number is a line with first derivative zero at a=0) For these plots we choose an irrational from the unit interval and then compute and plot its ρ_0 set for both the canonical and generic maps. One can see a structural variation between the two lines quite clearly; this variation reflects largely the shape of the fixed point horn for the particular map. It is natural that this largest horn should have the most effect on the others.



The irrational rotation number lines for $\rho_0 = \frac{\sqrt{2}}{15}$



These plots correspond to rotation numbers $\frac{\sqrt{2}}{15}$ and the golden mean, or $\frac{\sqrt{5}\ -\ 1}{2}$. In both cases the more generic function is the more densely plotted line. The data for the canonical map has been condensed along the a axis so that the invertibility points (the value of a for which each map loses invertibility) are the same to within .01. This allows realistic comparison between the line shapes. In both cases we see that the line for the generic function is farther from the nearest large (small q) horn; this is the expected result since the horns are wider for the generic map. It is also interesting to recall Hall's result that these lines have first derivative zero at a=0 while examining the second plot.

We have now discussed several differences between these two maps and plotted data which agreed with analytic work on the subject, but the differences are slight. This is becuase the fundamental structure is the same in both cases : rigid rotation plus some function having the same characteristics for both maps. There is, in fact a theorem relating these maps in certain cases.

Denjoy showed in 1932 that any map of this form with irrational rotation number is topologically conjugate to rigid rotation by the rotation number. (By this we mean that there exists a C^0 map sending one function to the other. In fact Denjoy's theorem is stronger; it does not mention a specific class of circle map.) Herman strengthened this result by showing that all C^3 circle diffeomorphisms except for a set of irrationals of measure zero are *smoothly* equivalent to the corresponding rigid rotation. These irrationals are the special class which can be approximated most quickly by rational numbers. (For additional details see [1])

REFERENCES

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[4] G. R. Hall, "Resonance Zones in Two Parameter Families of Circle Homeomorphisms," SIAM J. Math Anal., 15, 6, pp. 1075-1081.