Edge-Graceful Graphs

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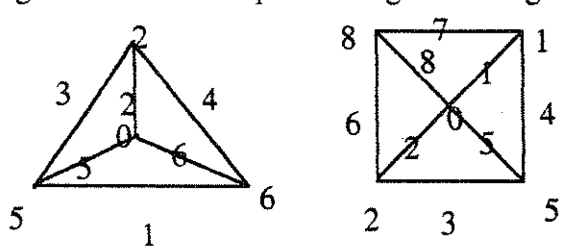
Graceful graphs

Let G be an undirected graph with no loops or multiple edges, p vertices, and q edges. The vertices of G are given by $\{v_1, v_2, ..., v_p\}$ and are labelled with distinct elements of $\{0, 1, 2, ..., q\}$. Suppose an edge has end vertices v_i & v_j, where v_i has label a_i & v_j has label a_j. Then, the label of the edge is $|a_i - a_j|$. If the vertices of G may be labeled so that these induced edge labels are distinct & form the set $\{1, 2, ..., q\}$, then G is said to be *graceful*. (Foreman, p. 1)

Figure 1. Examples of graceful graphs



Some of the results on graceful graphs are that trees with fewer than 5 end vertices, trees with fewer than 16 edges, and complete bipartite graphs are graceful. (Foreman, p. 7)

Edge-graceful graphs originated from graceful graphs. In fact, edge-gracefulness is a reversal of gracefulness. It involves labeling edges, rather than vertices, with an induced numbering consisting of sums of edge-labels, rather than differences of vertex labels.

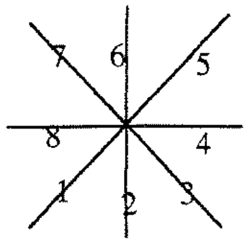
Edge-graceful graphs

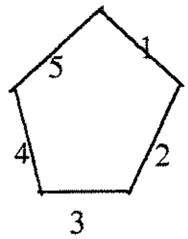
Edge-graceful graphs were defined by Sheng-Ping Lo in 1985. A graph G (p,q) is *edge-graceful* if there exists a bijective numbering of the edge set $\{e_i\}$ of G by $\{1, 2, ..., q\}$ so that the induced vertex labeling given by

$$\text{label vertex } v = \sum_{(e_i \text{ incident w/ } v)} \text{label } (e_i) \pmod{p}$$

is a bijection onto the set $\{0, 1, 2, ..., p-1\}$. (Foreman, p. 29)

Figure 2. Examples of edge-graceful graphs





(Foreman, p. 30)

Research has shown that star graphs, odd unicyclic graphs, regular complete k-partite graphs ($K_{n_1, n_2, \ldots, n_k}$), and complete graphs are some of the classes of graphs that are edge-graceful. A *star graph* is a graph with a vertex in the center that is adjacent to all other vertices. A graph G is *k-partite*, $k>1$, if it is possible to partition the set of vertices V(G) into k subsets $V_1, V_2, \ldots, V_n$ (partite sets) such that every element of of the set of edges E(G) joins a vertex of V_i to a vertex of V_j where $i \neq j$. A graph is *regular* when $n_1 = n_2 = \ldots = n_k$. A *complete graph*, K_p, has p vertices and an edge between any pair of vertices.

Thm. (Lo, 1985) If a graph G of p nodes and q edges is edge-graceful, then $p \mid (q^2 + q - p(p-1)/2)$.

Proof: Since G is edge-graceful, we have an edge-graceful labeling. Thus

$2(1 + 2 + \ldots + q) \equiv 0 + 1 + 2 + \ldots + (p-1) \pmod{p}$

$q2 + q \equiv p\,(p-1)/2 \pmod{p}$

Therefore, $p \mid (q^2 + q - p(p-1)/2)$.

Thm. (Lo, 1985) K_p is edge-graceful for p odd.

Thm. (Lee & Seah, 1990) The regular complete k-partite graph $K_{n1,n2,\ldots,nk}$ is edge-graceful if and only if n is odd, and k is either odd or a multiple of 4.

Partial Proof: By Lo's Thm, if a graph is edge-graceful, then $p \mid q^2 + q - p(p-1)/2$. Assume that $K_{n1,n2,\ldots,nk}$ is edge-graceful. For $K_{n1,n2,\ldots,nk}$, there are k partite sets of vertices, each with n vertices, so $p = nk$. Every vertex has degree of $n(k-1)$ and there are nk vertices. The number of edges is obtained by multiplying the degree for each vertex by the number of vertices and dividing by 2 since each edge will be counted twice, once on each vertex, so $q = n^2k(k-1)/2$.

So, $nk \mid (n^2k(k-1)/2)^2 + n^2k(k-1)/2 - nk(nk-1)/2$ must be true. When the right side is simplified and divided by nk, $n^3k(k-1)^2/4 - (n-1)/2$ is obtained, and it must be $\in \mathbf{Z}$.

Case 1. n even

$\rightarrow n^3k(k-1)^2$ even & divisible by 4 $\rightarrow (n-1)/2 \in \mathbf{Z} \rightarrow (n-1)$ even $\rightarrow$ n odd $\Rightarrow\Leftarrow$ the original assumption.

Case 2. n odd

$\rightarrow (n-1)/2 \in \mathbf{Z} \rightarrow n^3k(k-1)^2/4 \in \mathbf{Z}$

n^3 odd

let k be even $\rightarrow (k-1)^2$ odd $\rightarrow n^3k(k-1)^2/4$ is divisible by 4 if k is divisible by 4

let k be odd $\rightarrow$ (k-1)2 is even and divisible by 4.

Therefore, n must be odd, and k must be odd or a multiple of 4. ◊

What I have been working on

I have been trying to determine whether certain classes of graphs are edge-graceful or not edge-graceful. I often use Lo's theorem that states if edge-graceful, then $p \mid q^2 + q - p(p-1)/2$.

I also use a variation derived from $p \mid q^2 + q - p(p-1)/2$: $p(p+m) = 2(q^2+q)$ where m is odd.

pa = q2 + q - p(p-1)/2, where a $\in \mathbf{Z}$

pa + p(p-1)/2 = q2 + q

2ap + p(p-1) = 2 (q2 + q)

2ap + p2 - p = 2(q2 + q)

p2 + (2a -1)p = 2(q2 + q)

p (p + (2a-1)) = 2(q2 + q)

Since a $\in \mathbf{Z}$, 2a-1 must be odd.

Let m = 2a-1

So, p(p+m) = q2 + q - p(p-1)/2, where m is odd ◊

Using this variation, I made a chart starting with q = 3 and found all possible p for each q up to q = 41. I took each q and found $2(q^2 + q)$. Then, I found the prime factorization of $2(q^2 + q)$. Using the prime factorization, I was able to find pairs of factors p(p + m) where one was even and one was odd.

Table 1. Graphs with p vertices and q edges that may be edge-graceful.

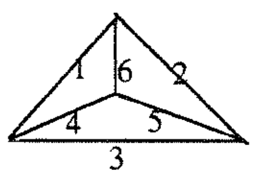
q	$2(q^2+q)$	p
3	24	1, 24, 3, 8
4	40	1, 40, 5, 8
5	60	1, 60, 5, 12, 3, 20, 4, 15
6	84	1, 84, 7, 12, 3, 28, 4, 21
7	112	1, 112, 7, 16
8	144	1, 144, 9, 16, 3, 48
9	180	1, 180, 3, 60, 5, 36, 9, 20, 15, 12, 45, 4
10	220	1, 220, 5, 44, 11, 20
11	264	1, 264, 3, 88, 11, 24, 33, 8
12	312	1, 312, 13, 24, 3, 104, 8, 39
13	364	1, 364, 91, 4, 13, 28, 7, 52
14	420	1, 420, 3, 140, 5, 84, 12, 35, 7, 60
15	480	1, 480, 3, 160, 5, 96, 15, 32
16	544	1, 544, 17, 32
17	612	1, 612, 3, 204, 9, 68, 17, 36, 51, 12
18	684	1, 684, 3, 228, 9, 76, 19, 36, 57, 12, 171, 4
19	760	1, 760, 5, 152, 19, 40, 95, 8
20	840	1, 840, 3, 280, 5, 168, 7, 120, 15, 56, 21, 40, 35, 24
21	924	1, 924, 3, 308, 7, 132, 11, 84, 21, 44, 33, 28, 77, 12

2 2	1 0 1 2	1, 1012, 11, 92, 23, 44, 253, 4
2 3	1 1 0 4	1, 1104, 3, 552, 23, 48, 69, 16
2 4	1 2 0 0	1, 1200, 3, 400, 5, 240, 15, 80, 25, 48, 75, 16
2 5	1 3 0 0	1, 1300, 5, 260, 13, 100, 65, 20, 25, 52
2 6	1 4 0 4	1, 1404, 3, 468, 9, 156, 13, 108, 117, 12, 27, 52, 351, 4, 39, 36
2 7	1 5 1 2	1, 1512, 3 504, 7, 216, 9, 168, 21, 72, 27, 56, 63, 24, 189, 8
2 8	1 6 2 4	1, 1624, 8, 203, 7, 232, 29, 56
2 9	1 7 4 0	4, 435, 3, 580, 5, 148, 29, 60, 15, 116, 20, 87, 12,145
3 0	1 8 6 0	4, 465, 5, 372, 3, 620, 31, 60, 12, 155, 20, 93, 124, 15
3 1	1 9 8 4	64, 31
3 2	2 1 1 2	3, 704, 11, 192, 33, 6 4
3 3	2 2 4 4	4, 561, 3, 748, 11, 204, 17, 132, 33, 68, 51, 44, 12, 187
3 4	2 3 8 0	5, 476, 7 340, 17, 140, 35, 68, 85, 28, 119, 20
3 5	2 5 2 0	3, 840, 5, 504, 7, 360, 9, 280, 15, 168, 21, 120, 35, 72, 45, 56, 63, 40, 24, 105
3 6	2 6 6 4	3, 888, 37, 72, 9, 296, 111, 24, 333, 8
3 7	2 8 1 2	19, 148, 37, 76, 703, 4

38	2964	3, 1040, 5, 624, 13, 240, 39, 76, 57, 52, 4, 741
39	3120	3, 1040, 5, 624, 13, 240, 39, 80,15, 208, 65, 48, 195, 16
40	3280	5, 656, 41, 80, 205, 16
41	3444	4, 861, 12, 287, 28, 123, 21, 164, 3, 1148, 7, 492, 41, 84

A *wheel* is the (n+1, 2n) graph which results from adding 1 central vertex & n edges connecting the new vertex to every vertex of a cycle C_n. Wheels are designated W_n, where *n* refers to the number of vertices in the outer cycle. I examined wheels for edge-gracefulness and found that W_3 is edge-graceful.

Figure 3. W_3



Corollary. Wheels are not edge-graceful except for W_3.

Proof: A wheel W has n+1 vertices and 2n edges. According to Lo, if a graph G with p vertices and q edges is edge-graceful, then $p \mid q^2 + q - p(p-1)/2$.

Assume W is edge-graceful.

$n+1 \mid (2n)^2 + 2n - (n+1)n/2$

$4n^2 + 2n - (n^2+n)/2$

$7n^2/2 + 3n/2$

div by n+1 $\rightarrow 7n/2 - 2 + 2/(n+1) \in \mathbf{Z}$

$\rightarrow 7n/2 + 2/(n+1) \in \mathbf{Z}$

$7n/2 \in \mathbf{Z}$ or halfway between integers

$\rightarrow 2/(n+1) \in \mathbf{Z}$ or halfway between integers

Case 1. $n = 1$

$7/2 + 1 = 9/2 \notin \mathbf{Z}$

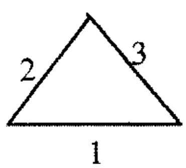
Case 2. $n = 3$

$21/2 + 1/2 = 11 \in \mathbf{Z}$
For $n > 3$, $2/(n+1) < 1/2$, so $7n/2 + 2/(n+1) \notin \mathbf{Z}$ for $n>3$. ◊

I examined complete graphs, $K_{p,}$ for edge-gracefulness. A complete (p, q) graph is a regular graph of degree p-1 having $q = p(p-1)/2$. Research on complete graphs and edge-gracefulness has been previously done by others.

K_3 is edge-graceful, which is a trivial case.

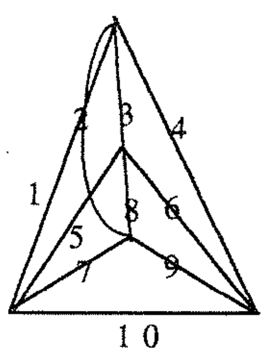
Figure 4. K_3



$K_4 = W_3$ is edge-graceful.

K_5 is edge-graceful.

Figure 5. K_5, which is obtained from $K_4 = W_3$ by adding a vertex and corresponding edges.



A *bicycle* consists of 2 cycles that intersect at a vertex, several edges, an edge, or are joined by a path. There are several types of bicycles, such as figure 8, theta, and handcuff.

Figure 6. Figure 8 graph
2 cycles joined at a vertex

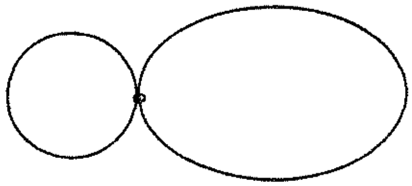


Figure 7. Theta graph
2 cycles joined at an edge or several edges

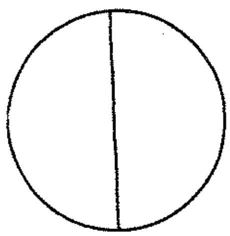
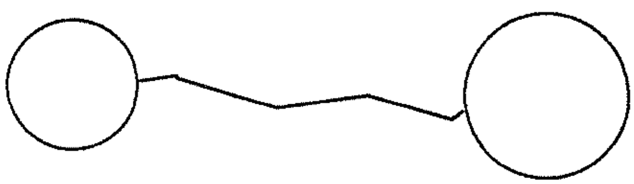


Figure 8. Handcuff graphs
2 cycles joined by a path



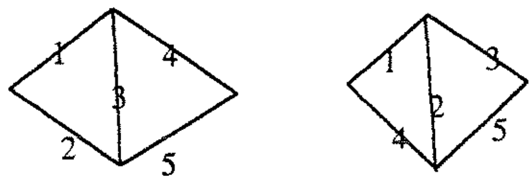
Corollary. Bicycles are not edge-graceful except for q=5.
Proof: Let B be any bicycle. B has q edges and q-1 vertices.
According to Lo, if a graph G with p vertices & q edges is edge-graceful, then $p \mid q^2 + q - p(p-1)/2$.
Assume B is edge-graceful, so
$q-1 \mid q^2 + q - (q-1)(q-2)/2$

$$q^2 + q - (q^2 - 3q + 2)/2 = q^2 + q - q^2/2 + 3q/2 - 1 = q^2/2 + 5q/2 - 1$$

divided by q-1 $\rightarrow q/2 + 3 + 2/(q-1) \in \mathbf{Z}$
$\rightarrow q/2 + 2/(q-1) \in \mathbf{Z}$
$q/2 \in \mathbf{Z}$ or halfway between two integers
$\rightarrow 2/(q-1)$ should be $\in \mathbf{Z}$ or halfway between two integers
Case 1. $q = 3$ (actually trivial, since bicycle has more than 3 edges)
$q/2 + 2/(q-1) = 3/2 + 1 = 5/2 \notin \mathbf{Z}$
Case 2. $q = 5$

$q/2 + 2/(q-1) = 5/2 + 1/2 = 3 \in \mathbf{Z}$
For $q>5$, $2/(q-1) < 1/2$, so $q/2 + 2/(q-1) \notin \mathbf{Z}$ for $q > 5$. ◊

Figure 9. Edge-graceful bicycle labelings for p=4, q=5



This is the only bicycle w/ 5 edges

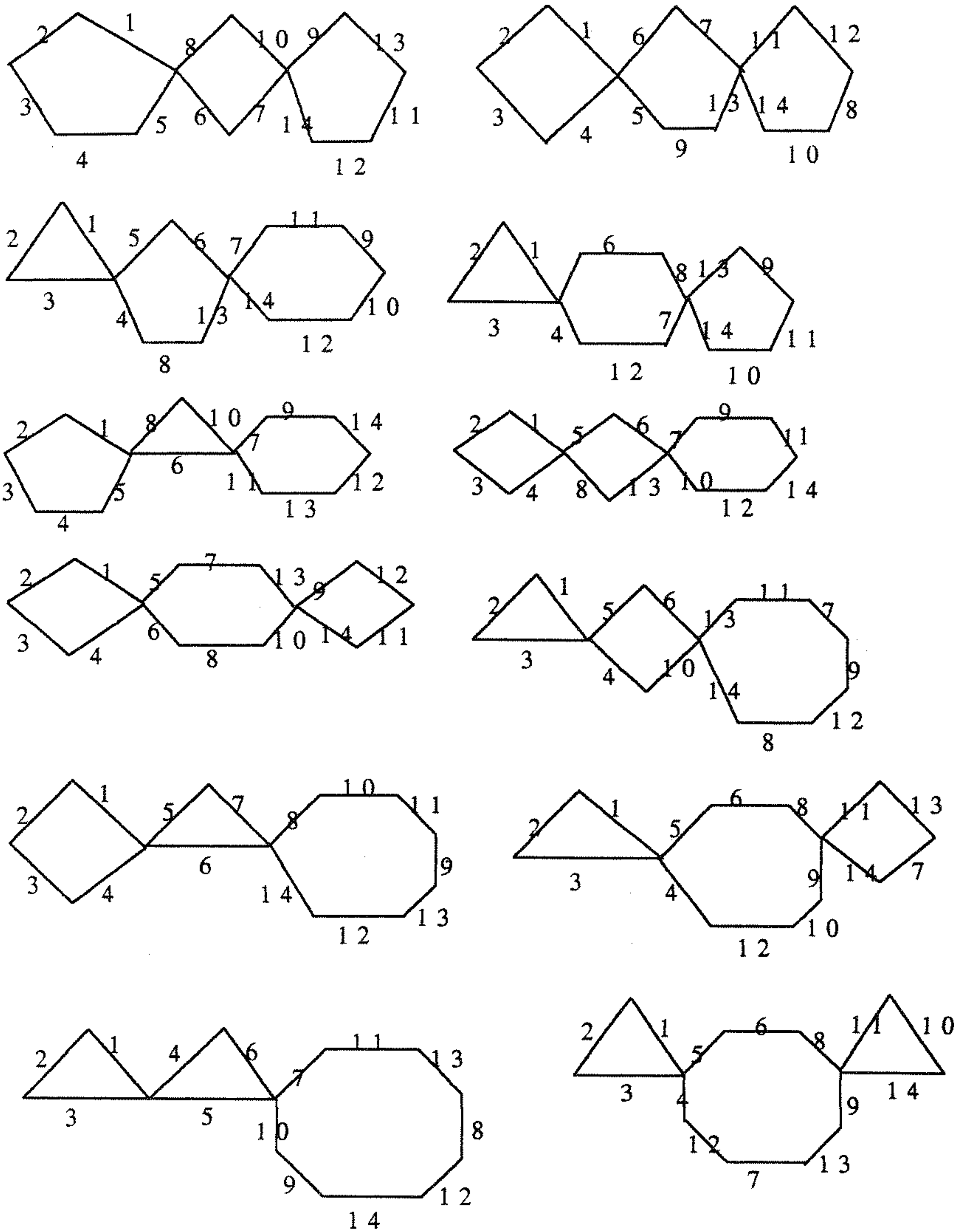
A *tricycle* consists of 3 cycles, rather than 2, with the definition similar to that of bicycles.

Corollary. Tricycles are not edge-graceful except for q= 6, 14.
Proof: Let T be any tricycle. T has q edges & q-2 vertices.
According to Lo, if a graph G with p vertices & q edges is edge-graceful, then $p \mid q^2 + q - p(p-1)/2$.
Assume T is edge-graceful.

$q-2 \mid q^2 + q - (q-2)(q-3)/2$
$\quad q^2/2 + 7q/2 - 3$
div by $q-2 \rightarrow q/2 + 9/2 + 6/(q-2) \in \mathbf{Z}$
$\rightarrow (q+9)/2 + 6/(q-2) \in \mathbf{Z}$
$(q+9)/2 \in \mathbf{Z}$ or halfway between integers
$\rightarrow 6/(q-2) \in \mathbf{Z}$ or halfway between integers
Case 1. $q = 4$
$(q+9)/2 + 6/(q-2) = 13/2 + 6/2 = 19/2$
Case 2. $q = 6$
$(q+9)/2 + 6/(q-2) = 15/2 + 3/2 = 9$
Case 3. $q = 8$
$(q+9)/2 + 6/(q-2) = 17/2 + 1 = 19/2$
Case 4. $q = 10$
$(q+9)/2 + 6/(q-2) = 19/2 + 3/4 = 41/4$
Case 5. $q = 14$
$(q+9)/2 + 6/(q-2) = 23/2 + 1/2 = 12$
For $q > 14$, $6/(q-2) < 1/2$, so $(q+9)/2 + 6/(q-2) \notin \mathbf{Z}$ for $q > 14$. ◊

The edge-graceful tricycle for $p = 4$, $q = 6$ is K_4.

Figure 10. Edge-graceful tricycles for p=12, q=14



-Complete n-partite graphs

Def. A graph G is n-partite, n>1, if it is possible to partition V(G) into n subsets V_1, V_2, ..., V_n (partite sets) such that every element of E(G) joins a vertex of V_i to a vertex of V_j, $i \neq j$

I examined bipartite graphs for edge-gracefulness. Based on Lo's theorem, $K_{3,6}$ may be edge-graceful.

$p = 9$, $q = 18$

$p \mid q^2 + q - p(p-1)/2$

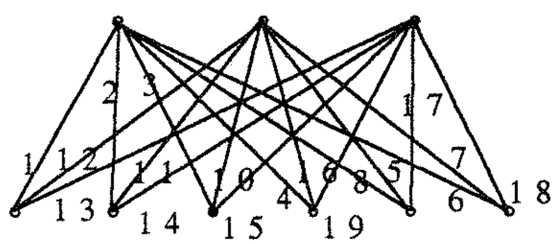
$9 \mid 18^2 + 18 - 9(8)/2$

$324 + 18 - 36 = 306$

$306/9 = 34$

I found an edge-graceful labeling for $K_{3,6}$.

Figure 11. $K_{3,6}$



Based on Lo's theorem, $K_{3n,6n}$ may be edge-graceful for n odd.

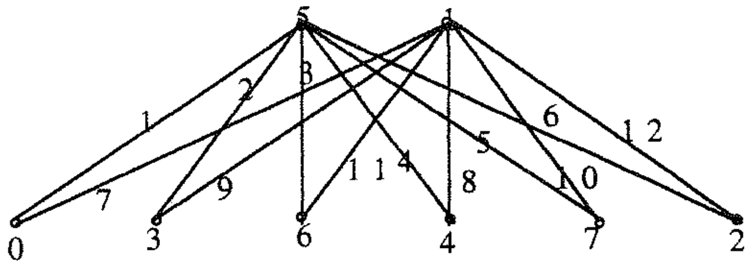
$p = 9n$, $q = 18n^2$

$9n \mid (18n^2)^2 + 18n^2 - 9n(9n-1)/2$

div. by $9n \rightarrow 36n^3 + 2n - (9n-1)/2 \in Z$

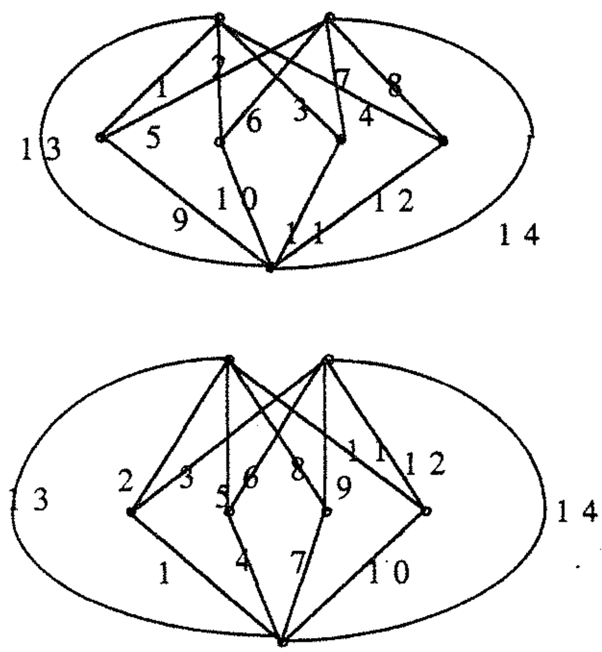
$\rightarrow (9n-1)/2 \in Z \rightarrow 9n-1$ even $\rightarrow 9n$ odd $\rightarrow n$ odd

Figure 12. $K_{2,6}$ is edge-graceful.



I examined tripartite graphs for edge-gracefulness. $K_{1,2,4}$ is edge graceful.

Figure 13. 2 different edge-graceful labelings for $K_{1,2,4}$



$K_{n,2n,4n}$ satisfies the theorem for n odd, so they may be edge-graceful.

$p = 7n$

vertices: n of deg 6n, 2n of deg 5n, 4n of deg 3n

$q = (6n^2 + 10n^2 + 12n^2)/2 = 14n^2$ edges

$7n \mid (14n^2)^2 + 14n^2 - 7n(7n-1)/2$

div by 7n $28n^3 + 2n - (7n-1)/2 \in \mathbf{Z}$

$\rightarrow (7n-1)/2 \in \mathbf{Z} \rightarrow 7n-1$ even $\rightarrow 7n$ odd $\rightarrow$ n odd

K_{n,n^2,n^3} satisfies the equation for n odd, so they may be edge-graceful.

$p = n + n^2 + n^3 = n(1 + n + n^2)$

vertices: n of deg $(n^2 + n^3)$, n^2 of deg $(n + n^3)$, n^3 of deg $(n + n^2)$

$q = [n(n^2 + n^3) + n^2(n + n^3) + n^3(n + n^2)]/2 = n^3 + n^4 + n^5 = n^3 (1 + n + n^2)$

$n(1 + n + n^2) \mid [n^3 (1 + n + n^2)]^2 + n^3 (1 + n + n^2) - n(1 + n + n^2)[n(1 + n + n^2) - 1]/2$

div by $n(1 + n + n^2)$

$n^5(1 + n + n^2) + n^2 - [n(1 + n + n^2) - 1]/2 \in \mathbf{Z}$

$\rightarrow [n(1 + n + n^2) - 1]/2 \in \mathbf{Z} \rightarrow n(1 + n + n^2) -1$ even $\rightarrow n(1 + n + n^2)$ odd

Let n be even $\rightarrow n + 1$ odd, n^2 even $\rightarrow (1 + n + n^2)$ odd $\rightarrow n(1 + n + n^2)$ even $\Rightarrow\Leftarrow$

Other tripartite graphs that may be edge-graceful (satisfy Lo's Thm where $p \mid q^2 + q - p(p-1)/2$)

$K_{1,3,4}$ $p = 8, q = 19$

$K_{1,1,10}$ $p = 12, q = 21$

$K_{1,2,9}$ $p = 12, q = 29$

$K_{1,5,6}$ $p = 12, q = 41$

$K_{1,3,9}$ $p = 13, q = 39$, $K_{n,3n,9n}$ for n odd